Massless Klein-Gordon equation in the Robertson-Walker spacetimes. *

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Abstract

We obtain, using the generalized derivative operators, the second order Casimir invariant operator associated to the Fantappié-de Sitter group, isomorphic to the 5-dimensional pseudorotation group, which is the group of motions admitted by the massless Robertson-Walker cosmological spacetimes.

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1 Introduction

The de Sitter spacetime is the curved spacetime which has been most studied by quantum field theorists because, together with the anti-de Sitter spacetime and Minkowski spacetime, it is the unique maximally symmetric curved spacetime [1, 2]. The symmetry group of de Sitter spacetime is the ten parameter group $SO(4,1)$ of homogeneous Lorentz transformations [3]. The group $SO(3,2)$ is the symmetry group of anti-de Sitter spacetime, and the Poincaré group is the symmetry group of Minkowski spacetime, both with ten parameter.

In this paper we obtain the generalized differential equation associated to the so called Fantappié-de Sitter group [4] using the methodology proposed by Arcidiacono, and generalize the metric of Beltrami [5] by introducing a parameter k such that the line element on the spatial sections is spherical in the de Sitter case, flat in the Minkowski case and hyperbolic in the anti-de Sitter case[1].

By using the second order Casimir invariant operator associated to the group we derive the Klein-Gordon wave equation, a result that generalizes a previous result obtained by the authors [6]. Finally, when the radius of the de Sitter or anti-de Sitter spacetime goes to infinity, we obtain again the results associated to the Minkowski spacetime [7].

This paper is organized as follows: in section 2 we discuss the massless Robertson-Walker spacetimes, that is, de Sitter spacetime, Minkowski spacetime and anti-de Sitter spacetime; in section 3 we discuss the relation between the two formulations i.e., how to pass from 5-dimensional Pitagorian metric to 4-dimensional Beltrami metric. In section 4 we present the so called Fantappié-de Sitter group and its invariants and in section 5 we obtain the second order Casimir invariant operator associated to this group.

2 Massless Robertson-Walker spacetimes

The Copernican Principle — the requirement of a homogeneous and isotropic space — is valid in the so called comoving coordinates. It is satisfied by the solutions of Einstein's field equations with cosmological constant Λ in the absence of matter and radiation, which are given as follows

- $\Lambda > 0$ (de Sitter spacetime).

$$ds^2 = R^2\{-d\tau^2 + \cosh^2\tau[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]\}.$$

[1] $k = 1, k = 0$ and $k = -1$, respectively.

This is the de Sitter spacetime, a space with constant curvature R, and may be visualized as the sphere of radius R in the space $\mathbb{R}_{(4,1)}$

$$-\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = R^2. \tag{1}$$

- $\Lambda = 0$ (Minkowski spacetime).

$$ds^2 = -d\tau^2 + dr^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

This is the Minkowski spacetime expressed in spherical coordinates, a space with null curvature.

- $\Lambda < 0$ (anti-de Sitter spacetime).

$$ds^2 = R^2\{-d\tau^2 + \cos^2\tau[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]\}.$$

This is the anti-de Sitter spacetime, a space with constant curvature $-R$, and may be visualized as the sphere of radius $-R$ in the space $\mathbb{R}_{(3,2)}$

$$-\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 = -R^2. \tag{2}$$

These three spacetimes are the only maximally symmetric spaces [1, 2].

3 Beltrami coordinates and derivatives

In this section we consider how to pass from 5-dimensional homogeneous coordinates $\xi = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4)$ to the Beltrami 4-dimensional coordinates $x = (x_0, x_1, x_2, x_3)$ of the de Sitter and anti-de Sitter spacetimes. We use the summation convention in this and the following sections, in which repeated Greek subscripts are summed over.

Making $\xi_0 \to i\xi_0$ in $\mathbb{R}_{(4,1)}$, and $\xi_1 \to i\xi_1$, $\xi_2 \to i\xi_2$, $\xi_3 \to i\xi_3$ in $\mathbb{R}_{(3,2)}$ the de Sitter spacetime and the anti-de Sitter spacetime — eq.(1) and eq.(2) — are formally represented as a sphere

$$\sum_{A=0}^{4} \xi_A \xi_A = (\xi_0)^2 + (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 + (\xi_4)^2 = R^2. \tag{3}$$

The relation between Beltrami coordinates x_μ and homogeneous coordinates is given by [4]

$$x_\mu = R\frac{\xi_\mu}{\xi_4} \quad \text{where} \quad \mu = 0,1,2,3. \tag{4}$$

Let us introduce the notation

$$A_k^2 = 1 + k\frac{-x_0^2 + x_1^2 + x_2^2 + x_3^2}{R^2}, \tag{5}$$

where $k = 1, 0$ or -1 is a parameter related to the de Sitter spacetime, Minkowski spacetime and anti-de Sitter spacetime, respectively. We can thus eliminate the ξ_4 coordinate and we have the following relations:

$$\xi_4 = \frac{R}{A_k} \quad \text{and} \quad \xi_\mu = \frac{x_\mu}{A_k}. \tag{6}$$

In these coordinates the de Sitter and anti-de Sitter line elements are

$$A_k^4 ds^2 = A_k^2 dx_\mu dx_\mu - kR^{-2}\left(x_\mu dx_\mu\right)^2 ,$$

where $x_0 = ict$, $R^2 A_k^2 = R^2 + k(r^2 + x_0^2)$ and $r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2$.

We note that when $k = 1$ the line element reduces to the line element of Beltrami metric [5].

To obtain the relations for the partial derivatives we consider a function $\varphi(\xi)$, a homogeneous function of degree N in all five variables $\xi = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4)$. Using Euler's theorem for homogeneous functions we have

$$\sum_A \xi_A \partial_A \varphi(\xi) = N\varphi(\xi) , \tag{7}$$

where we have put $\partial_A = \partial/\partial\xi_A$, with $A = 0,1,2,3,4$. Using the definition of homogeneous function we can write

$$\varphi\left(R\frac{\xi_0}{\xi_4}, ..., R\frac{\xi_4}{\xi_4}\right) = \left(\frac{R}{\xi_4}\right)^N \varphi(\xi) \tag{8}$$

and we finally get the following relation

$$R^N \varphi(\xi) = (\xi_4)^N \varphi(x, R) , \tag{9}$$

where the function in the right hand side is a function obtained from $\varphi(\xi_A)$ with the substitutions $\xi_4 \to R$ and $\xi_\mu \to x_\mu$.

Differentiating eq.(9) with respect to ξ_4 and ξ_μ we obtain respectively,

$$R\frac{\partial}{\partial\xi_4}\varphi(\xi) = A_k^{1-N}(N - x_\mu\partial_\mu)\varphi(x, R) \tag{10}$$

and

$$\frac{\partial}{\partial \xi_\mu}\varphi(\xi) = A_k^{1-N}\partial_\mu \varphi(x,R) , \tag{11}$$

where A_k is given by eq. (5) and $\partial_\mu \equiv \partial/\partial x_\mu$.

Introducing the function $\psi(x)$ defined by

$$\psi(x) = A_k^{-N}\varphi(x,R) \tag{12}$$

in the two equations above we can finally write the derivatives:

$$R\frac{\partial}{\partial \xi_4}\varphi(\xi) = \left(\frac{N}{A_k} - A_k x_\mu \partial_\mu\right)\psi(x) , \tag{13}$$

$$\frac{\partial}{\partial \xi_0}\varphi(\xi) = \left(A_k\partial_0 - k\frac{N}{A_k R^2}x_0\right)\psi(x) , \tag{14}$$

$$\frac{\partial}{\partial \xi_\nu}\varphi(\xi) = \left(A_k\partial_\nu + k\frac{N}{A_k R^2}x_\nu\right)\psi(x) , \tag{15}$$

where $\nu = 1,2,3$, and $\mu = 0,1,2,3$.

Then, we have solved the problem of passing from the 5-dimensional formulation, $\xi = (\xi_0,\xi_1,\xi_2,\xi_3,\xi_4)$, to the 4-dimensional spacetime formulation, $x = (x_0,x_1,x_2,x_3)$, i.e., in Cartesian coordinates, also called Beltrami coordinates. The relations expressed by eqs.(13), (14) and (15) are the link between the two formulations.

4 The Fantappié-de Sitter group

In this section we present the Fantappié-de Sitter group, the symmetry group of massless Robertson-Walker spacetimes, and write down its invariant operators in Beltrami coordinates.

The symmetry group of massless Robertson-Walker spacetimes is the pseudorotations group. With imaginary coordinates (placed in adequate form) it is called Fantappié-de Sitter group. The pseudorotations group preserves the equation $\sum \xi_A\xi_A = R^2$ with $\xi_0 \to i\xi_0$ in the de Sitter case and $\xi_1 \to i\xi_1,\ \xi_2 \to i\xi_2,\ \xi_3 \to i\xi_3$ in the anti-de Sitter case. Its generators satisfy [3]

$$-i\left[J_{\kappa\lambda}, J_{\mu\nu}\right] = \delta_{\kappa\nu}J_{\lambda\mu} - \delta_{\kappa\mu}J_{\lambda\nu} + \delta_{\lambda\mu}J_{\kappa\nu} - \delta_{\lambda\nu}J_{\kappa\mu} ,$$

$$-i\left[T_\lambda, J_{\mu\nu}\right] = \delta_{\lambda\mu}T_\nu - \delta_{\lambda\nu}T_\mu ,$$

$$-i\,[T_\mu, T_\nu] = -\frac{1}{R^2} J_{\mu\nu}\,,$$

where the Greek subscripts take values $0,1,2,3$ and $T_\mu = \frac{1}{R} J_{\mu 4}$. We note that as $R \to \infty$ we have

$$T_\mu \to p_\mu\,,$$

where p_μ is the four dimensional operator associated with the translations of Minkowski spacetime. In this way we obtain the Lie Algebra of the non homogeneous Lorentz group, the Poincaré group.

Introducing the correspondence

$$p_\mu \to -i\partial_\mu$$

we obtain a representation of the Fantappié-de Sitter group given by the 5-dimensional angular momentum operators

$$J_{AB} = -i\hbar \left(\xi_A \frac{\partial}{\partial \xi_B} - \xi_B \frac{\partial}{\partial \xi_A} \right) \equiv L_{AB}\,,$$

where $A, B = 0,1,2,3,4$. In terms of the Beltrami coordinates, these operators are given by

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \tag{16}$$

and

$$T_\lambda \equiv \frac{1}{R} L_{0\lambda} = A^2 p_\lambda + \frac{1}{R^2} x_\mu L_{\lambda\mu}. \tag{17}$$

In these expression the imaginary coordinates were maintained, and thus $A^2 = 1 + \sum_\mu x_\mu^2 / R^2$ with $\mu, \nu, \lambda = 0,1,2,3$.

We note that in the equations above (where T_μ are the analogous of the momentum operators in Minkowski spacetime) linear momentum and angular momentum are mixed in a unique tensor. This mixing is due to the fact that displacement transformations are the analogous of translations and therefore the energy-momentum operator is not conserved by the Fantappié-de Sitter group.

Now, we consider the explicit form in real coordinates of the ten operators. Introducing the T_0 operator, representing *temporal translations*, defined by

$$L_{04} \equiv R T_0 = -i\hbar \left(\xi_0 \frac{\partial}{\partial \xi_4} - \xi_4 \frac{\partial}{\partial \xi_0} \right)\,,$$

we have

$$T_0 = \hbar\sqrt{k}\left(\frac{\partial}{\partial x_0} - k\frac{x_0}{R^2}x_\mu\frac{\partial}{\partial x_\mu}\right). \tag{18}$$

The T_μ operators, representing *spatial translations*, are defined by

$$L_{\mu 4} \equiv RT_\mu = -i\hbar\left(\xi_\mu\frac{\partial}{\partial \xi_4} - \xi_4\frac{\partial}{\partial \xi_\mu}\right)$$

whence we obtain

$$T_\mu = \frac{i\hbar}{\sqrt{k}}\left(\frac{\partial}{\partial x_\mu} + k\frac{x_\mu}{R^2}x_\nu\frac{\partial}{\partial x_\nu}\right)$$

where $\mu = 1, 2, 3$.

Introducing the V_μ operators, which are related to the *center of mass inertia momentum*, given by

$$L_{0\mu} \equiv V_\mu = -i\hbar\left(\xi_0\frac{\partial}{\partial \xi_\mu} - \xi_\mu\frac{\partial}{\partial \xi_0}\right),$$

we have

$$V_\mu = k\hbar\left(x_0\frac{\partial}{\partial x_\mu} + x_\mu\frac{\partial}{\partial x_0}\right), \tag{19}$$

where $\mu = 1, 2, 3$.

Finally, we introduce L_λ operators, representing *spatial rotations*, defined by

$$L_{\mu\nu} \equiv L_\lambda = -i\hbar\left(\xi_\mu\frac{\partial}{\partial \xi_\nu} - \xi_\nu\frac{\partial}{\partial \xi_\mu}\right),$$

and we obtain

$$L_\lambda = -i\hbar\left(x_\mu\frac{\partial}{\partial x_\nu} - x_\nu\frac{\partial}{\partial x_\mu}\right), \tag{20}$$

where (μ, ν, λ) is any cyclic permutation of $(1, 2, 3)$ and in the above expressions $\hbar$ has its usual meaning.

We can write the two invariant operators of the Fantappié-de Sitter group, the so called Casimir invariant operators, using T_0, T_μ, V_μ and L_μ as follows:

$$I_2 = T^2 + T_0^2 + \frac{1}{R^2}\left(L^2 + V^2\right) = -M^2 \tag{21}$$

and

$$I_4 = \left(\vec{L}\cdot\vec{T}\right)^2 + \left(T_0\vec{L} + \vec{T}\times\vec{V}\right)^2 + \frac{1}{R^2}\left(\vec{L}\cdot\vec{V}\right)^2 = -N^2 \tag{22}$$

where M^2 and N^2 are constants.

We call eq.(21) the Klein-Gordon equation and we note that in the limit $R \to \infty$ we have

$$I_2 \to m^2 \qquad \text{and} \qquad I_4 \to m^2 s\,(s+1)$$

where m and s are respectively the rest mass and the spin that characterize the representations of the Poincaré group [3]. The representations of the Fantappié-de Sitter group are labeled by the eigenvalues of I_2 and I_4, which generalize the usual mass and spin. Yet, a particle in a de Sitter universe has not a well defined mass and spin but eigenvalues of the I_2 and I_4 invariant operators.

5 Second order Casimir invariant operator

In this section we introduce a spherical coordinate system (r, θ, ϕ) and obtain the explicit form of the second order Casimir invariant operator in these coordinates.

The spherical coordinates are $x_0 = x_0$, $x_3 = r\cos\theta$, $x_2 = r\sin\theta\sin\phi$ and $x_1 = r\sin\theta\cos\phi$, and we define the following operators:

$$\begin{aligned}
P_1 &= \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
P_2 &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
P_3 &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \\
\Omega &= x_0\frac{\partial}{\partial x_0} + r\frac{\partial}{\partial r}.
\end{aligned}$$

We can write for the translation operators:

$$
\begin{aligned}
T_0 &= \hbar\sqrt{k}\left(\frac{\partial}{\partial x_0} - k\frac{x_0}{R^2}\Omega\right) \\
T_1 &= \frac{i\hbar}{\sqrt{k}}\left(P_1 + k\frac{r\sin\theta\cos\phi}{R^2}\Omega\right) \\
T_2 &= \frac{i\hbar}{\sqrt{k}}\left(P_2 + k\frac{r\sin\theta\sin\phi}{R^2}\Omega\right) \\
T_3 &= \frac{i\hbar}{\sqrt{k}}\left(P_3 + k\frac{r\cos\theta}{R^2}\Omega\right).
\end{aligned}
$$

The inertial displacement operators are given by:

$$
\begin{aligned}
V_1 &= \hbar k\left(x_0 P_1 + r\sin\theta\cos\phi\frac{\partial}{\partial x_0}\right) \\
V_2 &= \hbar k\left(x_0 P_2 + r\sin\theta\sin\phi\frac{\partial}{\partial x_0}\right) \\
V_3 &= \hbar k\left(x_0 P_3 + r\cos\theta\frac{\partial}{\partial x_0}\right).
\end{aligned}
$$

The spatial rotation operators are given by:

$$
\begin{aligned}
L_1 &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\
L_2 &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\
L_3 &= -i\hbar\frac{\partial}{\partial\phi}.
\end{aligned}
$$

Finally, we obtain the explicit form for the second order Casimir invariant operator introducing the differential operators given above in eq.(21) and taking $x_0 = ct$. The second order Casimir invariant operator is then given by

$$I_2 \quad = \quad -\hbar^2 A_k^2 \left\{ k \left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \triangle \right] + \frac{1}{R^2}\mathcal{L} \right\}$$

where $A_k^2 = 1 + \dfrac{k}{R^2}\,(r^2 - c^2 t^2)$, $\triangle$ is the Laplacian in spherical coordinates and $\mathcal{L}$ is the operator

$$\mathcal{L} = t^2 \frac{\partial^2}{\partial t^2} + 2rt\frac{\partial^2}{\partial r \partial t} + r^2 \frac{\partial^2}{\partial r^2} + 2t\frac{\partial}{\partial t} + 2r\frac{\partial}{\partial r}.$$

We note that when $R \to \infty$, the operator I_2 reduces to the D'alembert wave operator, i.e.,

$$\lim_{R\to\infty} I_2 \equiv \Box = \hbar^2 \left(\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right).$$

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