A Comparative Analysis of the Monotone Iteration Method for Elliptic Problems

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Abstract

In this work, after a theoretical explanation of the Monotone Iteration Method, there are presented several numerical experiments with this method, when applied to solve some nonlinear Elliptic Equations. It is shown that, in some cases, uniqueness of solution can also be verified through the numerical implementation of the method. It is also presented its application to Cooperative Elliptic Systems. For all the examples Newton's method is also applied and a comparison between the Monotone Iteration Method and Newton's Method is made.

1 Introduction

The Monotone Iteration Method (MIM) has been widely used to show the existence of solutions of some elliptic equations and systems. It may be used

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whenever a maximum principle is available. Abstract versions of it have also been presented (see [1], [3], [4], among many others).

In this paper we make numerical experiments based on that method and we compare its performance with Newton's Method. Since the MIM depends on finding a super-solution, a sub-solution and a convenient constant α, a preliminary calculation is needed to implement it in each example. For Newton's Method this preparation is not needed.

A characteristic of the MIM is that, once a super-solution and a sub-solution are known, we have an a priori bound for the solution and the convergence of the method to some solution is supported by the theory, even if we start far away from the solution. On the other hand, Newton's Method may diverge or may converge to a uninteresting solution for instance; moreover, in the case of convergence, Newton's Method may be very sensitive to small changes in the initial approximation.

Even though our final task is the solution of Elliptic Equations, the MIM can also be applied to scalar equations. Section 2 consists of theoretical aspects for this case and we also present and analyse some numerical results obtained when we applied the MIM for solving a scalar equation $f(x) = 0$. Even in this simple situation the method shows interesting features. For instance, it detects numerically uniqueness of solution even in the presence of a $f(x)$ which is nonmonotone.

In the case of Elliptic Equations, presented in Section 3, we consider basically two cases. Examples 2 and 3 deal with the first one that consists of the equation with a forcing term $h(s,t)$. In the second case, presented in Example 3, the equation is autonomous. This equation has the trivial solution $u^*(s,t) \equiv 0$ and we are looking for nontrivial solutions. We make several tests and comparisons with which very interesting features of the MIM are detected . We also present, in Example 4, a case where we can verify the uniqueness of solution of an Equation, combining the theory with the numerical results found for the MIM.

In the last experiment of Section 3 we make a combination of the two methods. More precisely, we use the MIM to find a solution with approximation 10^{-2} and then, starting at this approximate solution, we use the MIM and Newton's Method to improve the approximation up to10^{-6}. For medium sized grids, we have verified that Newton's Method is faster. However, as the grid gets thinner (and we are closer to the exact solution of the elliptic equation), the MIM has a better performance in despite of the much larger number of iterations it has to perform.

In Section 4 we use again both methods to find a solution to a Cooperative

Elliptic System with two equations, given by the prompt-feedback reactor problem, which is a practical application that has similarity with models in ecology (see [4]).

2 The Case of a Scalar Equation

First of all we consider the question of solving a scalar equation

$$f(x) = 0 \tag{1}$$

and our theorem is the following:

Theorem 2.1. *Let $f : [c, d] \to R$ be a C^1 function such that $f(c) < 0, f(d) > 0$ and let $\alpha > 0$ be a constant such that $f'(x) < \alpha$ for any $x \in [c, d]$. Consider the iteration function*

$$\phi(x) = x - \frac{f(x)}{\alpha}$$

and let us define two sequences x_k and y_k in the following way:

$$x_0 = c \quad x_{k+1} = \phi(x_k), \qquad y_0 = d \quad y_{k+1} = \phi(y_k).$$

Then for any k we have

$$c < x_k \leq x_{k+1} \leq y_{k+1} \leq y_k < d,$$

x_k converges to the least solution of $f(x) = 0$ in the interval $[c, d]$ and y_k converges to the largest one.

Proof

The proof follows immediately from the monotonicity of ϕ on the interval $[c, d]$ and from the fact that the interval $[c, d]$ is invariant under ϕ because $\phi(c) = c - \dfrac{f(c)}{\alpha} > c$ and $\phi(d) = d - \dfrac{f(d)}{\alpha} < d$. ∎

Remarks

1. It can be proved that Theorem 2.1 holds if $f(x)$ is only Lipschitzian in the interval $[c, d]$.
2. The MIM has a similarity with the Modified Newton's Method because it uses a fix value α in the place of the derivative of the function in each step.

But here we use an a priori upper bound for the derivative instead of using its value in the initial approximation.
3. If x_k and y_k have the same limit, then the solution of $f(x) = 0$ in the interval $[c, d]$ is unique. So, uniqueness of solution of the equation can be verified numerically by the method introduced in Theorem 2.1, which is called Monotone Iteration Method (MIM), even if $f(x)$ is nonmonotone.

In order to compare the MIM with Newton's Method we considered the equation $f(x) \hat{=} x^3 + ax^2 + bx = 0$.
First we chose $a = -3$ so that $f''(1) = 0$. Defining $\psi(x) = x - \dfrac{f(x)}{f'(x)}$ and $h(x) = \psi(\psi(x))$, we then chose b in such way that $h(1) = 1$. This gives the following condition for a and b:

$$19a^6 - 171a^4b + 567a^2b^2 - 729b^3 = 0.$$

Since we have taken $a = -3$, a solution of this last equation is $b = 2.3611$. With this choice, the equation $f(x) = 0$ has $x = 0$ as the unique real solution. Moreover, an elementary calculation shows that $h'(1) = 0$ (because $f''(1) = 0$) and then there is an open interval I containing the number 1 in its interior such that if we take $x_0 \in I$ and define $x_{k+1} = h(x_k)$ then $\{x_k\}$ converges to 1. For instance, the interval $I = [0.94, 1.07]$ meets this condition. So if we take $x_0 \in [0.94, 1.07]$ and we consider the sequence $x_{k+1} = \psi(x_k)$ defined by Newton's method , then $\{x_{2k}\}$ converges to 1 and $\{x_{2k+1}\}$ converges to 1.5652. For this particular problem the conclusion is this:
1) for any real number x_0, if we choose α properly, then the MIM converges to the unique solution $f^*(x) \equiv 0$;
2) if we start with any $x_0 \in [0.94, 1.07]$ then the even and the odd elements of the sequence constructed by Newton's method have different limits, none of them being a solution of the problem.

3 Elliptic Equations

Now we consider the problem of solving the semilinear elliptic problem

$$-L(u) + f(x, u) = 0, \quad x \in \Omega \tag{2}$$

with boundary condition

$$Bu \hat{=} a\frac{\partial u}{\partial n} + bu = 0, \quad x \in \partial\Omega. \tag{3}$$

In (2) Ω is a bounded open set of the space R^N with boundary $\partial\Omega$ satisfying a Lipschitz condition, $f : \Omega \times R \to R$ is a continuous given function which is C^1 in u and

$$L(u) = \sum_{i=1,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial xj} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} - c_0(x)u(x)$$

is a second order uniformly elliptic operator with sufficient regular coefficients.
In (3) we take either $a = 1,\ b = 0$ (Neumann boundary condition) or $a = 0,\ b = 1$ (Dirichlet boundary condition).

Next we give the following

Definition. A C^2 function $v : \Omega \to R$ is a sub-solution of problem (2)-(3) if

$$-Lv(x) + f(x, v(x)) \le 0, \quad x \in \Omega, \quad Bv(x) \le 0, \quad x \in \partial\Omega.$$

A super-solution w is defined by taking the reverse inequalities.

Theorem 3.1. *Let $v(x)$ be a sub-solution of (2)-(3) and $w(x)$ be a super-solution of the same problem such that $v(x) \le w(x)$ for any $x \in \Omega$. Let $\alpha > 0$ be a constant such that $\alpha \ge \frac{\partial f(x,u)}{\partial u}$ for any (x, u) satisfying $v(x) \le u \le w(x)$ and $c_0(x) + \alpha > 0$ for any $x \in \Omega$.*
We define a sequence $\{v_k(x)\}$ in the following way: $v_0(x) = v(x)$ and $v_{k+1}(x)$ is the unique solution of

$$-Lv_{k+1}(x) + \alpha v_{k+1}(x) = \alpha v_k(x) - f(x, v_k(x)), \quad x \in \Omega$$

with boundary condition $Bv_{k+1}(x) = 0$ on $\partial\Omega$.
Similarly we define another sequence $\{w_k(x)\}$ by: $w_0(x) = w(x)$ and $w_{k+1}(x)$ is the unique solution of

$$-Lw_{k+1}(x) + \alpha w_{k+1}(x) = \alpha w_k(x) - f(x, w_k(x)), \quad x \in \Omega$$

with boundary condition $Bw_{k+1}(x) = 0$ on $\partial\Omega$.
Then for any k we have

$$v(x) \le v_k(x) \le v_{k+1}(x) \le w_{k+1}(x) \le w_k(x) \le w(x)$$

for any $x \in \Omega$, $\{v_k(x)\}$ and $\{w_k(x)\}$ converge to solutions $u_1(x)$ and $u_2(x)$ respectively of problem (2)-(3) satisfying $v(x) \le u_1(x) \le u_2(x) \le w(x)$ for

any $x \in \Omega$.
Moreover, any solution $u(x)$ of (2)-(3) such that $v(x) \le u(x) \le w(x)$ for any $x \in \Omega$ satisfies $u_1(x) \le u(x) \le u_2(x)$ for any $x \in \Omega$.

The proof of Theorem 3.1 is a consequence of the Maximum Principle and it can be found in [1], [3], [4].

The next result is useful to estimate the maximum and the minimum of solutions of some elliptic equations. For simplicicity, we consider a simpler class of elliptic operators $L(u)$.

Theorem 3.2. *Let $u(x)$ be a regular solution of the problem:*

$$\begin{array}{rcll} -Lu + \varphi(u) & = & 0 & x \in \Omega \\ u(x) & = & 0 & x \in \partial\Omega \end{array} \tag{4}$$

where

$$Lu = \Delta u + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i}.$$

If (x_0) is a point where the maximum of $u(x)$ is achieved then either we have $u(x_0) = 0$, or

$$\varphi(u(x_0)) \le 0.$$

Similarly for the minimum.

Proof

If $(x_0) \in \partial\Omega$, then $u(x_0) = 0$. If (x_0) is in the interior of Ω, then for each $1 \le i \le n$ we have:

$$\frac{\partial u}{\partial x_i}(x_0) = 0$$

$$\frac{\partial^2 u(x_0)}{\partial x_i^2} \le 0.$$

So,

$$\varphi(u(x_0)) = \Delta u(x_0) + \sum_{i=1}^{N} b_i(x_0) \frac{\partial u(u_0)}{\partial x_i} \le 0$$

and this proves the Theorem. ∎

3.1 Numerical Experiments

We start this Section presenting two examples where we consider boundary value problems of the form

$$-\Delta u + \varphi(u) \;=\; h(s,t) \tag{5}$$

with homogeneous Neumann boundary condition. In both cases, the domain is $\Omega = [0,1] \times [0,1]$ and we define $h(s,t)$ such that a solution of the problem is a given function $u^*(s,t)$. After discretization by central differences, (5) becomes a nonlinear system

$$F(y) \;=\; 0. \tag{6}$$

Example 1:
In this first case we take $\varphi(u) = 10sin(u),\;\; u^*(s,t) = \beta cos(\pi s)cos(\pi t)$ where $\beta = 0.5$. Then we have to solve:

$$\begin{aligned} -\Delta u + 10sin(u) &= -\Delta u^* + 10sin(u^*), \;\; \text{on} \;\; [0,1]\times[0,1] \\ \partial u/\partial n &= 0 \;\; \text{on} \;\; \partial\Omega. \end{aligned} \tag{7}$$

With these we can easily visualize numerically the behavior of both methods: Newton's method and the MIM.

To apply the MIM, after all the calculation we found $v_0 \equiv -1.0923$ as a constant sub-solution and $w_0 \equiv 1.0923$ as a constant super-solution for the problem. We also have that an iterative constant (α in Theorem 3.1) for the MIM is $\alpha = 10$.

¿From the periodicity of $\sin(u)$, we see that adding and subtracting multiples of 2π to v_0 and w_0 we have infinitely many constant sub-solutions and constant super-solutions for the problem: v_j and w_j respectively. Then if this problem has a solution, it will have infinite solutions (u_j^*), each one uniquelly determined by the corresponding pair v_j and w_j.

All the tests in this example were run in a SUN Ultra Creator using the software MatLab. We used a grid with 289 points which means 15 internal points in each axis and we have stopped the processes when $\|F(y_k)\|_\infty \leq 10^{-6}$.

The initial vectors of approximations were chosen in this way: $y_0 = (a_0, a_0, \cdots, a_0)^T$ where $a_0 = 1.1{+}2C\pi,\; C = 0,1,\ldots,5$, because 1.1 is a super-solution to the solution with all the components in the interval $[-0.5, 0.5]$ and with the other initial approximations we will obtain other solutions for the nonlinear system.

In order to compare the perfomance of MIM we used the same initial approximations for Newton's method. It is important to observe that these vectors may not be good choices for Newton's method since that, in some cases, they may not be in an appropriate neighborhood of u_j^*. This is an important difference between the two methods because in order to apply the MIM it is not necessary to start close to a solution of the problem.

Table 1 gives the notation used to represent the different solutions obtained in the next tests. The end points of the interval represent the minimum and the maximum values assumed by the entries of the approximate solution found.

s_j	Interval
s_1	$[-0.5, 0.5]$
s_2	$[1.8421, 4.4411]$
s_3	$[5.7832, 6.7832]$
s_4	$[8.1253, 10.7243]$
s_5	$[12.0664, 13.0664]$
s_6	$[14.4085, 17.0075]$
s_7	$[18.3496, 19.3496]$
s_8	$[20.6917, 23.2906]$
s_9	$[24.6327, 25.6327]$
s_{10}	$[26.9748, 29.5738]$
s_{11}	$[30.9159, 31.9159]$

Table 1: Notation for some approximate solutions to the nonlinear system.

a_0	MIM				Newton			
	s_j	IT	Time	Flops	s_j	IT	Time	Flops
1.1	s_1	6	0.3326	339591	s_1	5	1.3560	669376
$1.1 + 2\pi$	s_3	6	0.3201	339591	s_3	5	1.3697	669376
$1.1 + 4\pi$	s_5	6	0.3154	339591	s_5	5	1.3734	669376
$1.1 + 6\pi$	s_7	6	0.3452	339591	s_7	5	1.3674	669376
$1.1 + 8\pi$	s_9	6	0.3130	339591	s_9	5	1.3581	669376
$1.1 + 10\pi$	s_{11}	6	0.3134	339591	s_{11}	5	1.3597	669376

Table 2: Comparative results of MIM and Newton's Methods.

Remarks

1) In the case of Table 2, where both methods converged to the same solutions, the total iterations number of Newton's method is smaller than that

of the MIM. But the MIM is faster because it is cheaper in terms of the computational cost by iteration, considering the elapsed time and the number of flops performed. Such a difference is justified by the fact that, in each iteration of Newton's method it is solved a completely new linear system while in the MIM the matrix of coeficients is independent of the step and so, its factorization has to be done just once in the whole process.
2) The MIM requires an initial work to find a super and a sub-solution for the problem and also to find the iterative constant α; without them is it impossible to apply the MIM. In this example we obtained these constants very easily but in some cases it may happen to be not so easy to do this job.
3) After being calculated, the constants of the MIM represent an advantage for this method because they give a region where we guarantee the existence of a solution.

We have also analysed the performance of both methods for initial approximations between two super-solutions. For this, we have chosen some values between 1.1 and $1.1 + 2\pi$. Table 3 presents the results obtained. The solution called s_0 has its entries in the interval $[-13.0664, -12.0664]$ and we have kept the same notation for the other s_j solutions.

a_0	MIM				Newton			
	s_j	IT	Time	Flops	s_j	IT	Time	Flops
1.5	s_1	7	0.3471	367029	s_0	3	1.2520	402408
2.0	s_1	8	0.3645	394466	s_2	5	1.4212	669376
2.5	s_1	9	0.4018	421904	s_2	3	1.2198	402408
3.0	s_1	11	0.4868	476781	s_2	4	1.3470	535892
3.5	s_3	10	0.4434	449343	s_2	3	1.2294	402408
4.0	s_3	8	0.3693	394467	s_2	4	1.3527	535892
4.5	s_3	7	0.3309	367029	s_1	4	1.3479	535892
5.0	s_3	7	0.3645	367029	s_4	5	1.4031	669376
5.5	s_3	6	0.2961	339591	s_3	4	1.3126	535892
6.0	s_3	5	0.2571	312153	s_3	2	1.1332	268924
6.5	s_3	5	0.2603	312513	s_3	2	1.1117	268924

Table 3: Comparative results of `MIM` and `Newton's` Methods.

Remarks

1) Observe that for some solutions of the nonlinear system it was not possible to find a convergent sequence by the MIM since, for such solutions, it is not possible to find constant super and sub-solutions. In Table 3 we can see, for example, that the solution s_2, with entries in the interval $[1.8421, 4.4411]$

detected by Newton's method, was not found by the MIM.
2) Some initial approximations are not adequate for Newton's method because besides beeing not close enough to a solution x^*, they are in regions of ill-conditioning of the Jacobian matrix. In these cases, the sequence generated by Newton's method converged for other solutions, sometimes very far from the expected solution. This happens, for example, in the neighborhood of $a_0 = 1.5$. Small perturbations in this value made the solutions reached be very distinct from each other, as we can see in Table 4.

We ran the MIM with all these inital guesses and we always obtained the solution s_1.

a_0	`Newton`			
	$[a, b]$	`IT`	`Time`	`Flops`
1.51	$[-17.0075, -14.4084]$	5	1.4519	669376
1.53	$[-23.2906, -20.6917]$	5	1.3906	669376
1.55	$[-48.4234, -45.8244]$	4	1.2772	535892
1.57	$[-1.255E+03, -1.252E+03]$	4	1.2847	535892
1.59	$[52.1076, 54.7066]$	4	1.2863	535892

Table 4: Newton's Method with initial approximations close to 1.5.

Example 2 :
In this case we took $\varphi(u) = u^3+au^2+bu$, with $a = -3,\ b = 2.36110308062087$ and $u^*(s,t) = \beta cos(\pi s)cos(\pi t)$. For small values of β, the behavior of Newton's method is similar to its behavior for the unidimensional case of Section 2. We took for β the values $\beta = 5$ and several small values which will be given in a next Table. Then, the problem to be solved is :

$$\begin{array}{rcl} -\Delta u + u^3 + au^2 + bu & = & -\Delta u^* + u^{*3} + au^{*2} + bu^*, \quad \text{on} \ \ \Omega \\ \partial u/\partial n & = & 0 \ \ \text{on} \ \ \partial\Omega \end{array} \tag{8}$$

where $\Omega = [0, 1] \times [0, 1]$.

For this example the tests were run in a SUN Ultra1 Creator, using the software MatLab. We will now present some numerical results. All the problems were solved with dimension 289 and we have stopped the process when $||F(y_k)||_\infty < 10^{-6}$.

In Table 5, for each case we calculated a constant sub-solution, a constant super-solution and also a corresponding iterative constant α for the MIM. In this table, a_0 is the constant for all the entries of the initial approximation vector; `IT` is the total number of iterations performed and `TIME` represents the CPU time in seconds. In the column α we present the corresponding iterative constant α for the MIM and nc for Newton's method means that the method did not converge after 300 iterations and in all these cases the sequence of the even iterations converged to a constant vector with all the entries close or equal to 1 and the same to the sequence of odd iterations, to 1.5652. Observe also that when $\beta = 0$ the convergence of the MIM was to the solution $u^*(s,t) \equiv 0$.

We finish this example showing, in Table 6, the behavior of Newton's method for several values of β and for all of them we have started the iterations with $a_0 = 1$ and also with $a_0 = 0.95$, with the same results. Nonconvergence is indicated by nc and in all of these cases, the sequence oscillated again between values close or equal to 1 and close or equal to 1.5672.

β	a_0	MIM			Newton	
		α	IT	Time	IT	Time
5	-6	142	206	18.1095	6	1.8435
	7		211	18.3048	7	1.8558
0.01	-0.5	7	32	2.8933	4	1.4590
	1.0		45	3.9926	nc	–
0.045	1.0	10	67	5.9170	80	11.7094
0	1.0		67	5.8487	nc	–

Table 5: Behavior of Newton's Method and MIM for Example 2.

β	Newton	
	IT	Time
10^{-6}	nc	–
10^{-2}	nc	–
0.005	nc	–
0.002	nc	–
0.015	nc	–
0.04	nc	–
0.0425	nc	–
0.0449	62	9.2488
0.05	35	5.7022
0.09	15	2.9225

Table 6: Behavior of Newton's Method for other values of β.

Example 3 :
So far we have worked with problems that have a variational formulation and so they could be solved by other methods such as Gradient methods. In this Example we show the performance of Newton's Method and of the MIM for a problem with a convective term:

$$c_s \frac{\partial u}{\partial s} + c_t \frac{\partial u}{\partial t};$$

this problem does not have a variational formulation and so the Gradient methods can not be used. Now we work with the following problem:

$$\begin{array}{rcl} -Lu + \varphi(u) & = & 0, \text{ on } [0,1] \times [0,1] \\ u(s,t) & = & 0 \text{ on } \partial\Omega \end{array} \tag{9}$$

where

$$Lu = \Delta u - c_s \frac{\partial u}{\partial s} - c_t \frac{\partial u}{\partial t}$$

and

$$\varphi(u) = cu(u-5)(u-10)(u-15).$$

Observe that the problem has the trivial solution $u^*(s,t) \equiv 0$ and we are interested in calculating non trivial ones. The numerical results are presented in Tables 7 and 8. We used the constant $c = 0.1$ and several values for c_s, for c_t and for the initial vectors for which all the entries are equal to $a_0 \leq 10$, a super-solution to the MIM. These values were found after all the necessary calculation which gave also for the constant of the MIM, $\alpha = 40$. All the tests were run in a Pentium 166 microcomputer, using the software MatLab.
In the next tables:
It means the number of iterations performed until convergence;
$Time$ is the elapsed time until convergence;
Sol is the maximum value assumed by the approximated solution in the grid.
We have worked with 15 internal nodes, what makes the systems with dimension 225. The stopping criterion was

$$\|F(y_k)\|_\infty \leq 10^{-4}.$$

Nonconvergence (nc) means

$$\|F(y_k)\|_\infty > 10^{10}.$$

Notice that, according to Theorem 3.2, for any solution $u^*(s,t)$ we have that

$$u^*_{max} \leq 15,$$

because $\varphi(u) > 0$ for $u > 15$. In our experiments the vectors y_k assume positive large values, and this is also an indication of nonconvergence.

			Newton			MIM		
c_s	c_t	x_0	It	Time	Sol	It	Time	Sol
0.1	0.2	5	4	1.04	3.5978	19	1.86	3.5978
		6	4	1.05	3.5978	19	1.87	3.5978
		7	5	1.27	0.0000	19	1.87	3.5978
		8	-	-	nc	19	1.92	3.5978
		9	-	-	nc	20	2.02	3.5978
		10	14	3.46	3.5978	20	2.03	3.5978
0.4	-0.6	5	4	1.10	3.5901	19	1.81	3.5901
		6	4	1.10	3.5901	19	1.92	3.5901
		7	5	1.26	0.0000	20	2.03	3.5901
		8	-	-	nc	20	1.92	3.5901
		9	-	-	nc	20	2.03	3.5901
		10	14	3.46	3.5901	20	1.97	3.5901
0.6	-0.8	5	4	1.04	3.5822	19	1.82	3.5822
		6	4	1.04	3.5822	20	1.97	3.5822
		7	5	1.26	0.0000	20	1.98	3.5822
		8	-	-	nc	20	1.98	3.5822
		9	-	-	nc	20	1.92	3.5822
		10	15	3.74	3.5822	21	2.04	3.5822
2	-3	5	4	1.04	3.5053	24	2.25	3.5053
		6	4	1.05	3.5053	24	2.19	3.5053
		7	6	1.59	0.0000	24	2.25	3.5053
		8	-	-	nc	24	2.25	3.5053
		9	-	-	nc	24	2.20	3.5053
		10	18	4.51	3.5053	25	2.37	3.5053
5	-8	5	6	1.48	2.8917	52	4.72	2.5918
		6	5	1.26	2.8917	52	4.73	2.5918
		7	7	1.75	0.0000	52	4.73	2.5918
		8	-	-	nc	53	4.84	2.5918
		9	-	-	nc	53	4.84	2.5918
		10	-	-	nc	53	4.83	2.5918

Table 7: Behavior of Newton's Method and MIM for Example 3.

We have also checked that Newton's Method converged to the trivial solution $u^*(s,t) \equiv 0$ for every constant initial vector starting with $a_0 \in [6.801,\ 7.199]$, for all the choices of c_s and c_t that we have made. For all these values, the MIM converged to a solution whose maximum value is 3.5815.

Observe that when $c_s = c_t = 0$, we have a variational case. We exhibit this case in the separate Table 8 to emphasyze this fact. Again, for $a_0 = 7$ the convergence of Newton's method was to the trivial solution. Notice that, in the variational case, the trivial solution is uninteresting because it is a unstable stationary solution of an evolution equation.

			Newton			MIM		
c_s	c_t	x_0	It	Time	Sol	It	Time	Sol
0	0	5	4	0.82	3.5986	18	1.70	3.5986
		6	4	0.88	3.5986	19	1.81	3.5986
		7	5	1.32	0.0000	19	1.86	3.5986
		8	-	-	nc	19	1.81	3.5986
		9	-	-	nc	20	1.93	3.5986
		10	14	2.80	3.5986	20	1.93	3.5986

Table 8: Example 3, variational case.

The final remark in this Example is the analysis of Newton's Method for the following initial constant vectors for which we can not use the theory of the MIM: $x_0 = 1,\ 2,\ 3,\ 4$. We present in Table 9, only the cases where the results were different from the positive solution results already found for the corresponding values of c_s and c_t.

			Newton		
c_s	c_t	x_0	It	Time	Sol
0	0	1	11	2.91	0.0000
0.1	0.2	1	11	2.75	0.6060
0.4	-0.6	1	-	-	nc
0.6	-0.8	1	-	-	nc
2	-3	1	-	-	nc

Table 9: Newton's Method where the theory of the MIM does not apply. Observe that in these cases, the MIM converged to $Sol = 3.5986$.

Example 4 :
In this example we use the numerical results of the MIM to verify uniqueness

of solution. Here we solve the following problem:

$$\begin{aligned} -\Delta u + \varphi(u) &= 0, \quad \text{on } [0,1]\times[0,1] \\ u(s,t) &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{10}$$

where

$$\varphi(u) = cu(u-2)(u-3)(u-5).$$

For this problem, Theorem 3.2 assures the existence of at least one positive solution, provided that

$$\lambda_2 \leq \left.\frac{\partial \varphi}{\partial u}\right|_{u=0} < \lambda_1,$$

where λ_1 and λ_2 are the first and second eigenvalues of the Laplacian, respectively. We want then to verify whether such a solution is unique or not. It is easily seen that, choosing c according to the last condition and for k small enough, the first eigenfunction of the Laplacian,

$$u_1(s,t) = ksin(\pi s)sin(\pi t)$$

is a sub-solution for problem (10).

Using Theorem 3.2 again, we have that $u_c^* \equiv 5$ is a super-solution for this problem and we also have that every solution $u^*(s,t)$ is such that

$$u^*_{max} \leq 5.$$

The values found for c and k were:

$$0.6580 \leq c < 1.6449$$

and then

$$0 < k \leq 3.5.$$

We took $c = 1.15$, for which a constant α for the MIM is $\alpha = 35$ and some values for k, $k = 0.1,\ 0.2,\ 0.3$. The conclusions are the following:

1. Starting the MIM with the sub-solution $u_1(s,t)$, the sequences converged to a positive solution $u_p^*(s,t)$ whose maximum value in $[0,\ 1] \times [0,\ 1]$ is $u^*_{p_{max}} \leq 0.7$.

2. Starting the MIM with the constant super-solution $u_c(s,t) \equiv 5$ the sequence converged to the same solution $u_p^*(s,t)$ of item 1.

Since starting with a super-solution and with a sub-solution the convergence was to the same solution, by the remarks made after Theorem 2.1, we conclude that this problem has a unique positive solution.

This is a very interesting and special feature of the MIM: we can use its numerical results to conclude uniqueness (or nonuniqueness) of solution. Notice that this analysis is a very important and not such an easy task, in general.

Example 5 :
The last example of this Section shows another very important feature of the MIM: for the problems above, the MIM is a good tool to find an appropriate initial approximation for Newton's Method.

We have had used Newton's Method for problem (10) of Example 4 starting with the same sub-solutions $u_1(s,t)$ used there and in all the cases the sequences converged to the trivial solution $u^*(s,t) \equiv 0$. We also started Newton's Method with the constant super-solution $u_c(s,t) \equiv 5$ and the method did not converge. So we used a combination of the MIM and Newton's method, that consists of:

(i) running the MIM starting with $y_0 \equiv 5$, $\alpha = 35$ until $\|F(\bar{y}_k)\| \leq 10^{-2}$ (MINIC);

(ii) using the $\bar{y}_k$ of (i) as initial approximation, run Newton'sMethod (and also the MIM).(MIM/MIM for the combination MIM and MIM and MIM/NEW for the combination MIM and Newton).

Table 10 shows how this combination works. The stopping criterion for both methods was $\|F(\tilde{y}_k)\|_\infty \leq 10^{-6}$ and $Sol \equiv \|\tilde{y}_k\|_\infty$. Finally Dl means the number of interior points in each axis.

	MINIC		MIM/MIM			MIM/NEW		
Dl	Sol	It	Sol	It	Time	Sol	It	Time
3	0.6899	26	0.6890	35	0.17	0.6890	2	0.06
7	0.6728	28	0.6719	37	0.55	0.6719	2	0.11
9	0.6693	28	0.6684	38	0.82	0.6684	2	0.17
19	0.6641	29	0.6633	37	3.79	0.6633	2	0.88
24	0.6614	29	0.6606	37	6.37	0.6606	2	1.81
29	0.6631	29	0.6623	37	10.65	0.6623	2	13.51

Table 10: The combination of the MIM and Newton's Method.

Looking at the last line of Table 10 we can see that, when we increase the number of mesh points, the ellapsed time for the MIM turns out to be smaller then that of Newton's Method which is natural to happen because each Newton's iteration is much more expensive then the MIM's iterations. As we have pointed out, when we use the MIM, the matrix of the linear system that we have to solve at each step is the same.

4 Cooperative Elliptic Systems

In the case of elliptic systems we can prove a result similar to Theorem 3.1 provided we assume that the nonlinearity satisfies a cooperative condition. For 2×2 systems the problem is:

$$\begin{array}{rcl} -L(x,u) + f(x,u,v) & = & 0 \quad \text{on} \quad \Omega, \quad Bu = 0 \quad \text{on} \quad \partial\Omega \\ -M(x,v) + g(x,u,v) & = & 0 \quad \text{on} \quad \Omega, \quad Bv = 0 \quad \text{on} \quad \partial\Omega \end{array} \tag{11}$$

where Ω is as before, $f, g : \Omega \times R \times R :\to R$ are given continuous functions which are C^1 in u, v and Hölder continuous in x and

$$L(x,u) = \sum_{i=1,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} - c_0(x) u(x)$$

and

$$M(x,v) = \sum_{i=1,j=1}^{N} \tilde{a}_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \tilde{b}_i(x) \frac{\partial v}{\partial x_i} - \tilde{c}_0(x) v(x)$$

are given second order uniformly elliptic operators with sufficiently regular real coefficients and the boundary condition is as in Theorem 3.1.

If we define a vector $U = (u, v)$ then system (11) can be written as

$$-N(x,U) + H(x,U) = 0, \quad \text{on} \quad \Omega, \quad BU = 0 \quad \text{on} \quad \partial\Omega,$$

where $N(x,U) = (L(x,u), M(x,v))$, $H(x,U) = (f(x,u,v), g(x,u,v))$ and $BU = (Bu, Bv)$.

Definition. A C^2 function $\Phi : \Omega \to R \times R$ is a sub-solution of (11) if

$$-N(x,\Phi(x)) + H(x,\Phi(x)) \leq 0 \quad \text{on} \quad \Omega \quad \text{and} \quad B\Phi(x) \leq 0 \quad \text{on} \quad \partial\Omega$$

The inequality holds componentwise here.
A super-solution $\Psi(x)$ is defined in a similar way.

Theorem 4.1. *Suppose that $\Phi(x)$ and $\Psi(x)$ are sub-solution and super-solution, respectively, of (11) such that $\Phi(x) \leq \Psi(x)$ for any $x \in \Omega$. Suppose also that $\dfrac{\partial f(x,u,v)}{\partial v} \leq 0$ and $\dfrac{\partial g(x,u,v)}{\partial u} \leq 0$ for any (x,u,v) such that $\Phi(x) \leq U \leq \Psi(x)$. Let $\alpha > 0$ be a real number such that $\alpha + c_0(x) > 0$, $\alpha + \tilde{c}_0(x) > 0$ for any x in Ω and $\dfrac{\partial f(x,u,v)}{\partial u} \leq \alpha$, $\dfrac{\partial g(x,u,v)}{\partial v} \leq \alpha$ for any $(x,U) = (x,u,v)$ such that $\Phi(x) \leq U \leq \Psi(x)$.*
We define a sequence $\{\Phi_k(x)\}$ in the following way: $\Phi_0(x) = \Phi(x)$ and $\Phi_{k+1}(x)$ is the unique solution of

$$-L\Phi_{k+1}(x) + \alpha\Phi_{k+1}(x) = \alpha\Phi_k(x) - H(x, \Phi_k(x)) \quad x \in \Omega$$

with boundary condition $B\Phi_{k+1}(x) = 0$ on $\partial\Omega$.
Similarly we define another sequence $\{\Psi_k(x)\}$ by: $\Psi_0(x) = \Psi(x)$ and $\Psi_{k+1}(x)$ is the unique solution of

$$-L\Psi_{k+1}(x) + \alpha\Psi_{k+1}(x) = \alpha\psi_k(x) - H(x, \Psi_k(x)) \quad x \in \Omega$$

with boundary condition $B\Psi_{k+1}(x) = 0$ on $\partial\Omega$.
Then

$$\Phi(x) \leq \Phi_k(x) \leq \Phi_{k+1}(x) \leq \Psi_{k+1}(x) \leq \Psi_k(x) \leq \Psi(x)$$

for any $x \in \Omega$ and $\Phi_k(x)$ and $\Psi_k(x)$ converge to solutions $U_1(x)$ and $U_2(x)$ of system (11) satisfying $\Phi(x) \leq U_1(x) \leq U_2(x) \leq \Psi(x)$ for any $x \in \Omega$.
Moreover, any solution $U(x)$ of (11) such that $\Phi(x) \leq U(x) \leq \Psi(x)$ for any $x \in \Omega$ satisfies $U_1(x) \leq U(x) \leq U_2(x)$ for any $x \in \Omega$.

Proof

The proof of Theorem 4.1 follows exactly the proof of Theorem 3.1 because under the assumptions we have made, $-H(x,U)$ is monotone in the region $\Phi(x) \leq U \leq \Psi(x)$. ∎

Remark

In Theorem 4.1 we may allow a coupling also in the linear part but, in that case, a more complicate Maximum Principle is required (see [3]).

4.1 Numerical Experiments

Among others, a large class of examples of cooperative elliptic systems is made of the boundary value problems of the form

$$-\Delta U + \phi(U) \;=\; H(s,t), \quad (s,t) \in \Omega \tag{12}$$

with homogeneous Dirichlet boundary conditions. We will present one of such examples that is the prompt-feedback reactor problem. This problem has some similarity with various particular mutualism models in ecology, in the sense that the presence of each component contributes to faster growth rate of all other components (see [4]). We present some numerical experiments with the following two species mutualism model:

$$\begin{aligned} -\Delta u_1 - u_1(a - bu_1 + cu_2) &= 0, \\ -\Delta u_2 - u_2(e + fu_1 - gu_2) &= 0, \quad \text{on } \Omega, \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{13}$$

Here $a,\ b,\ c,\ d,\ e,\ f,\ g,$ are nonnegative constants, and u_1 and u_2 are the concentrations of mutualistic (cooperating) species. So, the interesting solutions are the positive ones.

In our first experiment we solved a system with a choice for the parameters that were taken satisfying the assumptions of the next Theorem (see also [4].

Let us recall that the first eigenvalue λ_1 of the operator $-\Delta$ with Dirichlet boundary condition is simple,positive and that the eigenfunction associated to it is strictly positive on Ω.

Theorem 4.2. *Suppose that a and $e > \lambda_1$, (the smallest eigenvalue of the Laplacian) that $b,\ g$ are positive and $c,\ f$ are nonnegative, satisfying*

$$bg > cf.$$

Then the boundary value problem (13) has a solution $\bar{u_1}(s,t), \bar{u_2}(s,t)$ with $\bar{u_i}(s,t) > 0,\ \ i = 1,\ 2$ for $(s,t) \in \Omega$.

Before proving Theorem 4.2, let us prove a Lemma that will be useful for studying uniqueness of solution of (13).

Lemma 4.1. *If $\lambda_1 < a$ and $\lambda_1 < e$ and $\phi(s,t)$ is the eigenfunction associated to λ_1 then there is an $\epsilon_0 > 0$ such that $(\epsilon\phi(s,t), \epsilon\phi(s,t))$ is a sub-solution of system (13) for $0 < \epsilon \le \epsilon_0$.*

Proof

In fact, $(\epsilon\phi(s,t), \epsilon\phi(s,t))$ to be a sub-solution of system (13) is equivalent to the two inequalities

$$\epsilon\lambda_1\phi(s,t) - \epsilon\phi(s,t)(a - b\epsilon\phi(s,t) + c\epsilon\phi(s,t)) \le 0$$

$$\epsilon\lambda_1\phi(s,t) - \epsilon\phi(s,t)(e + f\epsilon\phi(s,t) - g\epsilon\phi(s,t)) \leq 0$$

which are equivalent to

$$\lambda_1 - (a - b\epsilon\phi(s,t) + c\epsilon\phi(s,t)) \leq 0$$

$$\lambda_1 - (e + f\epsilon\phi(s,t) - g\epsilon\phi(s,t)) \leq 0$$

and these two inequalities are satisified if ϵ is positive and small and this proves the lemma. ∎

Proof of Theorem 4.2.
Under the assumptions of Theorem 4.2 and in view o Lemma 4.1, we know that $(\epsilon\phi(s,t), \epsilon\phi(s,t))$ is a sub-solution of system (13) for $0 < \epsilon \leq \epsilon_0$. Moreover, an easy calculation show that each component of the solution (u_0, v_0) of the linear system

$$a - bu + cv = 0$$

$$e + fu - gv = 0$$

is positive and then (u_0, v_0) is a constant super-solution of (13) and in view of Theorem 4.1, Theorem 4.2 is proved. ∎

For $a = 25,\ b = 5,\ c = 3,\ e = 22,\ f = 2,\ g = 6$, we used both Newton's Method and the MIM, starting with the constant super-solution $(u_0, v_0) = (9, 6.666...)$. For these values the constant α for the MIM was found to be $\alpha = 65$. Stopping the processes when $\|F(u_k, v_k)\|_\infty \leq 10^{-4}$ we verified convergence of both methods to a positive solution with $(u_{max}, v_{max}) \leq (2.626,\ 1.620)$.

The last experiment we have done was to use again numerical results of the MIM to help the identification of uniqueness of solution of this kind of Cooperative Elliptic Systems.

Next we analyse uniqueness of positive solution of system (13) and the the following a priori estimate is important.

Theorem 4.3. *Suppose $c < b$, $f < g$ and let $M = \max(\frac{a}{b-c}, \frac{e}{g-f})$. If $(u(s,t), v(s,t))$ is a positive solution of system then for any $(s,t) \in \Omega$ we have $u(s,t) \leq M$ and $v(s,t) \leq M$.*

Proof

Let $M_1 > 0$ and $M_2 > 0$ be the maximum of $u(s,t)$ and $v(s,t)$, respectively, and suppose $M_2 \leq M_1$. If we denote by (s_0, t_0) the point in the interior of Ω where $u(s_0,t_0) = M_1$ then $\Delta u(s_0,t_0) \leq 0$ and then from the first equation of (13) we see that

$$a - bu(s_0,t_0) + cv(s_0,t_0) \geq 0$$

and, since $v(s_0,t_0) \leq M_2 \leq M_1$ this las inequality gives $M_1 \leq \frac{a}{b-c}$. In a completely similar way, we show that if $M_1 \leq M_2$ then $M_2 \leq \frac{e}{g-f}$ and this proves the Theorem 4.3. ∎

Lemma 4.1 and Theorem 4.3 can be used to verify numerically whether the positive solution of system (13) is unique or not. In fact, let (u_0, v_0) be a constant positive super-solution such that $M \leq u_0$ and $M \leq v_0$, where M is the constant given by Theorem 4.3. Then, according to Theorem 4.3 any positive solution $(u(s,t), v(s,t))$ has to satisfy $u(s,t) \leq u_0$ and $v(s,t) \leq v_0$ for any $(s,t) \in \Omega$. Moreover, for any such a solution $(u(s,t), v(s,t))$ there is an $\epsilon > 0$ such that $(\epsilon\phi(s,t), \epsilon\phi(s,t)) \leq (u(s,t), v(s,t))$. So, in view of Theorem 4.1, system (13) has a unique positive solution if and only if the two solutions we get starting at (u_0, v_0) and $(\epsilon\phi(s,t), \epsilon\phi(s,t))$ with ϵ small enough are equal.

For the value of the parameters chosen above we have $M = 12.5$, $\epsilon_0 = 2.5$, $(u_0, v_0) = (13, 13)$ and the constant α can be taken as 65. In that case our numerical experiments comprove uniqueness of positive solution.

5 Conclusions

In this work we have presented some theoretical results and we have proven others for the MIM; we also have made several numerical experiments with this method, comparing it with Newton's method whose features are very well known [2]. Basically Newton's method can be used to solve any equation but in order to guarantee its convergence, we have to start at a point which is "close enough" to a solution and, sometimes, this may be difficult to accomplish because we do not know how to guess where the solution (or solutions) is. The results proved and the comparisons that we made here lead us to conclude that the MIM has some limitations and also several good features. These facts guarantee that both, the MIM and Newton's Method have their own merit. We think that the following points deserve to be emphasized:

(i) In order to apply the MIM we have to be able to find a sub-solution and a super-solution (this is not always easy) and the constant α (this is easy). So, in order to apply the MIM we always need an extra initial work. But once we have verified all these conditions, we can guarantee the existence of a solution and the convergence of the method, even if we are not close enough to a solution in the "Newton's sense".

(ii) The MIM works if a maximum principle is available. For instance, in the case of systems, this requires it to be cooperative. Of course this excludes many interesting problems.

(iii) In some cases, the MIM can be used to guarantee uniqueness (or nonuniqueness) of solution for certain type of problems.

(iv) Computationally speaking, the MIM is very cheap and so, even if we start at an approximation of the solution that is reasonable for both methods, its performance can be better than Newton's, mainly in the case of thin grids.

(v) In some cases, Newton's method can find some solutions that can not be detect (and found) by the MIM.

Our final remark is that several initial guesses that are good initial approximations for the MIM are very bad for Newton's method, either because they are far from a solution or because the Jacobian matrix is close to singular at these points. So, we found that another feature of the MIM is that it can be used to find a good initial guess for Newton's method which has been done in Example 5.

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