# VISUALIZING GEOMETRY WITH THE HELP OF "MATHEMATICA"

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Abstract. Among eight possible geometric structures on three-dimensional manifolds less studied from the differential geometric point of view are those modeled on the Heisenberg group $Heis^3$ and $PSL_2(R)$ - the group of all isometries of the Lobachevsky plane, see [Sc]. We consider their left-invariant metrics, define Levi-Civita connections and curvature tensors, define and solve equations of geodesic lines. Using "Mathematica" software package we also present drawings of geodesic lines and metric balls in these spaces.

## 1. Geometry of the Heisenberg group $Heis^3$

**1.1. Left-invariant metric, Levi-Civita connection and curvature tensor of $Heis^3$.** We begin with a well-known description of the Heisenberg group of dimension 3. Let $R^3$ be the euclidean space with coordinates $(x, y, z)$. Then the Heisenberg group $Heis^3$ is this space with the following multiplication rule:

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + (< x, y' > - < x', y >)), \tag{(1.1)}$$

where $<,>$ is a scalar product in $R^2 = \{(x, y)\}$. The element zero $0 = (0, ..., 0)$ is the unit of this group structure and the vector fields

$$X = (1, 0, -y), \quad Y = (0, 1, x), \quad T = (0, 0, 1) \tag{1.2}$$

are their left-invariant fields. We define the left-invariant metric on $Heis^3$ by taking $X, Y, T$ as **the orthonormal frame**.

**Definition 1.** Denote by $g$ the left-invariant metric on $Heis^3$ such that vector fields $X$, $Y$ and $T$ are orthonormal ones. The corresponding scalar product we denote as usual by ( , ).

Because due to (1.2) coordinate vectors are

$$\frac{\partial}{\partial x} = (X + yT), \qquad \frac{\partial}{\partial y} = (Y - xT), \quad \text{and} \quad \frac{\partial}{\partial z} = T,$$

and by our choice $\{X, Y, T\}$ is an orthonormal basis we arrive at the following formula for the metric tensor of our left-invariant metric in coordinates $(x, y, z)$:

$$g = \begin{pmatrix} 1 + y^2 & -xy & y \\ -xy & 1 + x^2 & -x \\ y & -x & 1 \end{pmatrix}. \tag{1.3}$$

The following was proved in [Mr1].

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**Proposition 1.** *For the covariant derivatives of the Riemannian connection of the left-invariant metric, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & T & -Y \\ -T & 0 & X \\ -Y & X & 0 \end{pmatrix}, \tag{1.4}$$

*where the* $(i,j)$*-element in the table above equals* $\nabla_{E_i}E_j$ *for our basis*

$$\{E_k, k=1,2,3\} = \{X, Y, T\}.$$

**1.2. Geodesic lines in** $Heis^3$**.** We find equations of geodesics issuing from $\mathbf{0}$=(0,0,0). Let $c(t)$ be such a geodesics with a natural parameter $t$, and its vector of velocity given by

$$\dot{c}(t) = \alpha(t)X(t) + \beta(t)Y(t) + \gamma(t)T. \tag{1.5}$$

Then the equation of a geodesic $\nabla_{\dot{c}(t)}\dot{c}(t) \equiv 0$ and our table of covariant derivatives (1.3) give:

$$(\alpha'(t) + 2\gamma\beta(t))X(t) + (\beta'(t) - 2\gamma\alpha(t))Y(t) + \gamma'(t)T = 0.$$

Thus we easily obtain the following equations for coordinates of the vector of velocity of the geodesic $c(t)$ in our left-invariant moving frame:

$$\begin{cases} \alpha'(t) + 2\gamma\beta(t) = 0 \\ \beta'(t) - 2\gamma\alpha(t) = 0 \end{cases} \quad \gamma'(t) = 0 \tag{1.6}$$

or

$$\begin{cases} (\alpha(t) + \beta(t))' - 2\gamma(\alpha(t) - \beta(t)) = 0 \\ (\alpha(t) - \beta(t))' + 2\gamma(\alpha(t) + \beta(t)) = 0 \end{cases} \quad \gamma'(t) = 0 \tag{1.7}$$

Because the parameter $t$ is natural we have

$$\alpha^2(t) + \beta^2(t)) + \gamma^2 \equiv 1,$$

and we could take $\gamma(t) \equiv \gamma$ where $|\gamma| \leq 1$ is the cos of the angle between $\dot{c}(0)$ and $T$-axe. For $|\gamma| = 1$ we have "vertical" geodesic, coinciding with $z$-axe, which is an integral line of the left-invariant vector field $T$. For $\gamma = 0$ our equations are linear. For $\gamma \neq 0$ after some easy computation one could find that:

$$\begin{cases} \alpha(t) = r\cos(2\gamma t + \phi) \\ \beta(t) = r\sin(2\gamma t + \phi) \end{cases} \tag{1.8}$$

where $r = \sqrt{\alpha^2 + \beta^2}$. To find in coordinates $\{x, y, z\}$ equations for geodesics $c(t) = (x(t), y(t), z(t))$ issuing from $\mathbf{0}$ note, that if

$$\dot{c}(t) = \alpha(t)X(t) + \beta(t)Y(t) + \gamma(t)T$$

and our left-invariant vector fields are

$$X = (1, 0, -y), \quad Y = (o, 1, x), \quad T = (0, 0, 1),$$

then

$$\frac{\partial}{\partial x} = X + yT, \quad \text{and} \quad \frac{\partial}{\partial y} = Y - xT,$$

so we easily have:

$$\begin{cases} \dot{x}(t) = \alpha(t) \\ \dot{y}(t) = \beta(t) \\ \dot{z}(t) = \gamma - \alpha(t)y(t) + \beta(t)x(t) \end{cases} \tag{1.9}$$

that after some computations gives the following equations for geodesics issuing from zero:

**Proposition 2.** *Geodesic lines issuing from zero* **0** *in the Heisenberg group* $Heis^3$ *satisfy to the following equations:*

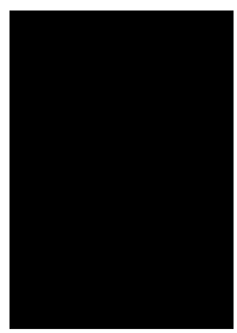
$$\begin{cases} x(t) = \frac{r}{2\gamma}(\sin(2\gamma t + \phi) - \sin(\phi)) \\ y(t) = \frac{r}{2\gamma}(\cos(\phi) - \cos(2\gamma t + \phi)) \\ z(t) = \frac{1+\gamma^2}{2\gamma} t - \frac{1-\gamma^2}{4\gamma^2}\sin(2\gamma t) \end{cases} \tag{1.10}$$

*for some numbers* $\phi$*,* $r$ *which could be defined from the initial condition* $\dot{c}(0) = (r\cos(\phi), r\sin(\phi), \gamma)$*; or if* $\gamma = 0$*, then they are "horizontal" and satisfy the following equations:*

$$\begin{cases} x(t) = \alpha(0)t \\ y(t) = \beta(0)t \\ z(t) = 0 \end{cases} \tag{1.11}$$

To find geodesics issuing from an arbitrary point and their equations it is sufficient to use left translation to this point and apply to the equation above the multiplication rule (1).

### 1.3. Computer generated pictures of geodesic lines and metric balls in $H^3$.



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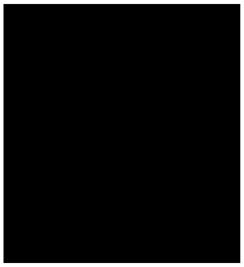
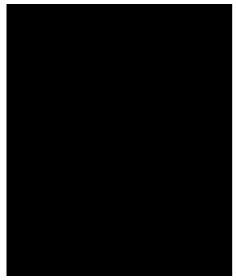
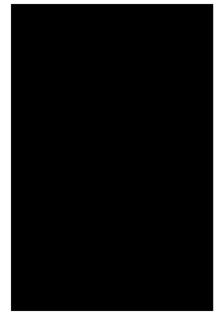
1. Two geodesic lines issuing from **0** in directions of $X$ and $Y$ axes for $t \in [-10, 10]$.
2. Exp-image of the $\{X, T\}$-ccordinate plane.

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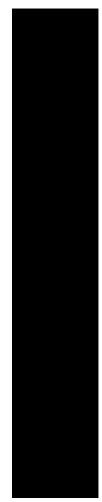

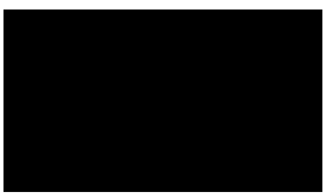

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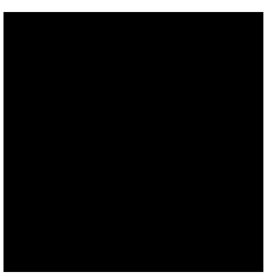
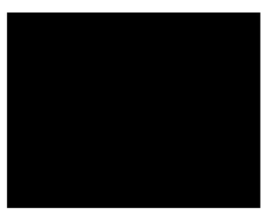

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1. Metric ball with the center at **0** and radius 1.
2. Metric ball with the center at **0** and radius 3.
3. Half of the ball of radius 5.
4. Focal point of the exponential map: singular point of the sphere with radius 5
5. Amplified singular point of the sphere of radius 5.
6. Amplified neighborhood of the singular point of the same sphere.
7. Singular point of the metric sphere of radius $t = 20$.

## II. On the geometry of $PSL_2(R)$

**2.1. Ball model of Lobachevsky plane and global cordinates in $\tilde{P}SL_2(R)$.**
In a standard way we identify the Lobachevsky hyperbolic plane $H^2$ with the unit disk $B^2 = \{z \in C| \, \|z\| < 1\}$ in the complex plane with the metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2},$$

where $z = x + iy$. In this coordinate system isometries can be described as transformations of $B^2$ of the form

$$z \to w(z) = -e^{i\phi} \frac{z - \alpha}{\bar{\alpha} z - 1} \quad \text{for} \quad \alpha \in B^2, 0 \le \phi \le 2\pi$$

(which is a composition of a parallel translation taking the point 0 to $\alpha$ and rotation on angle $\phi$). Then $(\alpha, \phi)$ for $\alpha \in B^2$ and $\phi \in S^1 = \{z \in C| \|z\| = 1\}$ are coordinates in the group $PSL_2(R)$ of all isometries of $H^2$. Respectively, in the universal cover $P\tilde{S}L_2(R)$ of $PSL_2(R)$ the system of global coordinates is $(\alpha, \phi)$, where $\phi \in R$. Below we denote by $I(\alpha, \phi)$ the isometry with coordinates $(\alpha, \phi)$ and also $\alpha = x + iy$, so that $(x, y, \phi)$ are our global coordinates in $P\tilde{S}L_2(R)$.

**2.2. Multiplication law in $P\tilde{S}L_2(R)$.**
In the coordinate system which we have introduced above the multiplication law giving coordinates of a composition of two isometries with coordinates $(\alpha, \phi)$ and $(\beta, \psi)$ can be defined in the following way. Because

$$I(\alpha, \phi) \circ I(\beta, \psi)(z) = -e^{\phi+\psi} \frac{\gamma}{\bar{\gamma}} \frac{z - \delta}{\bar{\delta} z - 1}$$

for $\gamma = \gamma(\alpha, \beta) = 1 + \alpha\bar{\beta}e^{-i\psi}$ and $\delta = \delta(\alpha, \beta) = (\beta + \alpha e^{-i\psi})/\gamma$, we deduce that

$$I(\alpha, \phi) \circ I(\beta, \psi) = I(\delta, \phi + \psi + 2Arg(\gamma))$$

which we simply denote as

$$(\alpha, \phi) \circ (\beta, \psi) = (\delta(\alpha, \beta), \phi + \psi + 2Arg(\gamma(\alpha, \beta))). \tag{2.1}$$

**2.3. Left-invariant vector fields, Lie algebra $psl_2(R)$ and metric tensor.** Take three coordinate vectors $X = \partial/\partial x$, $Y = \partial/\partial y$ and $T = \partial/\partial\phi$ of the coordinate system introduced above at the point $0 = (0, 0, 0)$, and applying the differential of the multiplication (2.1) by $(\alpha, \phi)$ from the left we obtain three left invariant vector fields $X(\alpha, \phi)$, $Y(\alpha, \phi)$ and $T(\alpha, \phi)$ correspondingly. As direct calculations show these vector fields in our coordinates $(x, y, \phi)$ are
(2.2)

$$X(x, y, \phi) = (1 - x^2 + y^2, -2xy, 2y), \quad Y(x, y, \phi) = (-2xy, 1 + x^2 - y^2, -2x) \quad \text{and} \quad T(x, y, \phi) = (y, -x, 1).$$

Also direct computations show that

$$[X, Y] = -4T \qquad [T, X] = Y \qquad [Y, T] = X,$$

or taking from now on $E_1 = X/2$, $E_1 = Y/2$ and $E_3 = T$ we have for the left-invariant vector fields $E_i, i = 1, 2, 3$

$$[E_1, E_2] = \lambda_3 E_3 \qquad [E_3, E_1] = \lambda_2 E_2 \qquad [E_2, E_3] = \lambda_1 E_1$$

where

$$\lambda_1 = \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = -1.$$

**Definition 2.** Denote by $g$ the left-invariant metric on $P\tilde{S}L_2(R)$ such that vector fields $E_i, i = 1,2,3$ are orthonormal ones. The corresponding scalar product we denote as usual by ( , ).

Because due to (2.2) coordinate vectors are

$$\frac{\partial}{\partial x} = \frac{1}{1-(x^2+y^2)}(2E_1 - 2yE_3), \qquad \frac{\partial}{\partial y} = \frac{1}{1-(x^2+y^2)}(2E_2 + 2xE_3),$$

and

$$\frac{\partial}{\partial \phi} = E_3 - y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{1}{1-(x^2+y^2)}(-2yE_1 + 2xE_2 + (1+x^2+y^2)E_3),$$

and by our choice $\{E_1, E_2, E_3\}$ is an orthonormal basis we arrive at the following formula for the metric tensor of our left-invariant metric in coordinates $(x, y, \phi)$:

$$(2.3) \quad g = \frac{1}{(1-(x^2+y^2))^2}\begin{pmatrix} 4+4y^2 & -4xy & -4y-2y(1+x^2+y^2) \\ -4xy & 4+4x^2 & 4x+2x(1+x^2+y^2) \\ -4y-2y(1+x^2+y^2) & 4x+2x(1+x^2+y^2) & (1+x^2+y^2)^2+4x^2+4y^2 \end{pmatrix}.$$

**2.4. Curvature tensor of $\tilde{P}SL_2(R)$.**

Applying well known formulas (see [M]) for the curvature tensor of the left-invariant metric $g$ we obtain the following. Ricci curvature of $g$:

$$(2.4) \qquad Ric(E_1) = -\frac{3}{2} \quad Ric(E_2) = -\frac{3}{2} \quad Ric(E_3) = \frac{1}{2},$$

scalar curvature

$$(2.5) \qquad \rho = -\frac{5}{2},$$

and for sectional curvatures $K_{ij}$ in two-dimensional directions generated by $\{E_i, E_j\}$

$$(2.6) \qquad K_{12} = -\frac{7}{4} \quad K_{13} = \frac{1}{4} \quad K_{23} = \frac{1}{4}.$$

Note, that there is a natural projection $\pi : P\tilde{S}L_2(R) \to B^2$ such that $\pi(x, y, \phi) = (x, y)$ which is a riemannian submersion (i.e., is an isometry in horizontal directions normal to the fibres of the projection $\pi$). Indeed, by the well-known O'Neill's formula (see [O'N]) we see that the sectional curvature of the base $H^2$ of this submersion equals

$$(2.7) \qquad K(H^2) = K_{12} + \frac{3}{4}\|[E_1, E_2]\|^2 = -\frac{7}{4} + \frac{3}{4} = -1.$$

**2.5. Cristoffel symbols of Levi-Civita connection.**

**Proposition 3, see [Mr2].** *For the covariant derivatives of the riemannian connection of the left-invariant metric $g$ defined above the following is true:*

$$(2.8) \qquad \nabla = \begin{pmatrix} 0 & -\frac{1}{2}E_3 & 2E_2 \\ \frac{1}{2}E_3 & 0 & 2E_1 \\ 3E_2 & -3E_1 & 0 \end{pmatrix},$$

*where the $(i, j)$-element in the table above equals $\nabla_{E_i} E_j$ for our orthonormal basis $E_i, i = 1, 2, 3$.*

**2.6. Equation of geodesics.**

First we find equations of geodesics issuing from $\mathbf{0}$=(0,0,0). Let $c(t)$ be such a geodesics with a natural parameter $t$, and its vector of velocity is given by

$$\dot{c}(t) = u'(t)E_1(c(t)) + v'(t)E_2(c(t)) + w'(t)E_3(c(t)). \tag{2.9}$$

Then the equation of a geodesic $\nabla_{\dot{c}(t)}\dot{c}(t) \equiv 0$ and our table of covariant derivatives (2.8) give:

$$(u''(t) - 5v'(t)w'(t))E_1(c(t)) + (v''(t) + 5u'(t)w'(t))E_2(c(t)) + w''(t)E_3(c(t)) = 0,$$

which easily gives

$$w'(t) \equiv w'(0) \tag{2.10}$$

and

$$u'(t) = u'(0)\cos(5w'(0)t) + v'(0)\sin(5w'(0)t) \quad \text{and } v'(t) = v'(0)\cos(5w'(0)t) - u'(0)\sin(5w'(0)t). \tag{2.11}$$

To find equations for the coordinates $(x(t), y(t), \phi(t))$ of the geodesic line $c(t)$ in our coordinate system $(x, y, \phi)$ recall that due to (2.2) we have

$$E_1 = (1 - x^2 + y^2)\frac{\partial}{\partial x} + (-2xy)\frac{\partial}{\partial y} + (2y)\frac{\partial}{\partial \phi}, E_2 = (-2xy)\frac{\partial}{\partial x} + (1 + x^2 - y^2)\frac{\partial}{\partial y} + (-2x)\frac{\partial}{\partial \phi},$$

and

$$E_3 = y\frac{\partial}{\partial x} + (-x)\frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}. \tag{2.12}$$

Therefore, (2.10)-(2.11) substituted in (2.9) gives

$$\dot{c}(t) = (u'(t)(1 - x^2 + y^2) + v'(t)(-2xy) + w'(0)y)\frac{\partial}{\partial x} +$$

$$(u'(t)(-2xy) + v'(t)(1 + x^2 - y^2) - w'(0)x)\frac{\partial}{\partial y} + (u'(t)(2y) + v'(t)(-2x) + w'(0))\frac{\partial}{\partial \phi}, \tag{2.13}$$

so that from $\dot{c}(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y} + \phi'(t)\frac{\partial}{\partial \phi}$ and that obviously $x'(0) = u'(0)$, $y'(0) = v'(0)$ and $\phi'(0) = w'(0)$ we arrive at the following equations

(2.14)
$$\begin{aligned}
\dot{x}(t) = & \ (1 - x^2(t) + y^2(t))(u'\cos(5w't) + v'\sin(5w't)) + (-2x(t)y(t))(v'\cos(5w't) - u'\sin(5w't)) + w'y(t) \\
\dot{y}(t) = & \ (-2x(t)y(t))(u'\cos(5w't) + v'\sin(5w't)) + (1 + x^2(t) - y^2(t))(v'\cos(5w't) - u'\sin(5w't)) - w'x(t)\,, \\
\dot{\phi}(t) = & \ (2y(t))(u'\cos(5w't) + v'\sin(5w't)) + (-2x(t))(v'\cos(5w't) - u'\sin(5w't)) + w'
\end{aligned}$$

where we drop argument 0 in $u'(0)$, $v'(0)$ and $w'(0)$ so that $(u', v'w')$ equals the unit vector $\dot{c}(0)$. First two equations are independent of the third one, hence we first concentrate on the equation for $\alpha(t) = (x(t)+iy(t))$. If we denote $u'\cos(5w't) + v'\sin(5w't) = \sqrt{1 - (w')^2}\cos(\tau)$, then $v'\cos(5w't) - u'\sin(5w't) = (\sqrt{1 - (w')^2})\sin(\tau)$ for $\tau = 5w't$, and due to $x^2 - y^2 = Re(\alpha^2)$ and $2xy = Im(\alpha^2)$ we find

$$(\dot{x}(t) + i\dot{y}(t)) = \dot{\alpha}(t) =$$

$$\sqrt{1-(w')^2}((1-Re(\alpha^2(t)))\cos(\tau)+(iIm(\alpha^2(t)))(i\sin(\tau)+(-iIm(\alpha^2(t)))\cos(\tau)+i(1+Re(\alpha^2(t)))\sin(\tau))$$

$$-(iw')(iIm(\alpha(t)))-(iw')Re(\alpha(t))=$$

$$\sqrt{1-(w')^2}((1-\alpha^2(t)))\cos(\tau)+(1+\alpha^2(t))(i\sin(\tau)))-iw'\alpha(t)=$$

$$\sqrt{1-(w')^2}(Exp^{i\tau}-\alpha^2(t)Exp^{-i\tau})-iw'\alpha(t). \tag{2.15}$$

Changing variable $t$ to $s=\sqrt{1-(w')^2}t$ last equality yields for

$$u(s)=\alpha(s/\sqrt{1-(w')^2})Exp^{i\frac{w's}{\sqrt{1-(w')^2}}}$$

the following equations

$$\dot{u}(s)=Exp^{isA}-u^2(s)Exp^{-isA}$$

where $A=6w'/\sqrt{1-(w')^2}$, which in turn gives Ricatti equation

$$\dot{r}(s)=1+(iA)r(s)-r^2(s)$$

for $r(s)=Exp^{iAs}u^{-1}(s)$. Finally, with the help of software package "Mathematica" we get

$$r(s)=\frac{1}{2}(Ai+(\sqrt{A^2-4})tg[-\frac{\sqrt{A^2-4}}{2}s+C_1]), \tag{2.16}$$

where $C_1=\pi/2$ due to initial conditions. Finally, we arrive at the following formula for the first two coordinates of the geodesic line:

$$\alpha(t)=\frac{Exp^{\frac{5w'}{\sqrt{1-(w')^2}}s}}{\frac{1}{2}(\frac{6w'i}{\sqrt{1-(w')^2}}+(\sqrt{\frac{40(w')^2-4}{1-(w')^2}})ctg[(\sqrt{10(w')^2-1})t])}. \tag{2.17}$$

Substituting $x(t)=Re(\alpha(t))$ and $y(t)=Im(\alpha(t))$ into the equation (2.14) for the third coordinate, we define $\phi(t)$, and with the help of "Mathematica" obtain the following pictures describing geodesic lines and metric balls in $PSL_2(R)$.

**2.7. Computer generated pictures of geodesic lines in $PSL_2(R)$.**

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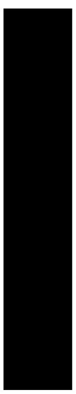

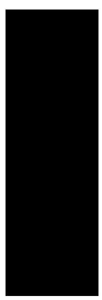

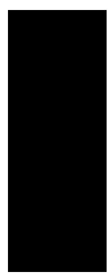

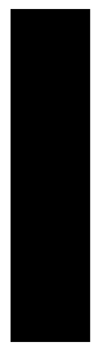

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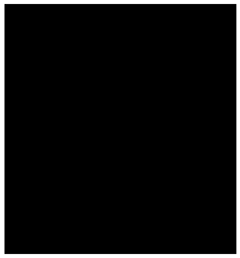

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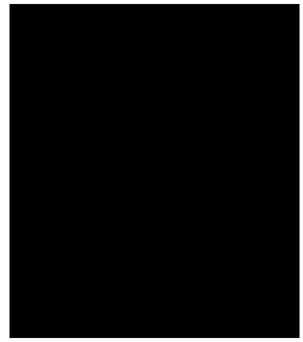

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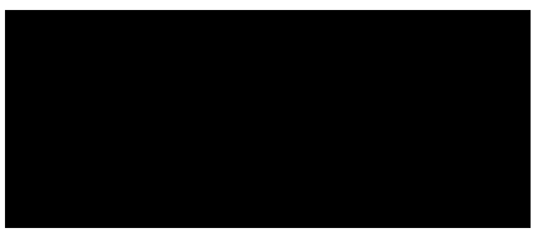

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1. Horizontal projection of the geodesic line issuing from **0** in the direction $(0.91, 0, 0.3)$.
2. Horizontal projection of the geodesic line issuing from **0** in the direction $(0.84, 0, 0.4)$.
3. Horizontal projection of the geodesic line issuing from **0** in the direction $(0.19, 0, 0.9)$.
4. Geodesic line issuing from **0** in the direction $(0.9975, 0, 0.05)$
5. Geodesic line issuing from **0** in the direction $(0.99, 0, 0.1)$
6. Geodesic line issuing from **0** in the direction $(0.91, 0, 0.3)$
7. Geodesic line issuing from **0** in the direction $(0.64, 0, 0.6)$
8. Metric ball of the radius 0.1.
9. Half of the sphere of the radius 1.
10. Part of the $\{X, \Phi\}$ coordinate plane in the neighborhood of the focal point.

## References

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