Nonexistence of invariant semigroups in affine symmetric spaces

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Abstract

Let (G, L, τ) be an affine symmetric space with G a simple Lie group, τ an involutive automorphism of G and L an open subgroup of the τ-fixed point group G^τ. It is proved here that the existence of a proper semigroup $S \subset G$ with $\mathrm{int} S \neq \emptyset$ and $L \subset S$ implies that (G, L, τ) is of Hermitian type, as conjectured by Hilgert and Neeb [4]. When S exists, it turns out that it leaves invariant an open L-orbit in a minimal flag manifold of G. A byproduct of our approach is an alternate proof of the maximality of the compression semigroup of an open orbit (see Hilgert and Neeb [3]).

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Introduction

Let G be a connected semi-simple Lie group and $\tau \neq 1$ an involutive automorphism of G. Semigroups containing the group G^τ of τ-fixed points,

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or rather an open subgroup L of G^{τ}, have been studied extensively in the literature, due mainly to their importance in harmonic analysis of causal symmetric spaces. We refer to [1], [2], [3], [4], [5], [6], [7], [8], [9] and references therein for the literature on semigroups and affine symmetric spaces. The theory developed so far highlights affine symmetric spaces of Hermitian – or regular – type, and the corresponding Ol'shanskiĭ semigroups. For this class there are available invariant cones in the Lie algebra which generate proper semigroups invariant under a subgroup of τ-fixed points. It turns out that a great deal of the theory is developed with the aid of the compression semigroups of the open L-orbits in the flag manifolds.

When studying these compression semigroups one is faced with a gap in the theory concerning the existence of proper semigroups containing L when the symmetric space is not of Hermitian type. Of course, it is known that only in the Hermitian case are there invariant cones in the Lie algebra. However, this is not an obstruction for the existence of an L-invariant semigroup having trivial Lie wedge.

The purpose of this paper is to fill this gap by proving that if (G, τ) is not of Hermitian type and S is a semigroup containing L then either $S = G$ or $\mathrm{int} S = \emptyset$. This result was conjectured before by Hilgert and Neeb [4] (see Conjecture III in [4] whose statement amounts to our formulation).

Precisely, we assume that G is a simple noncompact and connected Lie group having finite center, and prove that if $S = S_C$ is the compression semigroup of a subset C of a flag manifold of G (i.e., $S = \{g \in G : gC \subset C\}$), such that $L \subset S$, $S \neq G$ and $\mathrm{int} S \neq \emptyset$ then the pair (G, τ) is of Hermitian type and C is an open L-orbit having specific properties. Since any semigroup with nonempty interior is contained in a compression semigroup, the more general result stated above follows.

Our proof is based on previous results about the action of semigroups on flag manifolds. The main point in the proof is a relation established between two subgroups of the Weyl group $\mathcal{W}$, namely the subgroup $\mathcal{W}^{a}(\mathfrak{a})$ associated to the pair (G, τ) and the subgroup $\mathcal{W}(S)$ built from S. The subgroup $\mathcal{W}(S)$ is parabolic and hence has a fixed point in its action on the split subalgebra. This will imply the existence of a fixed point of the subgroup $\mathcal{W}^{a}(\mathfrak{a})$, ensuring that the affine symmetric space is of Hermitian type. In the course of the proof the subgroup $\mathcal{W}(S)$ is computed explicitly when S is an invariant compression semigroup with nonempty interior. From the knowledge of $\mathcal{W}(S)$ it is easy to show that the existing compression semigroups are maximal, a result already proved by Hilgert and Neeb [3].

1 Flag manifolds and Semigroups

This section is preparatory for the proof of the main results. We recall here the basic constructions and results about the action of semigroups on flag manifolds. Most of the results where proved elsewhere, so we just state them or outline their proofs. We refer the reader to [10], [11], [12], [13] for further details.

1.1 Flag manifolds

Let $\mathfrak{g}$ be a noncompact semi-simple Lie algebra. We view a flag manifold of $\mathfrak{g}$ as an orbit of a parabolic subalgebra of $\mathfrak{g}$ under the group $\mathrm{Aut}_0(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$. In the sequel we label a flag manifold by a subset of the set of simple roots as follows: Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition with Cartan involution θ and choose a maximal abelian subspace $\mathfrak{a}_0 \subset \mathfrak{p}$. Denote by Π the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a}_0)$, and select a simple system of roots $\Sigma_0 \subset \Pi$, with Π^+ the corresponding set of positive roots and $\mathfrak{a}_0^+$ the Weyl chamber. For a root $\alpha \in \Pi$ its root space is denoted by $\mathfrak{g}_\alpha$. Any parabolic subalgebra is conjugate under an inner automorphism of $\mathfrak{g}$ to a standard parabolic subalgebra

$$\mathfrak{b}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{b}^+,$$

with $\Theta \subset \Sigma_0$ a proper subset. In this expression $\mathfrak{b}^+$ stands for the minimal parabolic subalgebra $\mathfrak{b}^+ = \mathfrak{m} + \mathfrak{a}_0 + \mathfrak{n}^+$ where $\mathfrak{n}^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$ and $\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a}_0)$. Also, $\mathfrak{n}^-(\Theta)$ is the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle \Theta \rangle$, where $\langle \Theta \rangle$ is the set of positive roots generated by Θ. The flag manifold containing $\mathfrak{b}_\Theta$ is denoted by $\mathbb{B}_\Theta$. For the maximal flag manifold we write simply $\mathbb{B} = \mathbb{B}_\emptyset$. Recall that for any flag manifold $\mathbb{B}_\Theta$ there is a standard fibration $\pi : \mathbb{B} \to \mathbb{B}_\Theta$, which assigns to the minimal parabolic subalgebra $\mathfrak{b}$ the unique parabolic subalgebra $\pi(\mathfrak{b})$ in $\mathbb{B}_\Theta$ containing $\mathfrak{b}$.

Let G be a connected Lie group with finite center and Lie algebra $\mathfrak{g}$. The group acts on the flag manifolds of $\mathfrak{g}$ through the adjoint representation of G in $\mathfrak{g}$. We denote this representation by $g \cdot X$, $g \in G$, $X \in \mathfrak{g}$. Let N^+ be the nilpotent connected subgroup of G whose Lie algebra is $\mathfrak{n}^+$. In any flag manifold $\mathbb{B}_\Theta$ there is just one open and dense N^+-orbit, say σ_0. By an open Bruhat cell in $\mathbb{B}_\Theta$ we mean a subset of the form $g\sigma_0$, $g \in G$. Equivalently, the cell $g\sigma_0$ is the open and dense orbit of the group gN^+g^{-1}.

For our purposes it is relevant to have a description of the open cells in terms of incidence of parabolic subalgebras. Starting with $\mathbb{B}$, define an action of G on $\mathbb{B} \times \mathbb{B}$ by putting $g \cdot (\mathfrak{b}_1, \mathfrak{b}_2) = (g \cdot \mathfrak{b}_1, g \cdot \mathfrak{b}_2)$. Let $\mathfrak{b}^+$ be the standard minimal parabolic subalgebra defined above, and put $\mathfrak{b}^- = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}^-$, where $\mathfrak{n}^- = \sum_{\alpha \in \Pi^+} \mathfrak{g}_{-\alpha}$. Two parabolic subalgebras $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathbb{B}$ are said to be opposed (notation: $\mathfrak{b}_1 \pitchfork \mathfrak{b}_2$) in case $(\mathfrak{b}_1, \mathfrak{b}_2)$ belong to the G-orbit of $(\mathfrak{b}^+, \mathfrak{b}^-)$. This definition yields that the set of subalgebras opposed to $\mathfrak{b}^+$ is the orbit $N^+ \cdot \mathfrak{b}^-$, that is, the open cell σ_0. In fact, $\mathfrak{b}$ is opposed to $\mathfrak{b}^+$ if and only if $\mathfrak{b} = g \cdot \mathfrak{b}^-$ with $g \cdot \mathfrak{b}^+ = \mathfrak{b}^+$. This means that $g \in P$, the normalizer of $\mathfrak{b}^+$ in G. But it is well known that $P \cdot \mathfrak{b}^- = N^+ \cdot \mathfrak{b}^-$.

This characterization of σ_0 is carried over to arbitrary cells through the G-action. In general, given $\mathfrak{b} = g \cdot \mathfrak{b}^+ \in \mathbb{B}$ the set of subalgebras opposed to it is the cell $g\sigma_0$. Therefore, any open cell in $\mathbb{B}$ is the set of subalgebras opposed to a given minimal parabolic subalgebra $\mathfrak{b}$. In what follows we put $\sigma(\mathfrak{b})$ for the cell of subalgebras opposed to $\mathfrak{b}$.

Now, for a subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ denote by $\mathfrak{nil}(\mathfrak{c})$ its nilradical. We have $\mathfrak{nil}(\mathfrak{b}^+) = \mathfrak{n}^+$, $\mathfrak{nil}(\mathfrak{b}^-) = \mathfrak{n}^-$ and $\mathfrak{b}^+ \cap \mathfrak{nil}(\mathfrak{b}^-) = \mathfrak{b}^- \cap \mathfrak{nil}(\mathfrak{b}^+) = \{0\}$. Furthermore, given $n \in N^+$,

$$\left(n \cdot \mathfrak{b}^-\right) \cap \mathfrak{nil}\left(\mathfrak{b}^+\right) = n \cdot \left(\mathfrak{b}^- \cap \mathfrak{nil}\left(\mathfrak{b}^+\right)\right) = \{0\}$$

and $\mathfrak{b}^+ \cap \mathfrak{nil}(n \cdot \mathfrak{b}^-) = n \cdot (\mathfrak{b}^+ \cap \mathfrak{nil}(\mathfrak{b}^-)) = \{0\}$. In other words, $\mathfrak{b} \pitchfork \mathfrak{b}^+$ if and only if $\mathfrak{nil}(\mathfrak{b})$ and $\mathfrak{nil}(\mathfrak{b}^+)$ are transversal to $\mathfrak{b}^+$ and $\mathfrak{b}$, respectively.

Actually, $\mathfrak{b}^+ \cap \mathfrak{nil}(\mathfrak{b}) = \{0\}$ if and only if $\mathfrak{b} \cap \mathfrak{nil}(\mathfrak{b}^+) = \{0\}$, since the intersections are contained in the nilradicals of $\mathfrak{b}^+$ and $\mathfrak{b}$. Hence each one of these two transversalities conditions is necessary and sufficient for the subalgebras to be opposed. Through the adjoint representation, these conditions are carried over to arbitrary pairs of minimal parabolic subalgebras.

Proposition 1.1 *The minimal parabolic subalgebras* $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathbb{B}$ *are opposed if and only if* $\mathfrak{b}_1 \cap \mathfrak{nil}(\mathfrak{b}_2) = \{0\}$ *or* $\mathfrak{b}_2 \cap \mathfrak{nil}(\mathfrak{b}_1) = \{0\}$.

Therefore an open cell in $\mathbb{B}$ is the set of minimal parabolic subalgebras transversal to the nilradical of a fixed subalgebra $\mathfrak{b} \in \mathbb{B}$.

A similar picture holds for a flag manifold $\mathbb{B}_{\Theta}$. However, in general the transversality condition must be taken with respect to subalgebras in the flag manifold $\mathbb{B}_{\Theta^*}$ dual to $\mathbb{B}_{\Theta}$. Here the subset Θ^* defining $\mathbb{B}_{\Theta^*}$ is obtained as follows: Let $\mathcal{W}$ be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a}_0)$ and denote by $w_0 \in \mathcal{W}$ its principal involution, that is, the element of maximal length as a product of reflections with respect to the simple roots in Σ_0. Alternatively, w_0 is the only element of $\mathcal{W}$ such that $w_0(\Sigma_0) = -\Sigma_0$. Put $\iota = -w_0$. Then $\iota(\Sigma_0) = \Sigma_0$, so that it is an involutive automorphism of the Dynkin diagram of Σ_0. By definition $\Theta^* = \iota(\Theta)$.

It is well known and easy to prove from the standard decompositions that the nilradical of $\mathfrak{b}_{\Theta}$ is

$$\mathfrak{nil}(\mathfrak{b}_{\Theta}) = \mathfrak{n}_{\Theta}^{+} = \sum_{\alpha \in \Pi^{+} \setminus \langle \Theta \rangle} \mathfrak{g}_{\alpha}$$

(c.f. [15], page 80; [14], page 280). Define the subalgebra

$$\mathfrak{b}_{\Theta}^{-} = \theta(\mathfrak{b}_{\Theta}) = \mathfrak{n}^{+}(\Theta) \oplus \mathfrak{b}^{-},$$

where $\mathfrak{b}^{-}$ is as before the minimal parabolic subalgebra given by the negative roots and $\mathfrak{n}^{+}(\Theta)$ is the subalgebra spanned by the root spaces $\mathfrak{g}_{\alpha}$, $\alpha \in \langle \Theta \rangle$. We have $\mathfrak{b}_{\Theta}^{-} \in \mathbb{B}_{\Theta^*}$. In fact, let $\overline{w}_0$ be an inner automorphism of $\mathfrak{g}$ extending w_0 (e.g. $\overline{w}_0$ is a representative of w_0 in the normalizer $N_K(\mathfrak{a}_0)$ of $\mathfrak{a}_0$ in K, the compact group with Lie algebra $\mathfrak{k}$). Then $\overline{w}_0(\mathfrak{b}^{+}) = \mathfrak{b}^{-}$ and since $w_0(-\Theta^*) = \Theta$, it follows that $\overline{w}_0(\mathfrak{n}^{-}(\Theta^*)) = \mathfrak{n}^{+}(\Theta)$. Therefore, $\overline{w}_0(\mathfrak{b}_{\Theta^*}) = \mathfrak{b}_{\Theta}^{-}$, implying that $\mathfrak{b}_{\Theta}^{-} \in \mathbb{B}_{\Theta^*}$. Also, the nilradical of $\mathfrak{b}_{\Theta}^{-}$ is the image under $\overline{w}_0$ of $\mathfrak{nil}(\mathfrak{b}_{\Theta^*}) = \sum_{\alpha \in \Pi^{+} \setminus \langle \Theta^* \rangle} \mathfrak{g}_{\alpha}$. Hence we have

$$\mathfrak{nil}\left(\mathfrak{b}_{\Theta}^{-}\right) = \sum_{\alpha \in \Pi^{+} \setminus \langle \Theta \rangle} \mathfrak{g}_{-\alpha}.$$

This subalgebra is clearly transversal to $\mathfrak{b}_{\Theta}$.

Two subalgebras $\mathfrak{c}_1 \in \mathbb{B}_{\Theta}$ and $\mathfrak{c}_2 \in \mathbb{B}_{\Theta^*}$ are said to be opposed to each other (notation: $\mathfrak{c}_1 \pitchfork \mathfrak{c}_2$) in case the pair $(\mathfrak{c}_1, \mathfrak{c}_2) \in \mathbb{B}_{\Theta} \times \mathbb{B}_{\Theta^*}$ belongs to the G-orbit of $\left(\mathfrak{b}_{\Theta}, \mathfrak{b}_{\Theta}^{-}\right)$ under the componentwise action. Analogous to the minimal parabolic case the G-action carries over to pairs of subalgebras in $\mathbb{B}_{\Theta} \times \mathbb{B}_{\Theta^*}$ the incidence relations between $\mathfrak{b}_{\Theta}$ and $\mathfrak{b}_{\Theta}^{-}$.

Proposition 1.2 *The subalgebra* $\mathfrak{c}_1 \in \mathbb{B}_\Theta$ *is opposed to* $\mathfrak{c}_2 \in \mathbb{B}_{\Theta^*}$ *if and only if one of the two equivalent transversalities* $\mathfrak{c}_1 \cap \mathfrak{nil}(\mathfrak{c}_2) = \{0\}$ *or* $\mathfrak{c}_2 \cap \mathfrak{nil}(\mathfrak{c}_1) = \{0\}$ *is satisfied. Moreover, for* $\mathfrak{c} \in \mathbb{B}_{\Theta^*}$*, the subset*

$$\sigma(\mathfrak{c}) = \{\mathfrak{b} \in \mathbb{B}_\Theta : \mathfrak{b} \cap \mathfrak{nil}(\mathfrak{c}) = \{0\}\}$$

is an open cell in $\mathbb{B}_\Theta$*, and the map* $\mathfrak{c} \mapsto \sigma(\mathfrak{c})$ *gives a bijective correspondence between* $\mathbb{B}_{\Theta^*}$ *and the set of open cells in* $\mathbb{B}_\Theta$*.*

We discuss now the relation between opposed minimal parabolic subalgebras and split subalgebras. Given two opposed minimal parabolic subalgebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$, $\mathfrak{b}_1 \cap \mathfrak{b}_2$ contains a unique maximal abelian split subalgebra of $\mathfrak{g}$, that is, a subalgebra of the form $\mathfrak{a} = g \cdot \mathfrak{a}_0$ with $g \in G$. In this case one can select a Weyl chamber, say $\mathfrak{a}^+ \subset \mathfrak{a}$, such that for $H \in \mathfrak{a}^+$, $\mathfrak{b}_1$ is the attractor in $\mathbb{B}$ of $h = \exp H$ having $\sigma(\mathfrak{b}_2)$ as the corresponding stable manifold. This means that $h^k x \to \mathfrak{b}_1$, as $k \to +\infty$, for any $x \in \sigma(\mathfrak{b}_2)$. In the sequel we write $\mathfrak{a}^+(\mathfrak{b}_1, \mathfrak{b}_2)$ for the chamber coming from this construction, and put $A^+(\mathfrak{b}_1, \mathfrak{b}_2) = \exp(\mathfrak{a}^+(\mathfrak{b}_1, \mathfrak{b}_2))$.

Conversely, a minimal parabolic subalgebra is defined uniquely by a Weyl chamber $\mathfrak{a}^+$ contained in a maximal split subalgebra $\mathfrak{a}$. We denote this parabolic subalgebra by $\mathfrak{b}(\mathfrak{a}^+)$. Fixing $\mathfrak{a}^+$ one has a well defined isomorphism between $\mathcal{W}$, the Weyl group of the basic subalgebra $\mathfrak{a}_0$, and the Weyl group $\mathcal{W}(\mathfrak{a})$ of $\mathfrak{a}$. In fact, both groups are generated by reflections with respect to simple systems of roots having the same Dynkin diagram. Denote for a moment this isomorphism by ψ. If $w \in \mathcal{W}$ then $\psi(w)\mathfrak{a}^+$ is a Weyl chamber in $\mathfrak{a}$. In the sequel we write $\mathfrak{b}(\mathfrak{a}^+)w$ for the minimal parabolic subalgebra defined by $\psi(w)\mathfrak{a}^+$, that is, $\mathfrak{b}(\mathfrak{a}^+)w = \mathfrak{b}(\psi(w)\mathfrak{a}^+)$. This element of $\mathbb{B}$ is said to be the fixed point of type w for $A^+ = \exp(\mathfrak{a}^+)$.

For a given a parabolic subalgebra $\mathfrak{c}$ we denote by $N(\mathfrak{c})$ the connected subgroup whose Lie algebra is the nilradical $\mathfrak{nil}(\mathfrak{c})$ of $\mathfrak{c}$. Since $\mathfrak{nil}(\mathfrak{b}_\Theta^-) = \sum_{\alpha \notin \langle\Theta\rangle} \mathfrak{g}_{-\alpha}$ complements $\mathfrak{b}_\Theta$, the orbit of $N(\mathfrak{b}_\Theta^-)$ in $\mathbb{B}_\Theta$ containing $\mathfrak{b}_\Theta$ is open. The same remark holds for any subalgebra in $\mathbb{B}_\Theta$ opposed to $\mathfrak{b}_\Theta^-$, so that $\sigma(\mathfrak{b}_\Theta^-)$ is the open orbit of $N(\mathfrak{b}_\Theta^-)$ in $\mathbb{B}_\Theta$. This fact is carried over to arbitrary subalgebras $\mathfrak{c} \in \mathbb{B}_{\Theta^*}$ through the G-action: We have $\mathfrak{c} = g \cdot \mathfrak{b}_\Theta^-$ for some $g \in G$, hence $\mathfrak{nil}(\mathfrak{c}) = g \cdot \mathfrak{nil}(\mathfrak{b}_\Theta^-)$, $\sigma(\mathfrak{c}) = g\sigma(\mathfrak{b}_\Theta^-)$ and $N(\mathfrak{c}) = gN(\mathfrak{b}_\Theta^-)g^{-1}$, implying that $\sigma(\mathfrak{c})$ is the open orbit of $N(\mathfrak{c})$ in $\mathbb{B}_\Theta$.

Now, let $\pi : \mathbb{B} \to \mathbb{B}_\Theta$ and $\pi^* : \mathbb{B} \to \mathbb{B}_{\Theta^*}$ be the standard fibrations. Take $\mathfrak{b} \in \mathbb{B}$ and put $\mathfrak{c} = \pi^*(\mathfrak{b})$. Then $\mathfrak{nil}(\mathfrak{c}) \subset \mathfrak{nil}(\mathfrak{b})$, so that the open orbit of $N(\mathfrak{b})$ projects down onto the open orbit of $N(\mathfrak{c})$, implying that $\pi(\sigma(\mathfrak{b})) = \sigma(\mathfrak{c})$. For later reference we emphasize this fact.

Lemma 1.3 *If $\mathfrak{c} = \pi^*(\mathfrak{b})$ then $\sigma(\mathfrak{c}) = \pi(\sigma(\mathfrak{b}))$.*

Regarding the projection of opposed minimal parabolic subalgebras there is still the following fact, which is required later.

Lemma 1.4 *Let $\mathfrak{b} = \mathfrak{b}(\mathfrak{a}^+)$ be a minimal parabolic subalgebra built from the Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Let $\mathcal{W}(\mathfrak{a})$ be the Weyl group of $\mathfrak{a}$ and let w_0 be the principal involution of $\mathcal{W}(\mathfrak{a})$ with respect to $\mathfrak{a}^+$. Let $\pi : \mathbb{B} \to \mathbb{B}_\Theta$ be the projection onto a smaller flag manifold. Then for $w \in \mathcal{W}(\mathfrak{a})$ the fiber $\pi^{-1}\pi(\mathfrak{b}(w\mathfrak{a}^+))$ contains a subalgebra opposed to $\mathfrak{b}$ if and only if $\pi(\mathfrak{b}(w\mathfrak{a}^+)) = \pi(\mathfrak{b}(w_0\mathfrak{a}^+))$.*

Proof: The only subalgebra of the type $\mathfrak{b}(w\mathfrak{a}^+)$ opposed to $\mathfrak{b}$ is $\mathfrak{b}(w_0\mathfrak{a}^+)$. Let $N = N(\mathfrak{b})$ be the nilpotent subgroup whose Lie algebra is the nilradical of $\mathfrak{b}$. The subalgebras opposed to $\mathfrak{b}$ are the elements of the orbit $N \cdot \mathfrak{b}(w_0\mathfrak{a}^+)$. Take $w \in \mathcal{W}(\mathfrak{a})$ and suppose that for some $n \in N$, $n \cdot \mathfrak{b}(w_0\mathfrak{a}^+)$ and $\mathfrak{b}(w\mathfrak{a}^+)$ belong to the same fiber. Then $nw_0P_\Theta = wP_\Theta$, hence by the Borel-Kostant Theorem (see [15], Proposition 1.2.4.9), $w_0P_\Theta = wP_\Theta$, which means that $\mathfrak{b}(w\mathfrak{a}^+)$ and $\mathfrak{b}(w_0\mathfrak{a}^+)$ are in the same fiber. □

1.2 Semigroups

Let $S \subset G$ be a semigroup with nonempty interior. Recall that a control set for the action of S on $\mathbb{B}_\Theta$ is a subset $D \subset \mathbb{B}_\Theta$ which is maximal with the properties $D \subset \mathrm{cl}(Sx)$ for any $x \in D$, and there exists $x \in D$ such that $gx = x$ for some $g \in \mathrm{int}S$. It is known (see [10], [13]) that on any flag manifold there exists a unique invariant control set (i.e., $Sx \subset D$ for all $x \in D$) which is closed. Also, in $\mathbb{B}_\Theta$ there is just one control set which is S^{-1}-invariant. This set is open. Because of the natural order between control sets (see [11]) we call this open control set the minimal control set of S.

A subset $C \subset \mathbb{B}_\Theta$ is said to be admissible if the following conditions hold:

1. $C = \mathrm{cl}(\mathrm{int}C)$.

2. $C \subset \sigma(\mathfrak{c})$ for some $\mathfrak{c} \in \mathbb{B}_{\Theta^*}$.

The admissible subsets are the relevant invariant control sets of semigroups in G. In fact, we have the following fact proved in [13].

Proposition 1.5 *Let $S \subset G$ be a proper semigroup with* $\mathrm{int}S \neq \emptyset$*. Then there exists a unique subset $\Theta(S) \subset \Sigma_0$ such that*

1. *the invariant control set C of S in $\mathbb{B}_{\Theta(S)}$ is admissible, and*

2. *the invariant control set of S in $\mathbb{B}$ is $\pi^{-1}(C)$, where $\pi : \mathbb{B} \to \mathbb{B}_{\Theta(S)}$ is the standard fibration.*

In what follows we write $\mathbb{B}(S) = \mathbb{B}_{\Theta(S)}$ for the flag manifold whose existence and uniqueness is ensured by this proposition. Also, we let $\mathcal{W}(S)$ stand for the subgroup of $\mathcal{W}$ generated by the reflections with respect to the simple roots in $\Theta(S)$.

In the sequel we say that a semigroup S is of type $\Theta \subset \Sigma_0$ in case $\Theta(S) = \Theta$. We emphasize that if $\mathrm{int}S \neq \emptyset$ and S is a proper subsemigroup then S is of type Θ for some proper subset $\Theta \subset \Sigma_0$. At this point we remark that the type of S^{-1} is the dual $\Theta(S)^*$ of the type of S (see [12], Proposition 6.2).

In another direction, it was proved in [12], Proposition 4.2, that if $C \subset \mathbb{B}_\Theta$ is admissible then the compression semigroup

$$S_C = \{g \in G : gC \subset C\}$$

has nonempty interior in G and is of type Θ. Also, C is the invariant control set of S_C in $\mathbb{B}_\Theta$. Therefore a subset of $\mathbb{B}_\Theta$ is admissible if and only if it is the invariant control set of a semigroup with nonempty interior of type Θ.

A basic property of a semigroup S of type Θ is given by its intersection with the Weyl chambers. Recall that the set of transitivity D_0 of a control set D is defined by

$$D_0 = \{x \in D : x \in \mathrm{int}(S)\,x\}.$$

It is known that D_0 is dense in D and $gx \in D_0$ if $g \in S$ and $x \in D_0$ (see [13]). The following results were proved in [13].

Proposition 1.6 *Let S be a semigroup of type Θ and denote by C its invariant control set in $\mathbb{B}$. Also let $\mathfrak{a}$ be a maximal split subalgebra, $\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber, and put $A^+ = \exp \mathfrak{a}^+$. Suppose that $A^+ \cap \mathrm{int} S \neq \emptyset$. Then $\mathfrak{b}(\mathfrak{a}^+) w$ belongs to D_0 for some control set D, and $\mathfrak{b}(\mathfrak{a}^+) w \in C_0$ if and only if $w \in \mathcal{W}(S) = \mathcal{W}_{\Theta(S)}$.*

If $A^+ = \exp \mathfrak{a}^+$ is such that $A^+ \cap \mathrm{int} S \neq \emptyset$, we denote by $\mathcal{W}(S, A^+)$ the subgroup

$$\mathcal{W}\left(S, A^+\right) = \{w \in \mathcal{W}(\mathfrak{a}) : w A^+ w^{-1} \cap \mathrm{int} S \neq \emptyset\}, \tag{1}$$

where $\mathcal{W}(\mathfrak{a})$ is, as above, the Weyl group of $\mathfrak{a}$. By the previous proposition $\mathcal{W}(S, A^+)$ is isomorphic to $\mathcal{W}(S)$ under ψ.

Now, we develop a method of producing Weyl chambers meeting $\mathrm{int} S$ when S is a compression semigroup. For a subset $C \subset \mathbb{B}_\Theta$ its dual $C^* \subset \mathbb{B}_{\Theta^*}$ is defined by

$$C^* = \left\{\mathfrak{b} \in \mathbb{B}_{\Theta^*} : (\forall \mathfrak{b}' \in C), \mathfrak{b} \pitchfork \mathfrak{b}'\right\}.$$

Clearly, $C \subset \sigma(\mathfrak{b})$ for all $\mathfrak{b} \in C^*$. By definition, $C^* \neq \emptyset$ if C is admissible.

Proposition 1.7 *Let $C \subset \mathbb{B}_\Theta$ be admissible and denote by S_C its compression semigroup. Take $\mathfrak{c}_1 \in \mathrm{int} C$ and $\mathfrak{c}_2 \in C^*$. Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be minimal parabolic subalgebras such that $\pi(\mathfrak{b}_1) = \mathfrak{c}_1$ and $\pi^*(\mathfrak{b}_2) = \mathfrak{c}_2$. Suppose that $\mathfrak{b}_1$ is opposed to $\mathfrak{b}_2$. Then for all $h \in A^+(\mathfrak{b}_1, \mathfrak{b}_2)$ there exists a positive integer k such that $h^k \in \mathrm{int}(S_C)$.*

Proof: By Lemma 1.3, $\pi(\sigma(\mathfrak{b}_2)) = \sigma(\mathfrak{c}_2)$. Since $C \subset \sigma(\mathfrak{c}_2)$, it follows that $h^k x \to \mathfrak{c}_1$, as $k \to +\infty$, for all $x \in C$ and $h \in A^+(\mathfrak{b}_1, \mathfrak{b}_2)$. Now, the compactness of C implies that $h^k C \subset \mathrm{int} C$ for large k. Thus $h^k \in \mathrm{int}(S_C)$, as claimed. □

Remark: If $\mathfrak{c}_1 \in \mathbb{B}_\Theta$ and $\mathfrak{c}_2 \in \mathbb{B}_{\Theta^*}$ are opposed then for any $\mathfrak{b}_1 \in \mathbb{B}$ projecting into $\mathfrak{c}_1$ there exists $\mathfrak{b}_2 \in \mathbb{B}$ which projects into $\mathfrak{c}_2$ and is opposed to $\mathfrak{b}_1$. This is easily seen for the models $\mathfrak{b}_\Theta$ and $\mathfrak{b}_\Theta^-$. In this case $\mathfrak{b}_1 = \mathfrak{b}^+$ and $\mathfrak{b}_2 = \mathfrak{b}^-$ are opposed minimal parabolic subalgebras contained in $\mathfrak{b}_\Theta$ and $\mathfrak{b}_\Theta^-$, respectively. For general $\mathfrak{c}_1$ and $\mathfrak{c}_2$ we apply the adjoint action to the standard models and use the existence of an automorphism g such that $\mathfrak{c}_1 = g \cdot \mathfrak{b}_\Theta$ and $\mathfrak{c}_2 = g \cdot \mathfrak{b}_\Theta^-$. Therefore, the above proposition implies that every $x \in \mathrm{int} C$ is the attractor of some $h \in \mathrm{int}(S_C)$.

For later reference we note the following fact concerning the dual of an invariant control set.

Proposition 1.8 *Let S be a semigroup of type Θ and denote by C its invariant control set in $\mathbb{B}(S)$. Also, let D be the invariant control set of S^{-1} in the flag manifold $\mathbb{B}(S)^* = \mathbb{B}_{\Theta(S)^*}$ dual to $\mathbb{B}(S)$. Then its set of transitivity $D_0 \subset C^*$.*

Proof: From the general results about the duality operator we know that C^* is open and S^{-1}-invariant (see [12], Proposition 3.5 and Corollary 3.10). Hence $\mathrm{cl}\,(C^*)$ is invariant under S^{-1}, implying that $D \subset \mathrm{cl}\,(C^*)$. Now, D_0 is open and dense in D, so that $D_0 \cap C^* \neq \emptyset$. Since $D_0 = S^{-1}x$ for $x \in D_0$ and C^* is S^{-1}-invariant the result follows. □

We conclude this section with the proof of a sufficient condition for the maximality of a compression semigroup. With this condition it is easy to prove that the compression semigroups in affine symmetric spaces having nonempty interior are maximal.

Proposition 1.9 *Let $C \subset \mathbb{B}_\Theta$ be an admissible subset and denote by S_C its compression semigroup. Suppose that the union of the control sets of S_C in $\mathbb{B}_\Theta$ is dense in $\mathbb{B}_\Theta$. Then S_C is maximal of type Θ. In particular S_C is maximal in G if Θ is maximal in Σ_0.*

Proof: To say that S_C is maximal of type Θ means that if $T \subset G$ is a semigroup and $S_C \subset T$ properly then T is either G or of type $\Theta' \supset \Theta$ with $\Theta \neq \Theta'$. Let T be a semigroup containing S_C properly. Denote by E its invariant control set in $\mathbb{B}_\Theta$. We have that $C \subset E$. By definition of S_C, there exists $g \in T$ and $x \in C$ such that $gx \notin C$. Since $C = \mathrm{cl}\,(\mathrm{int}C)$ we can assume without loss of generality that $x \in \mathrm{int}C$. Hence, the assumption that the control sets are dense imply that $g\,(\mathrm{int}C)$ meets some control set of S_C different from C. Denote this control set by D. It follows that D is attainable by means of T from some point of $C \subset E$, so that $D \subset E$. Since the control sets in $\mathbb{B}_\Theta$ are the projections of the control sets in $\mathbb{B}$, this implies at once that T is not of type Θ. In fact, an application of Proposition 1.6 to both S_C and T shows that $\mathcal{W}\,(T)$ contains $\mathcal{W}\,(S_C)$ properly. □

Remark: Let E be as in the proof above. Then E is not contained in any open cell of $\mathbb{B}_\Theta$. In fact, suppose that $E \subset \sigma(\mathfrak{c})$ for some $\mathfrak{c} \in \mathbb{B}_{\Theta^*}$. Pick $\mathfrak{c}_1 \in \mathrm{int}C$. By Proposition 1.7 (and the remark following it) there are minimal parabolic subalgebras $\mathfrak{b}$ and $\mathfrak{b}_1$ projecting into $\mathfrak{c}$ and $\mathfrak{c}_1$, respectively, such that $A^+(\mathfrak{b}_1, \mathfrak{b})$ meets $\mathrm{int}(S_C)$. Since D is not the invariant control set of S_C, there exists $w \notin \mathcal{W}(S_C)$ such that the fixed point of type w of $A^+(\mathfrak{b}_1, \mathfrak{b})$ projects into D, contradicting the assumption that $D \subset E \subset \sigma(\mathfrak{c})$.

2 Affine symmetric spaces

Let (G, L, τ) be an affine symmetric space where G is a connected Lie group, τ an involutive automorphism of G and L an open subgroup of G^τ, the subgroup of τ-fixed points. The corresponding symmetric Lie algebra is denoted by $(\mathfrak{g}, \mathfrak{l}, \tau)$. We assume throughout that $\mathfrak{g}$ is simple and G has finite center. For the problem treated here the general case can be reduced to this one (see [4], Section I). Also, we avoid the trivial situation $\mathfrak{l} = \mathfrak{g}$ by assuming that $\tau \neq 1$.

Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$ be the decomposition into τ-eigenspaces, where $\mathfrak{q} = \{X \in \mathfrak{g} : \tau X = -X\}$. Also, let θ be a Cartan involution commuting with τ and denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition. Put $\mathfrak{k}_+ = \mathfrak{k} \cap \mathfrak{l}$, $\mathfrak{k}_- = \mathfrak{k} \cap \mathfrak{q}$, $\mathfrak{p}_+ = \mathfrak{p} \cap \mathfrak{l}$ and $\mathfrak{p}_- = \mathfrak{p} \cap \mathfrak{q}$. Then

$$\mathfrak{g} = \mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{p}_+ + \mathfrak{p}_-.$$

In order to describe the open orbits let $\mathfrak{a} \subset \mathfrak{p}$ be a τ-invariant maximal split subalgebra and put $\mathfrak{a}_\mathfrak{l} = \mathfrak{a} \cap \mathfrak{l}$ and $\mathfrak{a}_\mathfrak{q} = \mathfrak{a} \cap \mathfrak{q}$. For a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ denote by Π^+ the corresponding set of positive roots. Let $\mathfrak{b}(\mathfrak{a}^+)$ be the minimal parabolic subalgebra defined by $\mathfrak{a}^+$. It is well known that L has open orbits in $\mathbb{B}$. The following characterization of the open orbits was proved in [8], §3, Proposition 1.

Proposition 2.1 *Necessary and sufficient conditions for the L-orbit of $\mathfrak{b}(\mathfrak{a}^+)$ to be open are:*

1. $\mathfrak{a}_\mathfrak{q}$ *is maximal abelian in* $\mathfrak{p}_-$*, and*
2. Π^+ *is* $\mathfrak{q}$*-compatible, that is,* $\Pi^+ \setminus \mathfrak{a}_\mathfrak{q}^\perp$ *is* $-\tau$*-invariant.*

More important to us than this characterization of the Weyl chambers giving rise to open orbits, is the following relation between them given by the Weyl group action. Thus keeping fixed the τ-invariant maximal split subalgebra $\mathfrak{a}$, consider the groups

- $\mathcal{W}(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, the Weyl group of $\mathfrak{a}$.
- $\mathcal{W}_\tau(\mathfrak{a}) = \{w \in \mathcal{W}(\mathfrak{a}) : w\mathfrak{a}_\mathfrak{l} = \mathfrak{a}_\mathfrak{l}\}$.
- $\mathcal{W}^a(\mathfrak{a}) = N_{K^+}(\mathfrak{a})/Z_{K^+}(\mathfrak{a})$, where $K^+ = K \cap L$.

Here and in the sequel the notations $Z_U(\mathfrak{d})$ and $N_U(\mathfrak{d})$ stand for the centralizer and the normalizer of $\mathfrak{d}$ in U, respectively. The following inclusions hold

$$\mathcal{W}^a(\mathfrak{a}) \subset \mathcal{W}_\tau(\mathfrak{a}) \subset \mathcal{W}(\mathfrak{a}),$$

and the open orbits are enumerated by

Proposition 2.2 *Let $\mathfrak{a}^+$ be a Weyl chamber such that the L-orbit of $\mathfrak{b}(\mathfrak{a}^+)$ is open. Then the open L-orbits are the orbits of $\mathfrak{b}(w\mathfrak{a}^+)$ with $w \in \mathcal{W}_\tau(\mathfrak{a})$. Moreover, the orbits of $\mathfrak{b}(w_1\mathfrak{a}^+)$ and $\mathfrak{b}(w_2\mathfrak{a}^+)$, $w_1, w_2 \in \mathcal{W}_\tau(\mathfrak{a})$, agree if and only if $\mathcal{W}^a(\mathfrak{a})w_1 = \mathcal{W}^a(\mathfrak{a})w_2$.*

Proof: See [8], Theorem 3, and [4], Proposition II.10. □

Now, we can look at the semigroups. Start with a semigroup S containing L and such that int$S \neq \emptyset$ and $S \neq G$. Our purpose is to rule out the affine symmetric spaces where such a semigroup does not exist. For this we can assume without loss of generality that S is a compression semigroup. In fact, denote by C the invariant control set of S in $\mathbb{B}$. Since $S \neq G$, it follows that $C \neq \mathbb{B}$, hence the compression semigroup S_C of C is proper. Clearly $S \subset S_C$ so that $\mathrm{int}(S_C) \neq \emptyset$. Therefore, if L is contained in a proper semigroup with nonempty interior then it is contained in a compression semigroup of the same kind.

Thus we assume from now on that $L \subset S$ and S is the compression semigroup of its invariant control set $C \subset \mathbb{B}$. Let $\Theta = \Theta(S)$ be the type of S, and put $\pi : \mathbb{B} \to \mathbb{B}_\Theta$ and $\pi^* : \mathbb{B} \to \mathbb{B}_{\Theta^*}$ for the standard fibrations. The projection $\pi(C)$ is the invariant control set of S in $\mathbb{B}_\Theta$ and $C = \pi^{-1}(\pi(C))$.

By equivariance of π, $g \in G$ leaves C invariant if and only if $\pi(C)$ is invariant under g. Thus S coincides with the compression semigroup of $\pi(C)$.

Before proceeding we prove the following simple lemma.

Lemma 2.3 *Let D be a control set of S in $\mathbb{B}$. Then its set of transitivity D_0 contains an open L-orbit. Furthermore, any open orbit meeting D is contained in D_0.*

Proof: By definition $\mathrm{int} D \neq \emptyset$, $D \subset \mathrm{cl}(Sx)$ for all $x \in D$, and D is maximal with these properties. This maximality property – together with the fact that $L \subset S$ – implies that any L-orbit meeting D is contained in D. Moreover, the union of the open L-orbits is dense in $\mathbb{B}$, so that D contains some open L-orbit, say O. We have $D_0 \cap O \neq \emptyset$, because D_0 is dense in D. Since D_0 is S_C-invariant within D, it follows that $O \subset D_0$. □

In particular there are open L-orbits contained in C. Suppose that C contains the open orbit passing through $\mathfrak{b}(\mathfrak{a}^+)$ where the Weyl chamber $\mathfrak{a}^+$ satisfies the conditions of Proposition 2.1. The following lemma ensures that the subalgebra $\mathfrak{b}(w_0\mathfrak{a}^+)$ opposed to $\mathfrak{b}(\mathfrak{a}^+)$ belongs to the minimal control set C^- of S. Recall that C^- is open and S^{-1}-invariant. Also, $\pi^*(C^-)$ is the minimal control set of S in $\mathbb{B}_{\Theta^*}$.

Lemma 2.4 *Let w_0 be the principal involution with respect to $\mathfrak{a}^+$, i.e., $w_0\mathfrak{a}^+ = -\mathfrak{a}^+$. Then $\mathfrak{b}(w_0\mathfrak{a}^+) \in C_0^-$.*

Proof: Since $\mathcal{W}_\tau(\mathfrak{a})$ parametrizes the open L-orbits and C_0^- contains such an orbit, there exists $w \in \mathcal{W}_\tau(\mathfrak{a})$ such that $\mathfrak{b}(w\mathfrak{a}^+) \in C_0^-$. Write $\mathfrak{b}_w = \mathfrak{b}(w\mathfrak{a}^+)$. The projection $\pi(C)$ is the invariant control set of S in $\mathbb{B}_\Theta$. Also, $\pi^*(\mathrm{cl}C^-)$ is the invariant control set of S^{-1} in $\mathbb{B}_{\Theta^*}$, so that $\pi^*\left(C_0^-\right)$ is contained in the set of transitivity of the invariant control set of S^{-1} in $\mathbb{B}_{\Theta^*}$. Hence Proposition 1.8 implies that $\pi^*(\mathfrak{b}_w) \in \pi(C)^*$. Therefore, in the fiber $(\pi^*)^{-1}\pi^*(\mathfrak{b}_w)$ there exists $\mathfrak{b}$ opposed to $\mathfrak{b}(\mathfrak{a}^+)$. By Lemma 1.4 this implies that $\mathfrak{b}(w_0\mathfrak{a}^+) \in (\pi^*)^{-1}\pi^*(\mathfrak{b}_w)$. Hence $\mathfrak{b}(w_0\mathfrak{a}^+)$ belongs to the invariant control set of S^{-1} because this semigroup is of type Θ^*. Using the fact that the open orbits meeting C^- are contained in C_0^- we conclude that $\mathfrak{b}(w_0\mathfrak{a}^+) \in C_0^-$. □

Now, we apply propositions 1.7 and 1.8 in order to get a decisive information about the type Θ of S. By construction $\pi(\mathfrak{b}(\mathfrak{a}^+))$ belongs to the set of transitivity of the invariant control set of S in $\mathbb{B}_\Theta$. On the other hand the above lemma ensures that $\pi^*\left(\mathfrak{b}\left(w_0\mathfrak{a}^+\right)\right) \in \pi^*\left(C^-\right)_0$. Hence from Proposition 1.8 we get $\pi^*\left(\mathfrak{b}\left(w_0\mathfrak{a}^+\right)\right) \in \left(\pi\left(C\right)\right)^*$. Clearly, $\mathfrak{b}\left(\mathfrak{a}^+\right) \pitchfork \mathfrak{b}\left(w_0\mathfrak{a}^+\right)$. Since we are assuming that S is a compression semigroup, it follows by Proposition 1.7 that the chamber $A^+\left(\mathfrak{b}\left(\mathfrak{a}^+\right), \mathfrak{b}\left(w_0\mathfrak{a}^+\right)\right)$ meets $\mathrm{int}S$. Since by definition $A^+\left(\mathfrak{b}\left(\mathfrak{a}^+\right), \mathfrak{b}\left(w_0\mathfrak{a}^+\right)\right) = \exp\left(\mathfrak{a}^+\right)$, we have proved:

Proposition 2.5 *Suppose that the open orbit through $\mathfrak{b}\left(\mathfrak{a}^+\right)$ is contained in the invariant control set of S in $\mathbb{B}$. Then $A^+ \cap \mathrm{int}S \neq \emptyset$ where $A^+ = \exp\left(\mathfrak{a}^+\right)$.*

The fact that A^+ meets the interior of S permits to determine explicitly the subgroup $\mathcal{W}\left(S, A^+\right)$ and thus the flag manifold associated to S. In fact, by definition $\mathcal{W}\left(S, A^+\right)$ is the set of $w \in \mathcal{W}\left(\mathfrak{a}\right)$ such that $\mathfrak{b}\left(w\mathfrak{a}^+\right)$ belongs to the invariant control set C or, equivalently, the L-orbit of $\mathfrak{b}\left(w\mathfrak{a}^+\right)$ is contained in C.

Proposition 2.6 *Keeping the above notations, let $w_1, \ldots, w_s \in \mathcal{W}_\tau\left(\mathfrak{a}\right)$ be such that the L-orbits of $\mathfrak{b}\left(w_i\mathfrak{a}^+\right)$, $i = 1, \ldots, s$, are the open orbits contained in C. Then*

$$\mathcal{W}\left(S, A^+\right) = \bigcup_{i=1}^{s} \mathcal{W}^a\left(\mathfrak{a}\right) w_i.$$

In particular, $\mathcal{W}^a\left(\mathfrak{a}\right) \subset \mathcal{W}\left(S, A^+\right) \subset \mathcal{W}_\tau\left(\mathfrak{a}\right)$. Also, $\mathcal{W}^a\left(\mathfrak{a}\right)$ has a fixed point in $\mathfrak{a}$.

Proof: The first statement follows immediately from the previous proposition, the definition of $\mathcal{W}\left(S, A^+\right)$ (see (1)), and the characterization of the open orbits in Proposition 2.2. The existence of $\mathcal{W}^a\left(\mathfrak{a}\right)$-fixed points is a consequence of the fact that $\mathcal{W}\left(S, A^+\right)$ is parabolic with respect to $\mathfrak{a}^+$, i.e., is generated by the reflections with respect to the simple roots in a subset of the simple system of roots associated to $\mathfrak{a}^+$. Such a subgroup has a fixed point in $\mathfrak{a}^+$. □

This proposition implies that $\mathcal{W}_\tau\left(\mathfrak{a}\right) = \mathcal{W}\left(\mathfrak{a}\right)$. In fact, any control set contains at least one open L-orbit, which by Proposition 2.2 is the orbit of $\mathfrak{b}\left(w\mathfrak{a}^+\right)$ for some $w \in \mathcal{W}_\tau\left(\mathfrak{a}\right)$. However, $\mathfrak{b}\left(w_1\mathfrak{a}^+\right)$ and $\mathfrak{b}\left(w_2\mathfrak{a}^+\right)$ belong to the same control set if and only if $\mathcal{W}\left(S, A^+\right) w_1 = \mathcal{W}\left(S, A^+\right) w_2$.

Therefore the number of control sets is $|\mathcal{W}_\tau(\mathfrak{a})| / |\mathcal{W}(S, A^+)|$. But the general theory of control sets ensures that the number of control sets in $\mathbb{B}$ is $|\mathcal{W}(\mathfrak{a})| / |\mathcal{W}(S, A^+)|$ (see [13], Theorem 4.5). Hence $\mathcal{W}_\tau(\mathfrak{a}) = \mathcal{W}(\mathfrak{a})$.

Corollary 2.7 *If there exists a proper semigroup $S \supset L$ with* $\mathrm{int} S \neq \emptyset$ *then* $\mathcal{W}_\tau(\mathfrak{a}) = \mathcal{W}(\mathfrak{a})$, $\mathfrak{a}_\mathfrak{l} = \{0\}$ *and* $\mathfrak{a}_\mathfrak{q} = \mathfrak{a}$.

Proof: It was showed above that $\mathcal{W}_\tau(\mathfrak{a}) = \mathcal{W}(\mathfrak{a})$. To see that $\mathfrak{a}_\mathfrak{l} = 0$, recall first that $\mathfrak{g}$ is assumed to be simple, so that $\mathcal{W}(\mathfrak{a})$ is irreducible in $\mathfrak{a}$. Since $\mathfrak{a}_\mathfrak{l}$ is invariant under $\mathcal{W}_\tau(\mathfrak{a}) = \mathcal{W}(\mathfrak{a})$, it follows that $\mathfrak{a}_\mathfrak{l} = \{0\}$ or $\mathfrak{a}$. The second possibility is ruled out because in this case $\mathfrak{l} = \mathfrak{g}$. To see this observe that since the orbit of $\mathfrak{b}(\mathfrak{a}^+)$ is open, $\mathfrak{a}_\mathfrak{q}$ is maximal abelian in $\mathfrak{p}_-$. Hence $\mathfrak{a}_\mathfrak{l} = \mathfrak{a}$ implies that $\mathfrak{a}_\mathfrak{q} = \{0\}$, and thus $\mathfrak{p}_- = 0$ and $\mathfrak{p} = \mathfrak{p}_+$. This means that $\mathfrak{p} \subset \mathfrak{l}$. But $\mathfrak{p}$ generates $\mathfrak{g}$, so that $\mathfrak{l} = \mathfrak{g}$ if $\mathfrak{a}_\mathfrak{l} = \mathfrak{a}$. Therefore $\mathfrak{a}_\mathfrak{l} = \{0\}$ as claimed. □

We continue to keep fixed a Weyl chamber $\mathfrak{a}^+$ such that the L-orbit of $\mathfrak{b}(\mathfrak{a}^+)$ is open. The Lie algebra $\mathfrak{l}^a = \mathfrak{k}_+ + \mathfrak{p}_-$ is the subalgebra fixed by the involution $\tau\theta$, hence it is a reductive Lie algebra. The decomposition $\mathfrak{l}^a = \mathfrak{k}_+ + \mathfrak{p}_-$ is a Cartan decomposition of $\mathfrak{l}^a$. In view of the above corollary we assume that $\mathfrak{a}_\mathfrak{q} = \mathfrak{a}$. It is maximal abelian in $\mathfrak{p}_-$, and hence a maximal split subalgebra of $\mathfrak{l}^a$.

The symmetric pair $(\mathfrak{g}, \tau)$ is said to be regular or of Hermitian type provided $\mathfrak{z}(\mathfrak{l}^a) \cap \mathfrak{p}_- \neq 0$. It is now easy to prove our main result, namely that the existence of a semigroup containing L implies that $(\mathfrak{g}, \tau)$ is regular.

Theorem 2.8 *Suppose that there exists a proper semigroup $S \supset L$ with* $\mathrm{int} S \neq \emptyset$. *Then the symmetric pair $(\mathfrak{g}, \tau)$ is of Hermitian type.*

Proof: Let $(K_+)_0$ be the connected subgroup whose Lie algebra is $\mathfrak{k}_+$. Then $N_{(K_+)_0}(\mathfrak{a}) / Z_{(K_+)_0}(\mathfrak{a})$ is the Weyl group of the pair $(\mathfrak{l}^a, \mathfrak{a})$. Clearly, $N_{(K_+)_0}(\mathfrak{a}) \subset N_{K_+}(\mathfrak{a})$. Now, Proposition 2.6 implies that $N_{K_+}(\mathfrak{a})$ has a fixed point in $\mathfrak{a}$. Hence the Weyl group of $(\mathfrak{l}^a, \mathfrak{a})$ has a fixed point in $\mathfrak{a}$. Since $\mathfrak{l}^a$ is reductive, the fixed point set of the Weyl group of $(\mathfrak{l}^a, \mathfrak{a})$ is contained in the center $\mathfrak{z}(\mathfrak{l}^a)$ of $\mathfrak{l}^a$. Thus $\mathfrak{z}(\mathfrak{l}^a) \cap \mathfrak{p}_- \supset \mathfrak{z}(\mathfrak{l}^a) \cap \mathfrak{a} \neq 0$, that is, the affine symmetric space is regular. □

Next we improve this theorem by showing that the only possibility for C in the regular case is to be the closure of just one open orbit.

A basic fact about a regular pair is that the subspace $\mathfrak{c} = \mathfrak{z}(\mathfrak{l}^a) \cap \mathfrak{a}$ is one-dimensional and $\mathfrak{l}^a = \mathfrak{z}(\mathfrak{c})$ (see [4], Theorem V.1). Let $\Theta(\mathfrak{c}) \subset \Sigma$ be the subset of simple roots α such that $\alpha(\mathfrak{c}) = 0$. Denote by $\mathcal{W}^{\mathfrak{c}}$ the subgroup generated by the reflections with respect to the simple roots in $\Theta(\mathfrak{c})$. Since $\mathfrak{c}$ is one-dimensional the parabolic subalgebra $\mathfrak{b}_{\Theta(\mathfrak{c})}$ is maximal. Also, $\mathcal{W}^{\mathfrak{c}}$ is a maximal parabolic subgroup of $\mathcal{W}(\mathfrak{a})$. Put $L(\mathfrak{c}) = Z_G(\mathfrak{c})$ and $K(\mathfrak{c}) = L(\mathfrak{c}) \cap K$. The Lie algebra of $L(\mathfrak{c})$ is $\mathfrak{l}^a = \mathfrak{k}_+ + \mathfrak{p}_-$ and the Lie algebra of $K(\mathfrak{c})$ is $\mathfrak{k}_+$.

Lemma 2.9 *$\mathcal{W}^a(\mathfrak{a}) = \mathcal{W}(S, A^+) = \mathcal{W}^{\mathfrak{c}}$, where as above $A^+ = \exp(\mathfrak{a}^+)$ and $\mathfrak{a}^+$ is a basic chamber defining an open orbit contained in C.*

Proof: By [15], Lemma 1.2.4.5, $K(\mathfrak{c}) = Z_K(\mathfrak{a}) K(\mathfrak{c})_0$ where $K(\mathfrak{c})_0$ is the identity component of $K(\mathfrak{c})$. Also, Corollary 1.2.4.7 in [15] ensures that $\mathcal{W}^{\mathfrak{c}}$ is the Weyl group of $(\mathfrak{l}^a, \mathfrak{a})$, that is, $N_{K(\mathfrak{c})}(\mathfrak{a}) / Z_K(\mathfrak{a})$. Hence $\mathcal{W}^{\mathfrak{c}}$ is contained in $K(\mathfrak{c})_0$ in the sense that each $w \in \mathcal{W}^{\mathfrak{c}}$ has a representative in $K(\mathfrak{c})_0$. On the other hand $\mathfrak{k}_+$ is the Lie algebra of both $K(\mathfrak{c})$ and K_+, so that $K(\mathfrak{c})_0 \subset K_+$. It follows that $\mathcal{W}^{\mathfrak{c}} \subset \mathcal{W}^a(\mathfrak{a})$. Now, by Proposition 2.6, $\mathcal{W}^a(\mathfrak{a}) \subset \mathcal{W}(S, A^+)$. However, $\mathcal{W}^{\mathfrak{c}}$ is maximal parabolic in $\mathcal{W}(\mathfrak{a})$, so that $\mathcal{W}^a(\mathfrak{a}) = \mathcal{W}(S, A^+) = \mathcal{W}^{\mathfrak{c}}$. □

Joining this lemma with Proposition 2.6, we get easily that the only possibility for the invariant control set C is to be an open orbit.

Theorem 2.10 *Suppose that there exists a proper semigroup $S \supset L$ with $\mathrm{int} S \neq \emptyset$. Let C be the invariant control set of S in $\mathbb{B}$. Then $C = \mathrm{cl} O$, where O is an open L-orbit. Write $O = G \cdot \mathfrak{b}(\mathfrak{a}^+)$. Then one of the half-lines of $\mathfrak{c} = \mathfrak{z}(\mathfrak{l}^a) \cap \mathfrak{a}$ is contained in $\mathrm{cl}\mathfrak{a}^+$. Furthermore, the compression semigroup S_C is maximal.*

Proof: The fact that C contains just one open orbit follows from the above lemma and the characterization of the open orbits in Proposition 2.2. Since $\mathcal{W}(S, A^+)$ fixes the points in $\mathfrak{c}$, it follows that $\mathfrak{c}$ meets the closure of $\mathfrak{a}^+$ (see [13], Section 4). To see that S_C is maximal we note that the number of control sets (i.e., $|\mathcal{W}| / |\mathcal{W}(S)|$) equals the number of open L-orbits (i.e., $|\mathcal{W}| / |\mathcal{W}^a(\mathfrak{a})|$). Now, each control set contains at least one open orbit. Therefore, every open orbit is contained in a control set. This shows that the union of the control sets is dense. Hence the maximality of S_C follows from Proposition 1.9, and the remark that its type $\Theta(S)$ is a maximal subset. $\square$

Remark: In defining the orbit $O = G \cdot \mathfrak{b}(\mathfrak{a}^+)$ we can choose any Weyl chamber containing a given half-line $\mathfrak{c}^+$ of $\mathfrak{c}$, since $\mathcal{W}^{\mathfrak{c}}$ interchanges these chambers. Hence O is determined by $\mathfrak{c}^+$. On the other hand the semigroup S^{-1} also contains L, and it is not hard to see that the open orbit contained in the S^{-1}-invariant control set is given by $-\mathfrak{c}^+$.

Finally we mention that our results do not ensure the existence of proper semigroups with nonempty interior containing L. Clearly, this happens if and only if the compression semigroup of an open orbit like in Theorem 2.10 has nonempty interior. For the existence problem we refer to the literature on affine symmetric spaces. In particular we mention that it is proved in [4] that in case G complexifies to a simply connected group $G_{\mathbb{C}}$ then G^τ is contained in a compression semigroup of an open orbit, which is the same for every L.

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