A family of maximal noncontrollable Lie wedges with empty interior

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Abstract

Let S_k be the semigroup of matrices in $\mathrm{Sl}(n, \mathbb{R})$ having non-negative k-minors. Its Lie wedge $\mathcal{L}(S_k)$ generates $\mathfrak{sl}(n, \mathbb{R})$, is noncontrollable, and has empty interior if $1 < k < n-1$. Moreover, $\mathcal{L}(S_k)$ is maximal if k and n are even.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A Lie wedge $W \subset \mathfrak{g}$ is said to be noncontrollable (NC) in G provided the semigroup generated by $\exp(W)$ is not G. It is generating noncontrollable (GNC) if W generates $\mathfrak{g}$, as a Lie algebra and is noncontrollable. Also, W and is MGNC if it is maximal with these properties.

From the characterization of maximal semigroups provided by J. Lawson [3] it is known that if $\mathfrak{g}$ is solvable and G simply connected then a MGNC $W \subset \mathfrak{g}$ is a half-space bounded by a hyperplane subalgebra. In particular the MGNC Lie wedges have nonempty interior in $\mathfrak{g}$. A similar fact holds for the $\mathrm{Sl}(2, \mathbb{R})$: Up to conjugation there is just one Lie wedge in $\mathfrak{sl}(2, \mathbb{R})$ which is noncontrollable with respect to $\mathrm{Sl}(2, \mathbb{R})$, namely $W = \{(x_{ij}) : x_{12}, x_{21} \geq 0\}$ (see [2]), which has nonempty interior. On the other hand D. Mittenhuber [5] pointed out to the fact that for general semi-simple groups the situation is not so clear. There are pathologies: As is showed in [5] there are, already in rank one Lie groups, Lie semigroups whose Lie

*Research partially supported by CNPq grant n° 301060/94 − 0

wedge is MGNC but has empty interior in the Lie algebra $\mathfrak{su}(1,n)$. In this note we reinforce the discovery of [5] by providing an example of a Lie wedge in $\mathfrak{sl}(n,\mathbb{R})$, $n>2$ even, which has empty interior and is MGNC with respect to $\mathrm{Sl}(n,\mathbb{R})$.

1 Lie Wedges

Let $\{e_1,\ldots,e_n\}$ be the standard basis in $\mathbb{R}^n$. For each k multi-index $I=(1\leq i_1<\cdots<i_k\leq n)$ put $e_I=e_{i_1}\wedge\cdots\wedge e_{i_k}$. The set $\{e_I\}$ with I running through the k multi-indices is a basis of the k-fold exterior product $\bigwedge^k:=\bigwedge^k\mathbb{R}^n$. We endow $\bigwedge^k$ with the standard inner product $\langle\cdot,\cdot\rangle$, which makes $\{e_I\}$ an orthonormal basis. Denote by $\mathcal{O}_k\subset\bigwedge^k$ the positive orthant with respect to this basis:

$$\mathcal{O}_k=\{\sum a_Ie_I : a_I\geq 0\}.$$

Let $G=\mathrm{Sl}(n,\mathbb{R})$ and consider its natural representation on $\bigwedge^k$. Define the compression semigroup

$$S_k=\{g\in G : g\mathcal{O}_k\subset\mathcal{O}_k\}.$$

Of course $g\in S_k$ if and only if its matrix with respect to the standard basis of $\mathbb{R}^n$ has non-negative k-minors.

Let $\mathrm{Gr}_k^+(n)\subset\bigwedge^k$ stand for the subset of norm one decomposable vectors of $\bigwedge^k$.This notation is in accordance to the fact that $\mathrm{Gr}_k^+(n)$ is in bijection to the Grassmannian of oriented k-dimensional subspaces of $\mathbb{R}^n$. The compression semigroup S_k can be defined within $\mathrm{Gr}_k^+(n)$. In fact, put $C_k=\mathcal{O}_k\cap\mathrm{Gr}_k^+(n)$. Since $e_I\in\mathrm{Gr}_k^+(n)$, and $\mathcal{O}_k$ is the cone spanned by $\{e_I\}$ it follows that $gC_k\subset C_k$ if and only if $g\in S_k$.

If $k=1$ then S_k is the semigroup $\mathrm{Sl}^+(n,\mathbb{R})$ of matrices with positive entries. On the other hand S_{n-1} is conjugate to S_1^{-1}, as can be easily seen by identifying $\bigwedge^{n-1}$ with the dual $\left(\bigwedge^1\right)^*$.

Denote by $\mathcal{L}(S_k)\subset\mathfrak{sl}(n,\mathbb{R})$ the Lie wedge of S_k. Then $X\in\mathcal{L}(S_k)$ if and only if $\langle Xe_I,e_J\rangle\geq 0$ for all pair of k multi-indices $I\neq J$. From this one can get an explicitly description of $\mathcal{L}(S_k)$ for all k:

First the subalgebra $\mathfrak{a}$ of diagonal matrices is contained in $\mathcal{L}(S_k)$ for all k, because e_I is an eigenvector of all $H\in\mathfrak{a}$. Since $\mathcal{L}(S_k)$ is invariant under $\mathrm{Ad}(A)$, $A=\exp\mathfrak{a}$, it follows that if $X=(x_{ij})\in\mathcal{L}(S_k)$ then $x_{ij}E_{ij}\in\mathcal{L}(S_k)$ for all $i\neq j$. Here E_{ij} is the basic matrix with 1 in the i,j-entry and zero otherwise. Therefore, in order to get $\mathcal{L}(S_k)$ it is enough to check whether $\pm E_{ij}\in\mathcal{L}(S_k)$, for the various i,j with $i\neq j$. Consider the cases:

- If $k = 1$ it follows (and is well known) that $X = (x_{ij}) \in L(S_1)$ if and only if $x_{ij} \geq 0$ for $i \neq j$.

- Assume $k = n - 1$. Pick E_{ij} with $i < j$, and a multi-index I. If $i \in I$ or if $j \notin I$ then $E_{ij} e_I = 0$. Suppose then that $i \notin I$ and $j \in I$. Then $E_{ij} e_I$ is the k-fold exterior product of the basic elements with e_i in place of e_j. Thus let $F = (I \setminus \{j\}) \cup \{i\}$, with the indices in the right order. Then

$$E_{ij} e_I = (-1)^{j-i-1} e_F.$$

  Therefore $E_{ij} \in \mathcal{L}(S_{n-1})$ if $j - i$ is odd and $-E_{ij} \in \mathcal{L}(S_{n-1})$ if $j - i$ is even. By a similar reasoning with $j < i$ (or using the fact that $\mathcal{L}(S_k)$ is invariant under transposition), we get that $X = (x_{ij}) \in \mathcal{L}(S_{n-1})$ if and only if $x_{ij} \geq 0$ if $|i - j|$ is odd, and $x_{ij} \leq 0$ if $i \neq j$ and $|i - j|$ is even.

- Take k with $1 < k < n - 1$. Let E_{ij} be such that $1 < j - i < n - 1$ (the last inequality amounts to $(i, j) \neq (1, n)$). Let us check that $\pm E_{ij} \notin \mathcal{L}(S_k)$. Let I be a multi-index with $i \notin I$ and $j \in I$. Then

$$E_{ij} e_I = (-1)^{\nu(i,j,I)} e_F$$

  where $\nu(i, j, I)$ is the number of indices of the open interval $\operatorname{int}(i, j) = \{l : i < l < j\}$ that are in I, and $F = (I \setminus \{j\}) \cup \{i\}$, arranged in the proper order.

  Now we construct two multi-indices I, J, that do not contain i and contain j, such that $\nu(i, j, I) + \nu(i, j, J)$ is odd. This implies that $\pm E_{ij} \notin \mathcal{L}(S_k)$.

  Start with I. There are the possibilities:

  1. $\nu(i, j, I) = 0$. Then $j - 1 \notin I$ and $j - 1 \neq i$. Moreover, since $2 \leq k$, it follows that there is some $l \in I$ with either $l < i$ or $l > j$. Then take J to be $J = (I \setminus \{l\}) \cup \{j - 1\}$. It satisfies $i \notin J$, $j \in J$ and $\nu(i, j, J) = 1$.
  2. $\nu(i, j, I) > 0$. This means that there exists $l \in \operatorname{int}(i, j) \cap I$. There are still two possibilities:
     (a) There is some $p < i$ or $> j$ such that $p \notin I$. Then $J = (I \setminus \{l\}) \cup \{p\}$ is such that $\nu(i, j, J) = \nu(i, j, I) + 1$.
     (b) For all $p \notin I$, $i \leq p < j$. Then there exists an index $t \in I$ with either $t < i$ or $t > j$, because $(i, j) \neq (1, n)$. Since $k \leq n - 2$ there is $p \in \operatorname{int}(i, j)$ outside I. Replacing t by p, we get a multi-index J with $\nu(i, j, I) + \nu(i, j, J)$ is odd.

Therefore, $E_{ij} \notin \mathcal{L}(S_k)$.

Now, consider E_{1n}. The only possibility for $E_{1n}e_I$ to be different from zero is that I ends with n. In this case

$$E_{1n}e_I = (-1)^{k-1} e_F$$

where $F = (I \setminus \{n\}) \cup \{1\}$, arranged in the right order. Hence

1. If k is odd $E_{1n} \in \mathcal{L}(S_k)$.
2. If k is even $-E_{1n} \in \mathcal{L}(S_k)$.

Finally consider $E_{i,i+1}$. If $i \notin I$ and $i+1 \in I$ then $E_{i,i+1}e_I = e_F$ where $F = (I \setminus \{j\}) \cup \{i\}$ so that $E_{i,i+1} \in \mathcal{L}(S_k)$.

Taking transpositions in this last case we can sum up and state:

Proposition 1 *The Lie wedges of the semigroups S_k are*

1. $\mathcal{L}(S_1) = \{(x_{ij}) : x_{ij} \geq 0 \text{ if } i \neq j\}$
2. $\mathcal{L}(S_{n-1}) = \{(x_{ij}) : (-1)^{|i-j|-1} x_{ij} \geq 0 \text{ if } i \neq j\}$.
3. *If* $1 < k < n-1$ *then*
   $\mathcal{L}(S_k) = \{(x_{ij}) : x_{ij} \geq 0 \text{ if } |i-j| = 1,\ x_{ij} = 0 \text{ if } 2 < |i-j| < n-1,\ (-1)^{k-1} x_{ij} \geq 0 \text{ if } |i-j| = n-1\}$.

2 Maximality

Of course, $\mathcal{L}(S_k)$ has empty interior unless $k = 1$ or $n-1$. Moreover, $\mathcal{L}(S_k)$ is GNC for any k. Also,

Proposition 2 $\mathcal{L}(S_1)$ *is MGNC.*

Proof: Denote by T_1 the semigroup generated by $\mathcal{L}(S_1)$, $T_1 = \langle \exp(\mathcal{L}(S_1)) \rangle$. In the projective space $\mathbb{P}^{n-1}$ there are just two control sets for T_1, namely the subset C corresponding to the positive orthant $\mathcal{O}_1$, which is the invariant control set, and its complementary $\mathbb{P}^{n-1} \setminus C$, which is the minimal control set. To see this note that $[e_1]$ belongs to the invariant control set because e_1 is the principal eigenvector of some $H \in \mathcal{L}(S_1)$. Furthermore any $[x]$, with $x \in \mathcal{O}_1$ can be reached from $[e_1]$ by an element of T. Since C is T_1-invariant it follows that it is indeed the invariant control set. Similarly $[e_1]$ belongs to the invariant control set of T_1^{-1}. This invariant control set is a union of orthants because T_1 contains the subgroup A of diagonal matrices. Now, we show that any orthant, besides the positive one, is reachable from $[e_1]$ by T_1^{-1}. Put, for $x \in \mathbb{R}^{n-1}$,

$$v_x = \begin{pmatrix} 1 \\ x \end{pmatrix} \in \mathbb{R}^n.$$

If all entries of x are strictly negative then

$$g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in T_1,$$

and $g[e_1] = [v_x]$. Hence the corresponding orthant is reachable from $[e_1]$ under T_1^{-1}. On the other hand suppose that $y \in \mathbb{R}^{n-1}$ has at least two strictly negative entries, say y_r and y_s with $r < s$. For every $a \leq 0$, the matrix $1 + aE_{r+1,s+1}$ belongs to T_1^{-1}. Moreover, $(1 + aE_{r+1,s+1})\, v_y = v_z$, and z has the same entries as y except for the r position which is $y_r + ay_s$. Since $y_r, y_s < 0$, there exists $a < 0$ such that $y_r + ay_s > 0$. Hence we can apply an induction procedure to show that every orthant different from the positive one is reachable from $[e_1]$, by means of T_1^{-1}. Therefore, $\mathbb{P}^{n-1} \setminus C$ is the minimal control set of T_1.

Now, if $X \notin \mathcal{L}(S_1)$ then $\exp(t_0 X)\, C \cap (\mathbb{P}^{n-1} \setminus C) \neq \emptyset$ for some $t_0 > 0$. Hence the semigroup generated by $\exp(tX)$, $t \geq 0$, and T_1 is controllable in $\mathbb{P}^{n-1}$, showing that $\mathcal{L}(S_1)$ is MGNC. □

Note that $\mathcal{L}(S_{n-1}) = -h\mathcal{L}(S_1)\, h^{-1}$ where

$$h = \operatorname{diag}\{1, -1, 1, \ldots\}.$$

Hence $\mathcal{L}(S_{n-1})$ is MGNC as well.

On the other hand $\mathcal{L}(S_k)$ is properly contained in $\mathcal{L}(S_1)$ if k is odd and $\neq 1, n-1$. Hence for these values of k, $\mathcal{L}(S_k)$ is not MGNC. Also, if k is even and n is odd then $\mathcal{L}(S_k) \subset \mathcal{L}(S_{n-1})$, so that $\mathcal{L}(S_k)$ is not MGNC.

We proceed now to prove that if $k < n-1$ and n are even then $\mathcal{L}(S_k)$ is MGNC. Of course only maximality remains to be proved. Put

$$T_k = \langle \exp(\mathcal{L}(S_k)) \rangle.$$

For $X \in \mathfrak{sl}(n, \mathbb{R})$ denote by $T_k(X)$ the semigroup generated by T_k and $\exp(tX)$, $t \geq 0$. We must show that $T_k(X) = G$ if $X \notin \mathcal{L}(S_k)$.

In order to prove that a semigroup of $\mathrm{Sl}(n, \mathbb{R})$ is controllable it is enough to show that it is not of type r for any $r = 1, \ldots, n-1$. Here a semigroup being of type r means that its invariant control set in the Grassmannian $\mathrm{Gr}_r(n)$ is contained in the open Bruhat cells associated to the regular real elements in its interior. This implies that the semigroup leaves invariant a pointed cone in $\bigwedge^r$. Hence to check that a semigroup is not of type r it is enough to show that there exists $\xi \in \mathrm{Gr}_r^+(n)$ such that both $\pm\xi$ belong to an invariant control set of the semigroup in $\mathrm{Gr}_r^+(n)$ (see [6, Prop. 2.5]).

Now, the following fact about the invariant control sets of T_k is required.

Lemma 3 *There are at most two invariant control sets of T_k in $\mathrm{Gr}_r^+(n)$. One of them contains the basic vectors e_I, with I running though r-multi-indices..*

Proof: There is just one invariant control set of T_k in $\mathrm{Gr}_r(n)$. Since $\mathrm{Gr}_r^+(n)$ is a double covering of $\mathrm{Gr}_r(n)$, it follows that there are at most two invariant control sets in $\mathrm{Gr}_r^+(n)$.

There are different ways of proving that one of them contains the basic vectors. A quick way is by appealing to known properties of the semigroup $T = S_1 \cap \cdots \cap S_{n-1}$ of totally positive matrices (see Ando [1], Lusztig [4]). It is well known that T is infinitesimally generated and

$$\mathcal{L}(T) = \{(x_{ij}) : x_{ij} \geq 0 \text{ if } |i-j| = 1, x_{ij} = 0 \text{ if } |i-j| \geq 2\}.$$

It follows that $\mathcal{L}(T) \subset \mathcal{L}(S_k)$, and hence $T \subset T_k$ for any k. So that any T-invariant control set is contained in an invariant control set of T_k. Now, $\mathcal{O}_r$ is T-invariant. So that $C_r = \mathcal{O}_r \cap \mathrm{Gr}_r^+(n)$ contains an invariant control set of T (actually it can be proved that C_r is an invariant control set of T). It follows that C_r meets the interior of an invariant control set of T_k. However T_k contains the subgroup of diagonal matrices A. Also, an easy computation involving the eigenvalues of a diagonal matrix show that the basic vectors are contained in the closure of the A-orbit of any ξ in the interior of C_r. Therefore the basic vectors of $\bigwedge^r$ are contained in an invariant control set of T_k, concluding the proof. □

From this lemma we get easily that

Lemma 4 *T_k is not of type r if k is even and r is odd.*

Proof: Let $D \subset \mathrm{Gr}_k^+(n)$ be the invariant control set containing the basic vector. Since k is even, $-E_{1n} \in \mathcal{L}(T_k)$. On the other hand, the fact that r is odd implies that

$$E_{1n}(e_{n-r+1} \wedge \cdots \wedge e_n) = e_1 \wedge e_{n-r+1} \wedge \cdots \wedge e_{n-1}.$$

Now $\exp(-tE_{1n}) = 1 - tE_{1n}$. Hence

$$\lim_{t \to +\infty} \frac{1}{t} \exp(-tE_{1n})(e_{n-r+1} \wedge \cdots \wedge e_n) = -e_1 \wedge e_{n-r+1} \wedge \cdots \wedge e_{n-1}.$$

Thus the right hand side of this formula belongs to D, showing that $\pm\xi \in D$ for $\xi = e_1 \wedge e_{n-r+1} \wedge \cdots \wedge e_{n-1}$. Thus T_k is not of type r. □

Now, take $X \notin \mathcal{L}(S_k)$ and form the semigroup $T_k(X)$. In view of the above lemma, for the proof that $T_k(X) = G$ it remains only to check that $T_k(X)$ is not of type r for r even.

Since we are assuming that n is even, $r \neq n-1$. So that $\mathcal{L}(S_r) = \mathcal{L}(S_k)$, and hence $T_r = T_k$. By definition of S_k, it follows that the semigroup $\exp(tX)$, $t \geq 0$, does not leave $\mathcal{O}_r$ invariant. But $\mathcal{O}_r$ is the cone generated by the basic vectors. So that there exists a r-multi-index I and $t_0 > 0$ such that $\exp(t_0 X) e_I \notin \mathcal{O}_r$, that is, $\exp(t_0 X) e_I$ is in another orthant with respect to the standard basis. Now, use the fact that $A \subset T_k$ to see that $-e_J$ can be approximately reached from e_I by means of T_k. Therefore by Lemma 3 there is an invariant control set for T_k in $\mathrm{Gr}_r(n)$ containing $\pm e_J$. This shows that $T_k(X)$ is not of type r, for r even. Therefore $T_k(X) = \mathrm{Sl}(n, \mathbb{R})$, proving the

Proposition 5 *If n and k are even then $\mathcal{L}(S_k)$ is a MGNC Lie wedge with empty interior.*

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