A Family of Partition Identities Proved Combinatorially

José Plínio O. Santos
and
Paulo Mondek

Abstract

We present an infinite family of partitions identities that resembles one given by Bressoud in [2]. In his family Bressoud includes only the first of the two Rogers-Ramanujan identities. Our family is proved combinatorially and gives a description not only for the first but also for the second Rogers-Ramanujan identity.

1. Introduction

The Rogers-Ramanujan identities in its analytical forms are given by:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})} \tag{2}$$

where we are using the standard notation

$$(a)_0 = 1$$

and

$$(a)_n = (1-a)(1-aq)\dots(1-aq^{n-1}).$$

One well known arithmetical interpretation for these two identities is

"*The number of partitions of n with parts differing by at least 2 and greater than i equals the number of partitions of n in parts $\equiv \pm(i+1)(mod\ 5), i = 0, 1.$*"

We mention that there is an important generalization of this theorem given by Gordon [3] as follows:

"Let $B_{k,i}(n)$ denote the number of partitions of n of the form $(b_1, b_2, \ldots, b_s)$, where $b_j - b_{j+k-1} \geq 2$, and at most $i-1$ of the b_j equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i (mod\ 2k+1)$. Then $A_{k,i}(n) = B_{k,i}(n)$ for all n."

Another arithmetical interpretation for the Rogers-Ramanujan identities was given by Mondek and Santos [5] namely

"*The number of partitions of* m *in distinct parts of the form* $m = (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_s)$ *where* a_j *is odd and* b_i *even with* $a_j \not\equiv a_{j+1} (mod\ 4), a_n \equiv (2\lambda - 1)(mod\ 4)$ *equals to the number of partitions of* m *in parts* $\equiv \pm\lambda (mod\ 5), \lambda = 1, 2.$"

The general theorem that we are going to prove in the next section presents two combinatorial interpretations for each one of the Rogers-Ramanujan identities. The two interpretations for the first identity were given, previously, by Bressoud [2].

2. The Theorem

Let $a(1), a(2), \ldots, a(k)$ be a complete residue system modulo k with $0 \leq a(i) < k$ for all $i = 1, 2, \ldots, k$.

Theorem 1. For $\ell \geq 0, k \geq 2$ let $A(n,k,\ell)$ be the number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_s$ with $b_s > \ell$ and $b_j - b_{j+1} \geq k$ and $B(n,k,\ell)$ be the number of partitions of n in distinct parts that are greater than ℓ and for each $i, 1 \leq i < k$ the smallest part $\equiv \ell + a(i) + 1 (mod\ k)$ is $> k \left[\sum_{j=i+1}^{k} r_j \right] + \ell + a(i)$ where r_j is the number of parts $\equiv \ell + a(j) + 1 (mod\ k), 1 < j \leq k$.

Then $A(n,k,\ell) = B(n,k,\ell)$ and the generating function for $A(n,k,\ell)$ is given by:

$$\sum_{s=0}^{\infty} \frac{q^{\frac{s}{2}(ks-k+2)+\ell s}}{(q)_s}.$$

Proof. We consider a partition π enumerated by $A(n,k,\ell)$. We represent π with a modified Ferrers graph in which we indent each row by k nodes. Considering that each part is greater than ℓ we have a graph similar to:

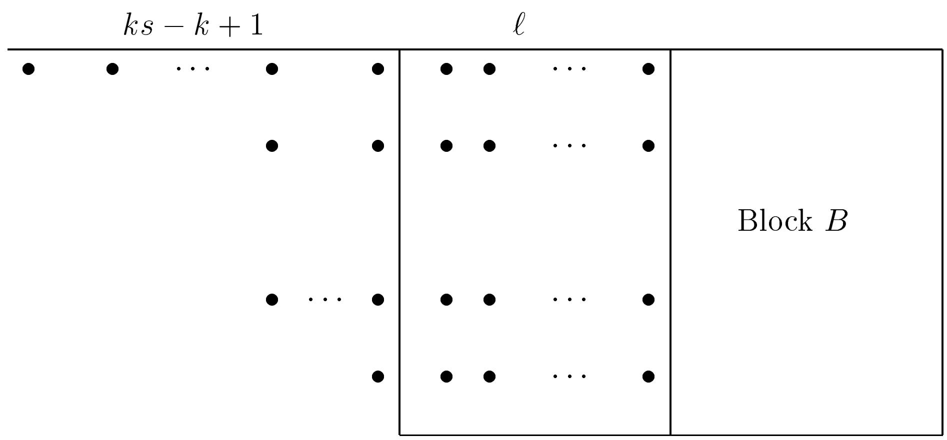

Figure 1

If we substract $\ell+1$ from $b_s, \ell+k+1$ from $b_{s-1}, \ell+2k+1$ from $b_{s-2}, \ldots, \ell+(s-1)k+1$ from b_1 we are left with a partition of

$$n-((\ell+1)+(\ell+1+k)+\cdots+\ell+1+(s-1)k)=n-s(\ell+1)-\frac{k}{2}s(s-1)$$

in at most s parts that is generated by

$$\frac{q^{\frac{s}{2}(ks-k+2)+\ell s}}{(q)_s}.$$

Then the generating function for $A(n,k,\ell)$ is

$$\sum_{s=0}^{\infty}\frac{q^{\frac{s}{2}(ks-k+2)+\ell s}}{(q)_s}.$$

Now in order to prove that $A(n,k,\ell)=B(n,k,\ell)$ we are going to construct a bijection between the two sets of partitions.

The part of the graph on the left of Block B has, from botton to top, respectively $\ell+1,\ \ell+k+1,\ \ell+2k+1,\ldots,\ \ell+(s-1)k+1$ nodes.

Lets assume that in Block B there are r_j parts $\equiv a(j)(mod\ k), 1\leq j\leq k$. Then, in π, there are r_j parts $\equiv \ell+a(j)+1(mod\ k)$.

Now we reorder the rows that are in Block B putting first the r_1 rows that are $\equiv a(1)(mod\ k)$ (in descending order), then the r_2 rows $\equiv a(2)(mod\ k)$ (in descending order), and so on, up to the r_k rows $\equiv a(k)(mod\ k)$. We observe that after reording the smallest part in the first r_1 rows is $> k(s-r_1)+\ell+a(1)$, that the smallest part in the next r_2 rows is $> k(s-r_1-r_2)+\ell+a(2)$ and that, in general, the smallest part among the r_i parts $\equiv \ell + a(i) + 1 (mod\ k)$ is $> k \left[s - \sum_{j=1}^{i} r_j \right] + \ell + a(i) =$ $k \sum_{j=i+1}^{k} r_j + \ell + a(i)$ since $s = r_1 + r_2 + \cdots + r_k$.

After reording, as described above, we consider the new rows as parts of a transformed partition that is enumerated by $B(n, k, \ell)$.

It is immediate from our construction that all parts are distinct and that this process is clearly reversible since that, given a partition enumerated by $B(n, k, \ell)$ we may order its parts by taking first the ones $\equiv \ell + 1 + a(1)(modk)$ (in descending order), after the ones $\equiv \ell + 1 + a(2)(mod\ k)$ (in descending order) and so on, up to the ones $\equiv \ell + 1 + a(k)(mod\ k)$ (in descending order). After reording as described above we represent the partition in a graphic form like the one in Figure 1. We finish this operation by reording only the rows that are in Block B. We have then a bijection between the partitions enumerated by $A(n, k, \ell)$ and $B(n, k, \ell)$.

□

To illustrate the procedure described in the theorem we consider $k = 9, \ell = 3, s = 6$; $a(1) = 2$; $a(2) = 0$; $a(3) = 4$; $a(4) = 8$; $a(5) = 6$; $a(6) = 7$; $a(7) = 3$; $a(8) = 1$ and $a(9) = 5$.

Let $71 + 50 + 39 + 27 + 15 + 5$ be an element enumerated by $A(207, 9, 3)$.

$$71 + 50 + 39 + 27 + 15 + 5 \leftrightarrow \left\{ \begin{array}{rcl} 71 & \leftrightarrow & \begin{array}{|c|c|c|}\hline 46 & 3 & 22 \\ \hline 37 & 3 & 10 \\ \hline 28 & 3 & 8 \\ \hline 19 & 3 & 5 \\ \hline 10 & 3 & 2 \\ \hline 1 & 3 & 1 \\ \hline \end{array} \\ 50 & \leftrightarrow & \\ 39 & \leftrightarrow & \\ 27 & \leftrightarrow & \\ 15 & \leftrightarrow & \\ 5 & \leftrightarrow & \end{array} \right. \leftrightarrow$$

$$\leftrightarrow \begin{array}{|c|c|c|}\hline 46 & 3 & 2 \\ \hline 37 & 3 & 22 \\ \hline 28 & 3 & 8 \\ \hline 19 & 3 & 10 \\ \hline 10 & 3 & 1 \\ \hline 1 & 3 & 5 \\ \hline \end{array} \quad \left. \begin{array}{rr} \leftrightarrow & 51 \\ \leftrightarrow & 62 \\ \leftrightarrow & 39 \\ \leftrightarrow & 32 \\ \leftrightarrow & 14 \\ \leftrightarrow & 9 \end{array} \right\} \leftrightarrow 62 + 51 + 39 + 32 + 14 + 9$$

that is an element enumerated by $B(207, 9, 3)$.

There are some special cases of great interest. If we take $k = 2,\ \ell = 0$, and $a(1) = 1, a(2) = 0$ (or $a(1) = 0, a(2) = 1$) in the theorem we have:

Corollary 1. The number of partitions of n with parts differing by at least 2 equals the number of partitions of n into distinct parts where the smallest even (odd) part is greater twice the number of odd (even) parts.

Considering, now, $k = 2,\ \ell = 1$ and $a(1) = 0, a(2) = 1$ (or $a(1) = 1, a(2) = 0$) we have:

Corollary 2. The number of partitions of n with parts greater than 1 and differing by at least 2 equals the number of partitions of n into distinct parts > 1 wherein each even (odd) part is larger than (2 plus) twice the number of odd (even) parts.

We mention that the two interpretation for the first Rogers-Ramanujan identity given in corollary 1 has been given, also, by Bressoud in [2].

In the next tables we list the bijection described in our theorem for $n = 15,\ k = 2,\ \ell = 0$ and $n = 15,\ k = 2,\ \ell = 1$, with $a(i) = i$.

$A(15,2,0)$		$B(15,2,0)$
15	$\longleftrightarrow$	15
$14+1$	$\longleftrightarrow$	$14+1$
$13+2$	$\longleftrightarrow$	$11+4$
$12+3$	$\longleftrightarrow$	$12+3$
$11+4$	$\longleftrightarrow$	$9+6$
$10+5$	$\longleftrightarrow$	$10+5$
$9+6$	$\longleftrightarrow$	$8+7$
$11+3+1$	$\longleftrightarrow$	$11+3+1$
$10+4+1$	$\longleftrightarrow$	$10+4+1$
$9+5+1$	$\longleftrightarrow$	$9+5+1$
$8+6+1$	$\longleftrightarrow$	$8+6+1$
$9+4+2$	$\longleftrightarrow$	$6+5+4$
$8+5+2$	$\longleftrightarrow$	$8+4+3$
$7+5+3$	$\longleftrightarrow$	$7+5+3$

Table 1

$A(15,2,1)$		$B(15,2,1)$
15	$\longleftrightarrow$	15
13+2	$\longleftrightarrow$	13+2
12+3	$\longleftrightarrow$	10+5
11+4	$\longleftrightarrow$	11+4
10+5	$\longleftrightarrow$	8 + 7
9+6	$\longleftrightarrow$	9 + 6
9+4+2	$\longleftrightarrow$	9+4+2
8+5+2	$\longleftrightarrow$	7+6+2
7+5+3	$\longleftrightarrow$	7+5+3

Table 2

References

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[2] Bressoud, D.M., *A New Family of Partition Identities*, Pacific Journal of Mathematics, vol. 77, nº 1, 1978.

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[5] Mondek, P. and Santos, J. P. O., *A New Combinatorial Interpretation for the Rogers-Ramanujan Identities.* Relatório Técnico nº 08/99 IMECC-UNICAMP, 1999.

IMECC-UNICAMP Cx.P. 6065
13083-970 - Campinas - SP - Brasil
email:josepli@ime.unicamp.br

CCET - UFMS Cx.P. 549
79070-900 - Campo Grande - MS - Brasil
email:mondek@hilbert.dmt.ufms.br