Right Eigenvalues Equation in Quaternion Quantum Mechanics

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Abstract

We discuss right 'complex' eigenvalues equations for n-dimensional matrix quaternionic and complex linear operators in quaternionic vector spaces. The quaternionic linear operators spectrum takes n-complex eigenvalues. A necessary and sufficient condition to diagonalize quaternionic matrix representations is given. We also extend our discussion to complex linear quaternionic operators and introduce left/right 'quaternionic' eigenvalues equations.

1 Introduction

After the fundamental works of Finkelstein et al. on Foundations of Quaternionic Quantum Mechanics [1] and Quaternionic Gauge Theories [2], in the last decade we have witnessed a renewed interest in algebrization and geometrization of Physical Theories by non commutative fields [3]. Among the numerous references on this subject we recall the important paper of Horwitz and Biedenharn [4], where the authors showed that the assumption of a complex scalar product, complex geometry [5], permits the definition of a suitable tensor product [6] between single-particle quaternionic wave functions. We also mention quaternionic applications in Special Relativity [7], Group Representations [8, 9, 10], Non Relativistic Scattering [11, 12], Electroweak [13] and Grand Unification Theories [14], Preonic Model [15]. A clear and detailed discussion of Quaternionic Quantum Mechanics is found in the recent book of Adler [16], where in the final chapter we find an interesting list of issues relating to the role of quaternions in Physics and possible topics for future developments.

The aim of this paper is to study the right complex eigenvalues problem in Quaternion Quantum Mechanics (QQM): Given quaternionic and complex linear operators on n-dimensional quaternionic vector spaces, we wish to determine a systematic way to obtain eigenvalues and the corresponding eigenvectors. In discussing such a problem, we find two obstacles: the first one is related to the difficulty in obtaining a suitable definition of determinant for quaternionic matrices, the second one is represented by the loss, for non-commutative fields, of the fundamental theorem of the Algebra. The lack of these tools, essentials in solving the eigenvalues problem in the complex world, make the problem over quaternionic fields a complicated puzzle. We overcome difficulties in approaching the eigenvalues problem in a quaternionic world by discussing eigenvalues equations for $2n$-dimensional complex matrices, which represent the 'complex' counterpart of n-dimensional quaternionic matrix operators. We shall show that quaternionic and complex linear operators, on quaternionic Hilbert space with quaternionic geometry, are diagonalizable if and only if the corresponding translated complex operators are diagonalizable. The spectral theorems, extended to quaternionic Hilbert spaces [1], are recovered in a more general context. We also give a practical method to diagonalize operators on finite dimension quaternionic vector spaces and to relate ani-hermitian to hermitian quaternionic operators.

This paper is organized as follows: In Section 2, we introduce basic notations and mathematical tools. In Section 3 we approach the right eigenvalues problem by discussing the eigenvalues spectrum for $2n$-dimensional complex matrices obtained by translating n-dimensional quaternionic matrix representations, and give a practical method to diagonalize quaternionic linear operators. We also discuss the right eigenvalues problem for complex linear quaternionic operators and analyze the results within a QQM with complex geometry. In Section 4, we introduce left/right quaternionic eigenvalues equations. In Appendix, we explicitly solve eigenvalues equations for two-dimensional quaternionic and complex linear operators. Our conclusions and out-looks are drawn in the final section.

2 Basic notations and mathematical tools

A quaternion, $q \in \mathbb{H}$, is expressed by four real quantities [17]

$$q = a + ib + jc + kd \; , \qquad a, b, c, d \in \mathbb{R} \tag{1}$$

and three imaginary units

$$i^2 = j^2 = k^2 = ijk = -1 \; .$$

The quaternion skew-field $\mathbb{H}$ is an associative but non-commutative algebra of rank 4 over $\mathbb{R}$, endowed with an involutory antiautomorphism

$$q \to \overline{q} = a - ib - jc - kd \; . \tag{2}$$

This conjugation implies a reversed order product, namely

$$\overline{pq} = \overline{q}\,\overline{p} \; , \qquad p \; , \; q \in \mathbb{H}.$$

Every nonzero quaternion is invertible, and the unique inverse is given by $1/q = \overline{q}/|q|^2$, where the quaternionic norm $|q|$ is defined by

$$|q|^2 = q\overline{q} = a^2 + b^2 + c^2 + d^2 \; .$$

Two quaternions q and p belong to the same eigenclass when the following relation

$$q = s^{-1}\, p\, s \; , \qquad s \in \mathbb{H} \; ,$$

is satisfied. It is easily verify that quaternions of the same eigenclass have the same real part and the same norm,

$$\mathrm{Re}(q) = \mathrm{Re}\,(s^{-1}\,p\,s) = \mathrm{Re}(p)\ , \qquad |q| = |s^{-1}\,p\,s| = |p|\ ,$$

consequently they have the same absolute value of the imaginary part. The previous equations can be rewritten in terms of unitary quaternions as follows

$$q = s^{-1}\,p\,s = \frac{\overline{s}}{|s|}\,p\,\frac{s}{|s|} = \overline{u}\,p\,u\ , \qquad u \in \mathbb{H}\ , \quad \overline{u}u = 1\ .$$

Likewise complex numbers can be constructed from real numbers by

$$z = \alpha + i\beta\ , \qquad \alpha, \beta \in \mathbb{R}\ ,$$

we can construct quaternions from complex numbers by adopting the so-called symplectic decomposition

$$q = z + jw\ , \qquad z, w \in \mathbb{C}\ .$$

Due to the non-commutative nature of quaternions we must distinguish between

$$q\vec{h} \quad \text{and} \quad \vec{h}q\ , \qquad \vec{h} \equiv (i, j, k)\ .$$

Thus, it is appropriate to consider left and right-actions for our imaginary units i, j and k. Let us define the operators

$$\vec{L} = (L_i, L_j, L_k)\,, \tag{3}$$

and

$$\vec{R} = (R_i, R_j, R_k)\,, \tag{4}$$

which act on quaternionic states in the following way

$$\vec{L}:\ \mathbb{H} \to \mathbb{H}\ , \qquad \vec{L}q = \vec{h}q \in \mathbb{H}\ , \tag{5}$$

and

$$\vec{R}:\ \mathbb{H} \to \mathbb{H}\ , \qquad \vec{R}q = q\,\vec{h} \in \mathbb{H}\ . \tag{6}$$

The algebra of left/right generators can be concisely expressed by

$$L_i^2 = L_j^2 = L_k^2 = L_iL_jL_k = R_i^2 = R_j^2 = R_k^2 = R_kR_jR_i = -\mathbb{1}\ ,$$

and by the commutation relations

$$[\,L_{i,j,k}\ ,\ R_{i,j,k}\,] = 0\ .$$

From these operators we can construct the following vector space

$$\mathbb{H}_L \otimes \mathbb{H}_R\,,$$

whose generic element will be characterized by left and right actions of quaternionic imaginary units i, j, k. In this paper we will work with two sub-spaces of $\mathbb{H}_L \otimes \mathbb{H}_R$, namely

$$\mathbb{H}^L \quad \text{and} \quad \mathbb{H}^L \otimes \mathbb{C}^R\ ,$$

whose elements are represented respectively by left actions of i, j, k

$$a + \vec{b} \cdot \vec{L} \quad \in \mathbb{H}^L \ , \qquad a, \vec{b} \in \mathbb{R} \ ,$$

and by left actions of i, j, k and right action of the only imaginary unit i

$$a + \vec{b} \cdot \vec{L} + cR_i + \vec{d} \cdot \vec{L} R_i \quad \in \mathbb{H}^L \otimes \mathbb{C}^R \ , \qquad a, \vec{b}, c, \vec{d} \in \mathbb{R} \ .$$

Let us now introduce two different types of operators: Operators $\mathbb{H}$-linear and $\mathbb{C}$-linear from the right. For simplicity of notation we denote

$$\mathcal{O}_{\mathbb{X}} : \ V_{\mathbb{H}} \to V_{\mathbb{H}} \ ,$$

to represent quaternionic operators right-linear on the $\mathbb{X}$-field. Operators which act only from the left on quaternionic Hilbert spaces, $V_{\mathbb{H}}$, represented by $\mathcal{O}_{\mathbb{H}}$ are obviously $\mathbb{H}$-linear from the right

$$\mathcal{O}_{\mathbb{H}}(|\psi > q) = (\mathcal{O}_{\mathbb{H}}|\psi >) \, q \ , \qquad q \in \mathbb{H} \ .$$

By considering the right action of the i-complex imaginary unit we obtain right $\mathbb{C}$-linear operators, $\mathcal{O}_{\mathbb{C}}$, which satisfy

$$\mathcal{O}_{\mathbb{C}}(|\psi > \lambda) = (\mathcal{O}_{\mathbb{C}}|\psi >) \, \lambda \ , \qquad \lambda \in \mathbb{C} \ .$$

Right $\mathbb{H}$ and $\mathbb{C}$ linear operators are $\mathbb{R}$-linear from the left. From now on, in order to lighten our presentation, we will use the terminology quaternionic and complex linear operators to indicate linear operators on $\mathbb{H}$ and $\mathbb{C}$ from the right.

The use of such left/right operators give new intersting opportunities in Quaternionic Group Theory [10]. Let us observe as follows: The so-called *symplectic* complex representation of a quaternion state $|\psi >$ can be represented by the following complex column matrix

$$|\psi > = |x > + j \, |y > \quad \leftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} \ . \tag{7}$$

The operator representation of L_i, L_j and L_k consistent with the above identification

$$L_i \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \ , \qquad L_j \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \ , \qquad L_k \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \ , \tag{8}$$

has been known since the discovery of quaternions. It permits any quaternionic number or matrix to be translated into a complex matrix, *but not necessarily vice-versa*. Eight real numbers are required to define the most general 2×2 complex matrix but only four are needed to define the most general quaternion. In fact since every (non-zero) quaternion has an inverse, only a subclass of invertible 2×2 complex matrices are identifiable with quaternions. Complex linear quaternionic operators complete the translation [18]. The right quaternionic imaginary unit

$$R_i \ \leftrightarrow \ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \ , \tag{9}$$

adds four additional degrees of freedom, obtained by matrix multiplication of the corresponding matrices,

$$R_i \ , \ L_i R_i \ , \ L_j R_i \ , \ L_k R_i \ ,$$

and so we have a set of rules for translating from any 2×2 complex matrices to complex linear operators, $\mathcal{O}_{\mathbb{C}}$.

3 The right complex eigenvalues problem in QQM

Before to discussing right quaternionic eigenvalues equations, we give the basic framework of Quaternionic Quantum Mechanics [16], where such equations find their natural application. First of all, due to non-commutative nature of quaternionic multiplication, we must specify whether the quaternionic Hilbert space is to be formed by right or by left multiplication of vectors by quaternionic scalars. The two different conventions give isomorphic versions of the Theory [19]. We approach the Adler convention of right multiplication by scalars, since this is the one appropriate to usual conventions of matrix operations and to Dirac bra and ket notations for state vectors [16].

The right eigenvalue equation for a generic quaternionic linear operator, $O_{\mathbb{H}}$, is written as

$$O_{\mathbb{H}}|\Psi> = |\Psi> q \ , \tag{10}$$

where

$$|\Psi> \in V_{\mathbb{H}} \ , \qquad q \in \mathbb{H} \ .$$

By adopting quaternionic scalar products in our quaternionic Hilbert spaces, $V_{\mathbb{H}}$, we find states in one to one correspondence with unit rays of the form

$$|\mathbf{r}> = \{|\Psi> u\} \tag{11}$$

where $|\Psi>$ is a unit normalized vector and u a quaternionic phase of magnitude unity. The state vector, $|\Psi> u$, corresponding to the same physical state $|\Psi>$, is an $O_{\mathbb{H}}$ eigenvector with eigenvalue $\overline{u}qu$

$$O_{\mathbb{H}}|\Psi> u = |\Psi> u\,(\overline{u}qu) \ .$$

So, quaternionic linear operators are characterized by an infinite eigenvalues spectrum

$$\{q \ , \ \overline{u}_1 q u_1 \ , \ \ldots \ , \ \overline{u}_i q u_i \ , \ \ldots\}$$

with u_i unitary quaternions. The corresponding eigenvectors set

$$\left\{|\Psi> \ , \ |\Psi> u_1 \ , \ \ldots \ , \ |\Psi> u_i \ , \ \ldots\right\}$$

represents a ray and so we can *characterize* our spectrum by choosing a ray rappresentative

$$|\psi> = |\Psi> u_\lambda \ .$$

For this state the right eigenvalues equation becomes

$$O_{\mathbb{H}}|\psi> = |\psi> \lambda \ , \tag{12}$$

with

$$|\psi> \in V_{\mathbb{H}} \ , \quad \lambda \in \mathbb{C} \ .$$

We now give a systematic method to determine the complex eigenvalues of quaternionic matrix representations for $O_{\mathbb{H}}$ operators.

3.1 Quaternionic linear operators and quaternionic geometry

In n-dimensional quaternionic vector spaces, $\mathbb{H}^n$, quaternionic linear operator, $O_{\mathbb{H}}$, are represented by $n \times n$ quaternionic matrices, $\mathcal{M}_n(\mathbb{H}^L)$, with elements in $\mathbb{H}^L$. Such quaternionic matrices admit $2n$-dimensional complex counterparts by the translation rules given in Eq. (8). Such complex matrices characterize a subset of the $2n$-dimensional complex matrices

$$\widetilde{M}_{2n}(\mathbb{C}) \subset M_{2n}(\mathbb{C}) \ .$$

The eigenvalues equation for $O_{\mathbb{H}}$ reads

$$\mathcal{M}_{\mathbb{H}}|\psi> = |\psi> \lambda \ , \tag{13}$$

where

$$\mathcal{M}_{\mathbb{H}} \in \mathcal{M}_n(\mathbb{H}^L) \ , \quad |\psi> \in \mathbb{H}^n \ , \quad \lambda \in \mathbb{C} \ .$$

• The one-dimensional eigenvalues problem

In order to introduce the reader to our general method of quaternionic matrix diagonalization, let us discuss one-dimensional right complex eigenvalues equations. In this case Eq. (13) becomes

$$Q_{\mathbb{H}}|\psi> = |\psi> \lambda \ , \tag{14}$$

where

$$Q_{\mathbb{H}} = a + \vec{b} \cdot \vec{L} \ \in \mathbb{H}^L \ , \quad |\psi> = |x> + j\,|y> \in \mathbb{H} \ , \quad \lambda \in \mathbb{C} \ .$$

By using the translation rules, given in Section 2, we find the following complex counterpart of Eq.(14)

$$\begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \ , \tag{15}$$

$$z = a + ib_1 \ , \quad w = b_2 - ib_3 \in \mathbb{C} \ .$$

The translated complex operator admits λ and λ^* as eigenvalues. To prove that, we take the complex conjugate of Eq.(15) and then apply a similarity transformation by the matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

In this way, we find

$$\begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix} \begin{pmatrix} -y^* \\ x^* \end{pmatrix} = \lambda^* \begin{pmatrix} -y^* \\ x^* \end{pmatrix} . \tag{16}$$

For $\lambda \neq \lambda^* \in \mathbb{C}$, we obtain the eigenvalues spectrum $\{\lambda, \lambda^*\}$. What happens when $\lambda \in \mathbb{R}$? In this case the eigenvalues spectrum will be determined by two equal eigenvalues λ. To show that, we remark that eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} \ , \quad \begin{pmatrix} -y^* \\ x^* \end{pmatrix} \ ,$$

associated to the same eigenvalues λ, are linearly independent. In fact,

$$\left\| \begin{matrix} x & -y^* \\ y & x^* \end{matrix} \right\| = |x|^2 + |y|^2 = 0 \qquad \text{if and only if } x = y = 0 \ .$$

So in the quaternionic world, by translation, we find two complex eigenvalues respectively λ and λ^*, associated to the following quaternionic eigenvectors

$$|\psi> \quad \text{and} \quad |\psi> j \quad \in \ |\mathbf{r}> \ .$$

The infinite quaternionic eigenvalue spectrum can be characterized by the complex eigenvalue λ and the ray rappresentative will be $|\psi>$. Thus, the problem of eigenvalues doubling in the complex translation is soon overcome. In the next Section, by using the same method, we will discuss eigenvalue equations in n-dimensional quaternionic vector spaces.

• The n-dimensional eigenvalues problem

Let us formulate two theorems which allow to generalize the previous results for quaternionic n-dimensional eigenvalues problems. The first theorem analyzes the eigenvalues spectrum of the $2n$-dimensional complex matrix $\widetilde{M}$, complex counterpart of n-dimensional quaternionic matrix $\mathcal{M}_{\mathbb{H}}$. The second one discusses linear independence for $\mathcal{M}_{\mathbb{H}}$ eigenvectors.

`T1 - THEOREM`

Let $\widetilde{M}$ be the complex counterpart of a generic $n \times n$ quaternionic matrix $\mathcal{M}_{\mathbb{H}}$. Its eigenvalues appear in conjugate pairs.

Let

$$\widetilde{M}\,|\phi_\lambda> = \lambda\,|\phi_\lambda> \tag{17}$$

be the eigenvalues equation for $\widetilde{M}$, where

$$\widetilde{M} \in M_{2n}(\mathbb{C}) \ , \quad |\phi_\lambda> = \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} \in \mathbb{C}^{2n} \ , \quad \lambda \in \mathbb{C} \ .$$

By taking the complex conjugate of Eq. (17),

$$\widetilde{M}^*|\phi_\lambda>^* = \lambda^*|\phi_\lambda>^* \ ,$$

and applying a similarity transformation by the matrix

$$S = \mathbb{1}_n \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ,$$

we obtain

$$S\widetilde{M}^*S^{-1}S|\phi_\lambda>^* = \lambda^* S|\phi_\lambda>^* \ . \tag{18}$$

From the blocks structure of the complex matrix $\widetilde{M}$ it is easily checked that

$$S\widetilde{M}^*S^{-1} = \widetilde{M},$$

and consequently Eq. (18) reads

$$\widetilde{M}\,|\phi_{\lambda^*}> = \lambda^*|\phi_{\lambda^*}> \ , \tag{19}$$

where

$$|\phi_{\lambda^*}>=S|\phi_\lambda>^*=\begin{pmatrix}-y_1^*\\x_1^*\\\vdots\\-y_n^*\\x_n^*\end{pmatrix}.$$

Let us show that the eigenvalues appear in conjugate pairs (this implies a double multiplicity for real eigenvalues). To do it, we need to prove that $|\phi_\lambda>$ and $|\phi_{\lambda^*}>$ are linearly independent. In order to demonstrate the linear independence of such eigenvectors, trivial for $\lambda\neq\lambda^*$, we observe that linear dependence, possible in the case $\lambda=\lambda^*$, should require

$$\left\|\begin{matrix}x_i & -y_i^*\\ y_i & x_i^*\end{matrix}\right\|=|x_i|^2+|y_i|^2=0\quad i=1,...,n\ ,$$

verified only for null eigenvectors. The linear independence of $|\phi_\lambda>$ and $|\phi_{\lambda^*}>$ ensures an even multiplicity for real eigenvalues ■.

We shall use the results of the first theorem to obtain informations about the $\mathcal{M}_{\mathbb{H}}$ right complex eigenvalues spectrum. Due to non-commutativity nature of the quaternionic field we cannot give a suitable definition of determinant for quaternionic matrices and consequently we cannot write a characteristic polynomial $P(\lambda)$ for $\mathcal{M}_{\mathbb{H}}$. Another difficulty it is represented by the right position of the complex eigenvalue λ.

`T2 - THEOREM`

$\mathcal{M}_{\mathbb{H}}$ admits n linearly independent eigenvectors on $\mathbb{H}$ if and only if its complex counterpart $\widetilde{M}$ admits $2n$ linearly independent eigenvectors on $\mathbb{C}$.

Let

$$\left\{|\phi_{\lambda_1}>\ ,\ |\phi_{\lambda_1^*}>\ ,\ ...\ ,\ |\phi_{\lambda_n}>\ ,\ |\phi_{\lambda_n^*}>\right\}\tag{20}$$

be a set of $2n$ $\widetilde{M}$-eigenvectors, linearly independent on $\mathbb{C}$, and

$$\alpha_i\ ,\ \beta_i\qquad i=1,...,n$$

be generic complex coefficients. By definition

$$\sum_{i=1}^{n}\left(\alpha_i|\phi_{\lambda_i}>+\beta_i|\phi_{\lambda_i^*}>\right)=0\quad\Leftrightarrow\quad\alpha_i=\beta_i=0\ .\tag{21}$$

By translating the complex eigenvectors set (20) in quaternionic formalism we find

$$\left\{|\psi_{\lambda_1}>\ ,\ |\psi_{\lambda_1^*}>\ ,\ ...\ ,\ |\psi_{\lambda_n}>\ ,\ |\psi_{\lambda_n^*}>\right\}.\tag{22}$$

By eliminating the eigenvectors, $|\psi_{\lambda_i^*}>=|\psi_{\lambda_i}>j$, corresponding for complex eigenvalues to ones with negative imaginary part, linearly dependent with $|\psi_{\lambda_i}>$ on $\mathbb{H}$, we obtain

$$\left\{|\psi_{\lambda_1}>\ ,\ ...\ ,\ |\psi_{\lambda_n}>\right\}.$$

This set is formed by n linearly independent vectors on $\mathbb{H}$. In fact, by taking an arbitrary quaternionic linear combination of such vectors, we have

$$\sum_{i=1}^{n}\left[|\psi_{\lambda_i}>(\alpha_i+j\beta_i)\right]=\sum_{i=1}^{n}\left(|\psi_{\lambda_i}>\alpha_i+|\psi_{\lambda_i^*}>\beta_i\right)=0\quad\Leftrightarrow\quad\alpha_i=\beta_i=0\ .\tag{23}$$

Note that Eq. (23) represents the quaternionic counterpart of Eq. (21) ∎.

The $\mathcal{M}_{\mathbb{H}}$ complex eigenvalues spectrum is thus obtained by taking from the $2n$ dimensional $\widetilde{M}$-eigenvalues spectrum

$$\{\lambda_1 \ , \ \lambda_1^* \ , \ \dots \ , \ \lambda_n \ , \ \lambda_n^*\},$$

the reduced n-dimensional spectrum

$$\{\lambda_1 \ , \ \dots \ , \ \lambda_n\} \ .$$

We stress here the fact that, the choice of positive, rather than negative, imaginary part is a simple convention. In fact, from the quaternionic eigenvectors set (22), we can extract different sets of quaternionic linearly independent eigenvectors

$$\left\{ [|\psi_{\lambda_1} > \ \text{or} \ |\psi_{\lambda_1^*} >] \ , \ \dots \ , \ [|\psi_{\lambda_n} > \ \text{or} \ |\psi_{\lambda_n^*} >] \right\} \ ,$$

and consequently we have a free choice in characterizing the n-dimensional $\mathcal{M}_{\mathbb{H}}$-eigenvalues spectrum. A direct consequence of the previous theorems, is the following corollary.

`T2 - COROLLARY`

Two $\mathcal{M}_{\mathbb{H}}$ quaternionic eigenvectors with complex eigenvalues, λ_1 and λ_2, with $|\lambda_1| \neq |\lambda_2|$, are linearly independent on $\mathbb{H}$.

Let

$$|\psi_{\lambda_1} > (\alpha_1 + j\beta_1) + |\psi_{\lambda_2} > (\alpha_2 + j\beta_2) \tag{24}$$

be a quaternionic linear combination of such eigenvectors. By taking the complex translation of Eq. (24), we obtain

$$\alpha_1|\phi_{\lambda_1} > + \beta_1|\phi_{\lambda_1^*} > + \alpha_2|\phi_{\lambda_2} > + \beta_2|\phi_{\lambda_2^*} > \ . \tag{25}$$

The set of $\widetilde{M}$-eigenvectors

$$\left\{ |\phi_{\lambda_1} > \ , \ |\phi_{\lambda_1^*} > \ , \ |\phi_{\lambda_2} > \ , \ |\phi_{\lambda_2^*} > \right\}$$

is linear independent on $\mathbb{C}$. In fact, the first theorem, T-1, ensures linear independence between eigenvectors associated to conjugate pairs of eigenvalues, and the condition $|\lambda_1| \neq |\lambda_2|$ $(\Rightarrow \lambda_1 \neq \lambda_2)$ complete the proof by assuring the linear independence between

$$\left\{ |\phi_{\lambda_1} > \ , \ |\phi_{\lambda_1^*} > \right\}$$

and

$$\left\{ |\phi_{\lambda_2} > \ , \ |\phi_{\lambda_2^*} > \right\} \ .$$

Thus the linear combination in Eq. (25), complex counterpart of Eq. (24), is null if and only if $\alpha_{1,2} = \beta_{1,2} = 0$, and consequently the quaternionic linear eigenvectors $|\psi_{\lambda_1} >$ and $|\psi_{\lambda_2} >$ are linear independent on $\mathbb{H}$ ∎.

- **A brief discussion about the spectrum choice**

What happens on the eigenvalues spectrum when we have two simultaneous diagonalizable quaternionic linear operators? We show that for complex operators the choice of a common quaternionic eigenvectors set reproduce in QQM the standard results of Complex Quantum Mechanics (CQM). Let

$$\mathcal{A}_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} E \qquad \text{and} \qquad \mathcal{A}_2 = \frac{\hbar}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{26}$$

be anti-hermitian complex operators associated respectively to energy a nd spin. In CQM, the corresponding eigenvalues spectrum is

$$\left\{ iE \; , \; iE \right\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{ i\frac{\hbar}{2} \; , \; -i\frac{\hbar}{2} \right\}_{\mathcal{A}_2} \; , \tag{27}$$

and physically we can describe a particle with positive energy E and spin $\frac{1}{2}$. What happens in QQM with quaternionic geometry? The complex operators in Eq. (26) also represent two-dimensional quaternionic linear operators and so we can translate them in the complex world and then extract the eigenvalues spectrum. By following the method given this Section, we find the following eigenvalues

$$\left\{ iE \; , \; iE \; , \; iE \; , \; iE \right\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{ i\frac{\hbar}{2} \; , \; -i\frac{\hbar}{2} \; , \; i\frac{\hbar}{2} \; , \; -i\frac{\hbar}{2} \right\}_{\mathcal{A}_2} \; ,$$

and adopting the positive imaginary part convention we extract

$$\left\{ iE \; , \; iE \right\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{ i\frac{\hbar}{2} \; , \; i\frac{\hbar}{2} \right\}_{\mathcal{A}_2} \; .$$

It seems that we lose the physical meaning of $\frac{1}{2}$-spin particle with positive energy. How can we recover the different sign in the spin eigenvalues? The solution to this apparent puzzle is represented by the choice of a *common* quaternionic eigenvectors set. In fact, we observe that the previous eigenvalues spectra are related to the following eigenvectors sets

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \; \begin{pmatrix} 0 \\ j \end{pmatrix} \right\}_{\mathcal{A}_2} \; .$$

By fixing a common eigenvectors set

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} \; , \tag{28}$$

we recover the standard results of Eq. (27). Obviously,

$$\begin{aligned}
\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} &\to \left\{ +iE \, , +iE \right\}_{\mathcal{A}_1} , \; \left\{ +i\frac{\hbar}{2} \, , -i\frac{\hbar}{2} \right\}_{\mathcal{A}_2} , \\
\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ j \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} &\to \left\{ +iE \, , -iE \right\}_{\mathcal{A}_1} , \; \left\{ +i\frac{\hbar}{2} \, , +i\frac{\hbar}{2} \right\}_{\mathcal{A}_2} , \\
\left\{ \begin{pmatrix} j \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} &\to \left\{ -iE \, , +iE \right\}_{\mathcal{A}_1} , \; \left\{ -i\frac{\hbar}{2} \, , -i\frac{\hbar}{2} \right\}_{\mathcal{A}_2} , \\
\left\{ \begin{pmatrix} j \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ j \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} &\to \left\{ -iE \, , -iE \right\}_{\mathcal{A}_1} , \; \left\{ -i\frac{\hbar}{2} \, , +i\frac{\hbar}{2} \right\}_{\mathcal{A}_2} ,
\end{aligned}$$

represent equivalent choices. Thus, in this particular case, the different possibilities in choosing our quaternionic eigenvectors set will give the following outputs

$$\begin{array}{llllll}
\text{Energy}: & +E\,,\;\; +E & \text{and} & \frac{1}{2}\text{-spin}: & \uparrow\;,\;\downarrow \\
& +E\,,\;\; -E & \text{and} & & \uparrow\;,\;\uparrow \\
& -E\,,\;\; +E & \text{and} & & \downarrow\;,\;\downarrow \\
& -E\,,\;\; -E & \text{and} & & \downarrow\;,\;\uparrow
\end{array}$$

Thus, we can also describe a $\frac{1}{2}$-spin particle with positive energy by re-interpretating spin up/down negative energy as spin down/up positive energy solutions

$$-E\;,\;\uparrow(\downarrow)\quad\rightarrow\quad E\;,\;\downarrow(\uparrow)\;.$$

• A practical rule for diagonalization

We know that $2n$-dimensional complex operators, are diagonalizable if and only if they admit $2n$ linear independent eigenvectors. It is easy to demonstrate that the diagonalization matrix for $\widetilde{M}$

$$\widetilde{S}_{2n}\widetilde{M}\widetilde{S}_{2n}^{-1}=\widetilde{M}^{diag}\;,$$

is given by

$$\widetilde{S}_{2n}=\texttt{Inverse}\left[\begin{pmatrix}
x_1^{(\lambda_1)} & x_1^{(\lambda_1^*)} & \dots & x_1^{(\lambda_n)} & x_1^{(\lambda_n^*)}\\
y_1^{(\lambda_1)} & y_1^{(\lambda_1^*)} & \dots & y_1^{(\lambda_n)} & y_1^{(\lambda_n^*)}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
x_n^{(\lambda_1)} & x_n^{(\lambda_1^*)} & \dots & x_n^{(\lambda_n)} & x_n^{(\lambda_n^*)}\\
y_n^{(\lambda_1)} & y_n^{(\lambda_1^*)} & \dots & y_n^{(\lambda_n)} & y_n^{(\lambda_n^*)}
\end{pmatrix}\right]\;. \tag{29}$$

Such a matrix is in the same subset of $\widetilde{M}$, i.e. $\widetilde{S}_{2n}\in\widetilde{M}_{2n}(\mathbb{C})$. In fact, by recalling the relationship between $|\phi_\lambda>$ and $|\phi_{\lambda^*}>$, we can rewrite the previous diagonalization matrix as

$$\widetilde{S}_{2n}=\texttt{Inverse}\left[\begin{pmatrix}
x_1^{(\lambda_1)} & \text{-}y_1^{*\,(\lambda_1)} & \dots & x_1^{(\lambda_n)} & \text{-}y_1^{*\,(\lambda_n)}\\
y_1^{(\lambda_1)} & x_1^{*(\lambda_1)} & \dots & y_1^{(\lambda_n)} & x_1^{*\,(\lambda_n)}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
x_n^{(\lambda_1)} & \text{-}y_n^{*\,(\lambda_1)} & \dots & x_n^{(\lambda_n)} & \text{-}y_n^{*\,(\lambda_n)}\\
y_n^{(\lambda_1)} & x_n^{*\,(\lambda_1)} & \dots & y_n^{(\lambda_n)} & x_n^{*\,(\lambda_n)}
\end{pmatrix}\right]\;. \tag{30}$$

The linear independence of the $2n$ complex eigenvectors of $\widetilde{M}$ guarantees the existence of $\widetilde{S}_{2n}^{-1}$ and the isomorphism between the group of $n\times n$ invertible quaternionic matrices $\mathsf{GL}(n,\mathbb{H})$ and the complex counterpart group $\widetilde{\mathsf{GL}}(2n,\mathbb{C})$ ensures $\widetilde{S}_{2n}^{-1}\in\widetilde{M}_{2n}(\mathbb{C})$. So, the quaternionic n-dimensional matrix which diagonalizes $\mathcal{M}_\mathbb{H}$

$$\mathcal{S}_{\mathbb{H}\,n}\mathcal{M}_\mathbb{H}\mathcal{S}_{\mathbb{H}\,n}^{-1}=\mathcal{M}_\mathbb{H}^{diag}\;,$$

can be directly obtained by translating Eq. (30) in

$$\mathcal{S}_{\mathbb{H}\,n}=\texttt{Inverse}\left[\begin{pmatrix}
x_1^{(\lambda_1)}+jy_1^{(\lambda_1)} & \dots & x_1^{(\lambda_n)}+jy_1^{(\lambda_n)}\\
\vdots & \ddots & \vdots\\
x_n^{(\lambda_1)}+jy_n^{(\lambda_1)} & \dots & x_n^{(\lambda_n)}+jy_n^{(\lambda_n)}
\end{pmatrix}\right]\;. \tag{31}$$

In translating complex matrices in quaternionic language, we remember that an appropriate mathematical notation should require the use of the left/right quaternionic operators $L_{i,j,k}$ and R_i. In this case, due to the particular form of our complex matrices,

$$\widetilde{M}\ ,\ \widetilde{S}_{2n}\ ,\ \widetilde{S}_{2n}^{-1} \in \widetilde{M}_{2n}(\mathbb{C})\ ,$$

their quaternionic translation is performed by left operators and so we use the simplified notation i, j, k instead of L_i, L_j, L_k.

This diagonalization quaternionic matrix is strictly related to the choice of a particular set of quaternionic linear independent eigenvectors

$$\left\{ |\psi_{\lambda_1}>\ ,\ ...\ ,\ |\psi_{\lambda_n}> \right\}\ .$$

So, the diagonalized quaternionic matrix reads

$$\mathcal{M}_{\mathbb{H}}^{diag} = \text{diag}\ \{\lambda_1\ ,\ ...\ ,\ \lambda_n\}\ .$$

The choice of a different quaternionic eigenvectors set

$$\left\{ \left[|\psi_{\lambda_1}>\ \text{ or } |\psi_{\lambda_1^*}>\right]\ ,\ ...\ ,\ \left[|\psi_{\lambda_n}>\ \text{ or } |\psi_{\lambda_n^*}>\right] \right\}\ ,$$

will give a different diagonalization matrix and consequently a different diagonalized quaternionic matrix

$$\mathcal{M}_{\mathbb{H}}^{diag} = \text{diag}\ \{[\lambda_1 \text{ or } \lambda_1^*]\ ,\ ...\ ,\ [\lambda_n \text{ or } \lambda_n^*]\}\ .$$

In conclusion,

$$\mathcal{M}_{\mathbb{H}} \quad \text{diagonalizable} \quad \Leftrightarrow \quad \widetilde{M} \quad \text{diagonalizable}\ ,$$

and the diagonalization quaternionic matrix can be easily obtained from the quaternionic eigenvectors set.

3.2 Complex linear operators and complex geometry

In this Section, we discuss right eigenvalues equation for complex linear operators. In n-dimensional quaternionic vector spaces, $\mathbb{H}^n$, complex linear operator, $O_{\mathbb{C}}$, are represented by $n \times n$ quaternionic matrices, $\mathcal{M}_n(\mathbb{H}^L \otimes \mathbb{H}^R)$, with elements in $\mathbb{H}^L \otimes \mathbb{H}^R$. Such quaternionic matrices admit $2n$-dimensional complex counterparts which recover the *full* set of $2n$-dimensional complex matrices, $M_{2n}(\mathbb{C})$. It is immediate to check that quaternionic matrices $\mathcal{M}_{\mathbb{H}} \in \mathcal{M}_n(\mathbb{H}^L)$ are characterized by $4n^2$ real parameters and so a natural translation gets the complex matrix $\widetilde{M}_{2n}(\mathbb{C}) \subset M_{2n}(\mathbb{C})$, whereas a generic $2n$-dimensional complex matrix $M \in M_{2n}(\mathbb{C})$, characterized by $8n^2$ real parameters needs to double the $4n^2$ real parameters of $\mathcal{M}_{\mathbb{H}}$. By allowing right-action for the imaginary units i we recover the missing real parameters. So, the $2n$-dimensional complex eigenvalues equation

$$M|\phi> = \lambda|\psi>\ , \tag{32}$$

with

$$M \in M_{2n}(\mathbb{C})\ , \quad |\phi> \in \mathbb{C}^{2n}\ , \quad \lambda \in \mathbb{C}\ ,$$

becomes, in quaternionic formalism,

$$\mathcal{M}_{\mathbb{C}}|\psi> = |\psi>\lambda\ , \tag{33}$$

where

$$\mathcal{M}_{\mathbb{C}} \in \mathcal{M}_n(\mathbb{H}^L \otimes \mathbb{H}^R) \ , \quad |\psi> \in \mathbb{H}^n \ , \quad \lambda \in \mathbb{C} \ .$$

The right position of the complex eigenvalue λ is justified by the translation rule

$$i\mathbb{1}_{2n} \quad \leftrightarrow \quad R_i\mathbb{1}_n \ .$$

By solving the complex eigenvalue problem of Eq. (32), we find $2n$ eigenvalues and we nave no possibilities to classify or characterize such a complex eigenvalues spectrum . Is it possible to extract a suitable quaternionic eigenvectors set? What happens when the complex spectrum is characterized by $2n$ *different* complex eigenvalues? To give satisfactory answers to these questions we must adopt a complex geometry [4, 5]. In this case

$$|\psi> \quad \text{and} \quad |\psi> j$$

represent orthogonal vectors and so we cannot kill the eigenvectors $|\psi> j$. So, for n-dimensional quaternionic matrices $\mathcal{M}_{\mathbb{C}}$ we must consider the *full* eigenvalues spectrum

$$\left\{ \lambda_1 \ , \ \dots \ , \ \lambda_{2n} \right\} \ . \tag{34}$$

The corresponding quaternionic eigenvectors set is then given by

$$\left\{ |\psi_{\lambda_1}> \ , \ \dots \ , \ |\psi_{\lambda_{2n}}> \right\} \ , \tag{35}$$

which represents the quaternionic translation of the M-eigenvectors set

$$\left\{ |\phi_{\lambda_1}> \ , \ \dots \ , \ |\phi_{\lambda_{2n}}> \right\} \ . \tag{36}$$

In conclusion within a Quaternionic Quantum Mechanics with complex geometry [20] we find for quaternionic linear operators, $M_{\mathbb{H}}$, and complex linear operators, $M_{\mathbb{C}}$, a $2n$-dimensional complex eigenvalues spectrum and consequently $2n$ quaternionic eigenvectors. Let us now give a practical method to diagonalize complex linear operators. Complex $2n$-dimensional matrices, M, are diagonalizable if and only if admit $2n$ linear independent eigenvectors. The diagonalizable matrix can be written in terms of M-eigenvectors as follows

$$S_{2n} = \texttt{Inverse} \left[\begin{pmatrix} x_1^{(\lambda_1)} & x_1^{(\lambda_2)} & \dots & x_1^{(\lambda_{2n-1})} & x_1^{(\lambda_{2n})} \\ y_1^{(\lambda_1)} & y_1^{(\lambda_2)} & \dots & y_1^{(\lambda_{2n-1})} & y_1^{(\lambda_{2n})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{(\lambda_1)} & x_n^{(\lambda_2)} & \dots & x_n^{(\lambda_{2n-1})} & x_n^{(\lambda_{2n})} \\ y_n^{(\lambda_1)} & y_n^{(\lambda_2)} & \dots & y_n^{(\lambda_{2n-1})} & y_n^{(\lambda_{2n})} \end{pmatrix} \right] \ . \tag{37}$$

This matrix admits a quaternionic counterpart [18] by complex linear operators

$$\mathcal{S}_{\mathbb{C}\,n} = \texttt{Inverse} \left[\begin{pmatrix} q_{+-,1}^{[1,2]} - q_{-+,1}^{[1,2]} iR_i & \dots & q_{+-,1}^{[2n-1,2n]} - q_{-+,1}^{[2n-1,2n]} iR_i \\ \vdots & \ddots & \vdots \\ q_{+-,n}^{[1,2]} - q_{-+,n}^{[1,2]} iR_i & \dots & q_{+-,n}^{[2n-1,2n]} - q_{-+,n}^{[2n-1,2n]} iR_i \end{pmatrix} \right] \ , \tag{38}$$

where

$$q_{+-,i}^{[m,n]} = \frac{x_i^{(\lambda_m)} + y_i^{(\lambda_n)\,*}}{2} + j\,\frac{y_i^{(\lambda_m)} - x_i^{(\lambda_n)\,*}}{2} \ ,$$

and

$$q_{-+,i}^{[m,n]} = \frac{x_i^{(\lambda m)} - y_i^{(\lambda n)\,*}}{2} + j\,\frac{y_i^{(\lambda m)} + x_i^{(\lambda n)\,*}}{2}\ .$$

To simplify the notation we use i j, k instead of L_i, L_j, L_k. The right operator R_i indicates the right action of the imaginary unit i. The diagonalized quaternionic matrix reproduces the quaternionic translation of the complex matrix

$$M^{diag} = \text{diag}\,\{\lambda_1\ ,\ \dots\ ,\ \lambda_{2n}\}$$

into

$$\mathcal{M}_{\mathbb{C}}^{diag} = \text{diag}\left\{\frac{\lambda_1+\lambda_2^*}{2} + \frac{\lambda_1-\lambda_2^*}{2i}\,R_i\ ,\ \dots\ ,\ \frac{\lambda_{2n-1}+\lambda_{2n}^*}{2} + \frac{\lambda_{2n-1}-\lambda_{2n}^*}{2i}\,R_i\right\}\ . \tag{39}$$

4 Quaternionic eigenvalues equation

By working with quaternions we have different possibilities to write eigenvalues equations. In fact, in solving such equations, we could consider quaternionic or complex, left or right eigenvalues. In this Section, we briefly introduce the problematic inherent to quaternionic eigenvalues equations and emphasize the main difficulties present in such an approach.

4.1 Right quaternionic eigenvalues equation

As remarked in Section 3, the most general right eigenvalues equation for quaternionic linear operators, $\mathcal{O}_{\mathbb{H}}$, reads

$$\mathcal{M}_{\mathbb{H}}|\tilde{\psi}> = |\tilde{\psi}>q\ , \qquad q\in\mathbb{H}\ .$$

Such an equation can be converted into a right complex eigenvalues equation by rephasing the quaternionic eigenvalues, q,

$$\mathcal{M}_{\mathbb{H}}|\tilde{\psi}>u = |\tilde{\psi}>u\,\overline{u}qu = |\tilde{\psi}>\lambda\ , \qquad \lambda\in\mathbb{C}\ .$$

In discussing right quaternionic equations for complex linear operators,

$$\mathcal{M}_{\mathbb{C}}|\psi> = |\psi>q\ , \tag{40}$$

due to the presence of the right imaginary unit i, R_i, we cannot apply such a similarity transformation on q. Right complex immaginary units, R_i, have no a well defined hermiticity within a QQM with quaternionic geometry. Nevertheless, by adopting a complex geometry, we recover the anti-hermiticity of such an operator

$$\int_{\mathbb{C}} d\tau < \phi|R_i|\psi> = -\int_{\mathbb{C}} d\tau\ [R_i|\phi>]^\dagger\,|\psi>\ .$$

Within a QQM with complex geometry, a generic anti-hermitian operator must satisfy

$$\int_{\mathbb{C}} d\tau < \phi|\mathcal{A}_{\mathbb{C}}|\psi> = -\int_{\mathbb{C}} d\tau[\mathcal{A}_{\mathbb{C}}|\phi>]^\dagger|\psi>\ . \tag{41}$$

We can immediately find a constraint on our $\mathcal{A}_{\mathbb{C}}$-eigenvalues by putting in the previous equation $|\phi> = |\psi>$,

$$\int_{\mathbb{C}} d\tau < \psi|\psi>q_\psi = -\int_{\mathbb{C}} d\tau q_\psi^\dagger < \psi|\psi> \Rightarrow q_\psi = i\alpha_\psi + jw_\psi\ , \tag{42}$$

$$\alpha_\psi \in \mathbb{R} \; , \quad w_\psi \in \mathbb{C} \; .$$

Note that the purely quaternionic nature of the $\mathcal{A}_\mathbb{C}$- eigenvalues is save because $< \psi|\psi >$ represents a real quantity and thus commutes with q_ψ. We know that an important property must be satisfied for anti-hermitians operators, namely eigenvectors $|\phi >$ and $|\psi >$ associated to different eigenvalues $q_\phi \neq q_\psi$, have to be orthogonal. By combining Eq. (41) with the constraint (42) we find

$$\int_\mathbb{C} d\tau < \phi|\psi > q_\psi = \int_\mathbb{C} d\tau \, q_\phi < \phi|\psi > \; .$$

The different position of the quaternionic eigenvalues require a complex projection, $(q)_\mathbb{C}$, to recover the standard result

$$q_{\psi,\phi} \longrightarrow \lambda_{\psi,\phi} \; \in \mathbb{C} \; .$$

In conclusion, a consistent QQM with complex geometry needs of right complex eigenvalues equations.

4.2 Left quaternionic eigenvalues equation

What happens for left quaternionic eigenvalues equations? In solving such equations for quaternionic and complex linear operators,

$$\begin{cases} \mathcal{M}_\mathbb{H}|\tilde{\psi} > = \tilde{q}\,|\tilde{\psi} > \; , \\ \mathcal{M}_\mathbb{C}|\tilde{\psi} > = \tilde{q}\,|\tilde{\psi} > \; , \qquad \tilde{q} \in \mathbb{H} \; , \end{cases}$$

we have not a sistematic way to approch the problem. In this case, due to the presence of left quaternionic eigenvalues (translated in complex formalism by two-dimensional matrices), the translation trick does not apply and so we must solve directly the problem in the quaternionic world.

In discussing left quaternionic eigenvalues equations, we underline the difficulty hidden in diagonalizing such operators. Let us suppose that the matrix representations of our operators be digonalized by a matrix $\mathcal{S}_{\mathbb{H}/\mathbb{C}}$

$$\mathcal{S}_\mathbb{H}\, \mathcal{M}_\mathbb{H}\, \mathcal{S}_\mathbb{H}^{-1} = \mathcal{M}_\mathbb{H}^{diag} \qquad \text{and} \qquad \mathcal{S}_\mathbb{C}\, \mathcal{M}_\mathbb{C}\, \mathcal{S}_\mathbb{C}^{-1} = \mathcal{M}_\mathbb{C}^{diag} \; .$$

The eigenvalues equation will be modified in

$$\begin{cases} \mathcal{M}_\mathbb{H}^{diag}\mathcal{S}_\mathbb{H}|\tilde{\psi} > = \mathcal{S}_\mathbb{H}\tilde{q}\mathcal{S}_\mathbb{H}^{-1}\, \mathcal{S}_\mathbb{H}\,|\tilde{\psi} > \; , \\ \mathcal{M}_\mathbb{C}^{diag}\mathcal{S}_\mathbb{C}|\tilde{\psi} > = \mathcal{S}_\mathbb{C}\tilde{q}\mathcal{S}_\mathbb{C}^{-1}\, \mathcal{S}_\mathbb{C}\,|\tilde{\psi} > \; , \end{cases}$$

and now, due to the non-commutative nature of $\widetilde{q}$,

$$\mathcal{S}_{\mathbb{H}/\mathbb{C}}\, \widetilde{q}\, \mathcal{S}_{\mathbb{H}/\mathbb{C}}^{-1} \neq \widetilde{q} \; .$$

So, we can have operators with the same left quaternionic eigenvalues spectrum but no similarity transformation relating them. This is explicitly show in Appendix B for a bi-dimensional quaternionic linear operator. Let us now analyze other difficulties in solving left quaternionic eigenvalues equation. Hermitian quaternionic linear operators satisfay

$$\int d\tau < \widetilde{\phi}|\mathcal{H}_\mathbb{H}|\widetilde{\psi} > = \int d\tau \, [\mathcal{H}_\mathbb{H}|\widetilde{\phi} >]^\dagger|\widetilde{\psi} > \; .$$

By putting $|\widetilde{\phi}>=|\widetilde{\psi}>$ in the previous equation we find constraints on the quaternionic eigenvalues $\widetilde{q}$

$$\int d\tau < \widetilde{\psi}|\, \widetilde{q}\, |\widetilde{\psi}> = \int d\tau < \widetilde{\psi}|\, \widetilde{q}^{\dagger}\, |\widetilde{\psi}> \ .$$

From this equation we cannot exract the conclusion that $\tilde{q}$ must be real, $\widetilde{q}=\widetilde{q}^{\dagger}$. In fact

$$\int d\tau < \widetilde{\psi}|(\widetilde{q}-\widetilde{q}^{\dagger})|\widetilde{\psi}> = 0$$

could admit quaternionic solutions for $\tilde{q}$ (see Appendix B). So, the first complication is represented by the possibility to find hermitian operators with quaternionic eigenvalues. In discussing physical problems, we overcame this problem by choosing anti-hermitian operators. Now,

$$\int d\tau < \widetilde{\phi}|\mathcal{A}_{\mathbb{H}}|\widetilde{\psi}> = -\int d\tau\, [\mathcal{A}_{\mathbb{H}}|\widetilde{\phi}>]^{\dagger}|\widetilde{\psi}> \ ,$$

will imply for $|\widetilde{\phi}>=|\widetilde{\psi}>$

$$\int d\tau < \widetilde{\psi}|(\widetilde{q}+\widetilde{q}^{\dagger})|\widetilde{\psi}> = 0 \ .$$

In this case, $(\widetilde{q}+\widetilde{q}^{\dagger})$ represents a real quantity and so commutes with $|\widetilde{\psi}>$, giving

$$\widetilde{q} = i\alpha + jw \ .$$

We could work with anti-hermitian operators and choose $|\widetilde{q}|$ as physical output. A particular case of left/right eigenvalues equation for a two-dimensional anti-hermitian operators is presented in Appendix B. The result emphasizes an important difference between such operators. Left and right eigenvalues have different absolute value and so cannot represent the same physical quantity.

5 Conclusions

The study undertaken in this paper demonstrates the possibility to construct a pratical method to diagonalize quaternionic and complex linear operators on quaternionic vector spaces. Quaternionic eigenvalues equations have to be 'right' eigenvalues equations. As shown in our paper, the choice of a right position for quaternionic eigenvalues is fundamental in searching for a diagonalization method. A left position of quaternionic eigenvalues gives unwished surprises : Operators with the same eigenvalues which are not related by similarity transformation, Hermitian operators with quaternionic eigenvalues; etc.

Quaternionic linear operators in n-dimensional vector spaces take infinite spectra of quaternionic eigenvalues. Neverthless, the *complex translation trick* ensures that such spectra are related by similarity transformations and this gives the possibility to choose n rappresentative complex eigenvalues to perform calculations,

$$\left\{ \lambda_1 \ , \ \ldots \ , \ \lambda_n \right\} \ .$$

The complete set of quaternionic eigenvalues spectra can be generated from this complex eigenvalues spectrum,

$$\left\{ \overline{u}_1 \lambda_1 u_1 \ , \ \ldots \ , \ \overline{u}_n \lambda_n u_n \right\}$$

Such a simmetry is broken when we consider more diagonalizable operators. In this case the freedom in constructing the eigenvalue spectrum for the first operator, and consequently the free choice in determining an eigenvectors basis, will fix the eigenvalues spectrum for the other operators.

The powerfull of the complex traslation trick gives the possibility to study general properties for quaternionic and complex linear operators. The last ones play an important role with a QQM with complex geometry by reproducing the standard complex results in reduced quaternionic vector spaces [20]. Our method of diagonalization becomes very important in the resolution of quaternionic differential equations [21] and consequently in the study of quaternionic potentials in the Schröedinger equation [22].

Mathematical topics to be developed are represented by the discussion of eigenvalues equations for real linear quaternionic operators, $\mathcal{O}_{\mathbb{R}}$, operators characterized by right actions of the quaternionic immaginary units i, j, k and by a detailed discussion about left quaternionic eigenvalues equations.

We conclude by remarking an important difference between the structure of an anti-hermitian operator in complex and in quaternionic Quantum mechanics. In complex Quantum Mechanics, we can always trivially relate an anti-hermitian operator, $\mathcal{A}$ to an hermitian operator, $\mathcal{H}$, by removing a factor i

$$\mathcal{A} = i\,\mathcal{H}\ .$$

In quaternionic Quantum Mechanics, we must take care. For example,

$$\mathcal{A} = \begin{pmatrix} -i & 3j \\ 3j & i \end{pmatrix} ,$$

is an anti-hermitian operator, nevertheless, $i\mathcal{A}$ does not represent an hermitian operator. The reason is simple: In writing the spectral representations for $\mathcal{A}$ and $\mathcal{H}$ we take

$$\mathcal{A} = \sum_a |a\rangle\, ai\, \langle a| \qquad \text{and} \qquad \mathcal{H} = \sum_a |a\rangle\, a\, \langle a| \qquad a \in \mathbb{R}\ .$$

Due to the non-commutative nature of $|a\rangle$, we cannot extract the complex imaginary unit i. Our approach to quaternionic eigenvalues equations contains a pratical method to find eigenvectors $|a\rangle$ and eigenvalues ia and consequently solves the problem to determine, given a quaternionic anti-hermitian operator, the corresponding hermitian operator. An easy computation show that

$$ia = \{2i\,,\, 4i\} \qquad \text{and} \qquad |a\rangle = \left\{ \begin{pmatrix} i \\ j \end{pmatrix} , \begin{pmatrix} k \\ 1 \end{pmatrix} \right\}\ .$$

So, the hermitian operator related to $\mathcal{A}$ is

$$\mathcal{H} = \begin{pmatrix} 3 & k \\ -k & 3 \end{pmatrix}\ .$$

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Appendix A
Two dimensional right complex eigenvalues equations

In this Appendix we explicitly solve the right eigenvalues equations for quaternionic, $\mathcal{O}_{\mathbb{H}}$, and complex $\mathcal{O}_{\mathbb{C}}$, linear operators, in two-dimensional quaternionic vector spaces.

• Quaternionic linear operators

Let

$$\mathcal{M}_{\mathbb{H}} = \begin{pmatrix} i & j \\ k & i \end{pmatrix} \tag{43}$$

be the quaternionic matrix representation associated to a quaternionic linear operator in a bi-dimensional quaternionic vector space. Its complex counterpart reads

$$\widetilde{M} = \begin{pmatrix} i & 0 & 0 & -1 \\ 0 & -i & 1 & 0 \\ 0 & -i & i & 0 \\ -i & 0 & 0 & -i \end{pmatrix} .$$

In order to solve the right eigenvalues problem

$$\mathcal{M}_{\mathbb{H}} |\psi> = |\psi> \lambda \ , \qquad \lambda \in \mathbb{C} \ ,$$

let us determine the $\widetilde{M}$-eigenvalues spectrum. From the constraint

$$\det \left[\widetilde{M} - \lambda \mathbb{1}_4 \right] = 0 \ ,$$

we find for the $\widetilde{M}$-eigenvalues the following solutions

$$\left\{ \lambda_1 \ , \ \lambda_1^* \ , \ \lambda_2 \ , \ \lambda_2^* \right\}_{\widetilde{M}} = \left\{ 2^{\frac{1}{4}} e^{i\frac{3}{8}\pi} \ , \ 2^{\frac{1}{4}} e^{-i\frac{3}{8}\pi} \ , \ -2^{\frac{1}{4}} e^{-i\frac{3}{8}\pi} \ , \ -2^{\frac{1}{4}} e^{i\frac{3}{8}\pi} \right\}_{\widetilde{M}} .$$

The $\widetilde{M}$-eigenvectors set is given by

$$\left\{ \begin{pmatrix} -1 + i\lambda_1 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \begin{pmatrix} 0 \\ -1 - i\lambda_1^* \\ -1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 - i\lambda_1^* \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} -1 - i\lambda_1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}_{\widetilde{M}} .$$

The $\mathcal{M}_{\mathbb{H}}$-eigenvalues spectrum is soon obtained from that one of $\widetilde{M}$. For example by adopting the positive imaginary part convention we find

$$\left\{ \lambda_1 \ , \ \lambda_2 \right\}_{\mathcal{M}_{\mathbb{H}}} = \left\{ 2^{\frac{1}{4}} e^{i\frac{3}{8}\pi} \ , \ - 2^{\frac{1}{4}} e^{-i\frac{3}{8}\pi} \right\}_{\mathcal{M}_{\mathbb{H}}} , \tag{44}$$

and the corresponding quaternionic eigenvectors set reads

$$\left\{ \begin{pmatrix} -1 + i\lambda_1 \\ j \end{pmatrix} , \begin{pmatrix} j(1 - i\lambda_1^*) \\ 1 \end{pmatrix} \right\}_{\mathcal{M}_{\mathbb{H}}} . \tag{45}$$

The quaternionic matrix which diagonalizes $\mathcal{M}_{\mathbb{H}}$ is

$$\mathcal{S}_{\mathbb{H}} = \texttt{Inverse}\left[\begin{pmatrix} -1+i\lambda_1 & j(1-i\lambda_1^*) \\ j & 1 \end{pmatrix}\right] = -\frac{1}{2\,|\lambda_1|^2}\begin{pmatrix} i\lambda_1^* & j\left[i\lambda_1+|\lambda_1|^2\right] \\ k\lambda_1^* & i\lambda_1-|\lambda_1|^2 \end{pmatrix} \ . \qquad (46)$$

As remarked in the paper, we have infinite possibilities of diagonalization

$$\left\{\overline{u}_1\lambda_1 u_1 \ , \ \overline{u}_2\lambda_2 u_2\right\} \ .$$

Equivalent diagonalized matrices can be obtained from

$$\mathcal{M}_{\mathbb{H}}^{diag} = \text{diag}\left\{\lambda_1 \ , \ \lambda_2\right\}$$

by performing a similarity trasformation

$$\mathcal{U}^{-1}\mathcal{M}_{\mathbb{H}}^{diag}\,\mathcal{U} = \mathcal{U}^{\dagger}\mathcal{M}_{\mathbb{H}}^{diag}\,\mathcal{U} \ ,$$

and

$$\mathcal{U} = \text{diag}\left\{u_1 \ , \ u_2\right\} \ .$$

The diagonalization matrix given in Eq. (46) becames

$$\mathcal{S}_{\mathbb{H}} \ \rightarrow \ \mathcal{U}^{\dagger}\mathcal{S}_{\mathbb{H}} \ .$$

• Complex linear operators

Let

$$\mathcal{M}_{\mathbb{C}} = \begin{pmatrix} -iR_i+j & -kR_i+1 \\ -kR_i-1 & iR_i+j \end{pmatrix} \qquad (47)$$

be the quaternionic matrix representation associated to a complex linear operator in a bi-dimensional quaternionic vector space. Its complex counterpart is

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \ .$$

The right complex eigenvalues problem

$$\mathcal{M}_{\mathbb{C}}|\psi> = |\psi>\lambda \ , \qquad \lambda\in\mathbb{C} \ ,$$

can be solved by determining the M-eigenvalues spectrum

$$\left\{\lambda_1 \ , \ \lambda_2 \ , \ \lambda_3 \ , \ \lambda_4\right\}_{M/\mathcal{M}_{\mathbb{C}}} = \left\{2 \ , \ -2 \ , \ 2i \ , \ -2i\right\}_{M/\mathcal{M}_{\mathbb{C}}} \ . \qquad (48)$$

Such eigenvalues also determine the $\mathcal{M}_{\mathbb{C}}$-eigenvalues spectrum. The $\mathcal{M}_{\mathbb{C}}$-eigenvectors set is obtained by translating the complex M-eigenvectors set

$$\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} , \begin{pmatrix}0\\1\\1\\0\end{pmatrix} , \begin{pmatrix}1\\-i\\i\\1\end{pmatrix} , \begin{pmatrix}-i\\1\\-1\\-i\end{pmatrix}\right\}_M \ ,$$

in quaternionic formalism

$$\left\{ \begin{pmatrix} 1 \\ \text{-}j \end{pmatrix} \; , \; \begin{pmatrix} j \\ 1 \end{pmatrix} \; , \; \begin{pmatrix} 1+k \\ i+j \end{pmatrix} \; , \; \begin{pmatrix} j-i \\ k-1 \end{pmatrix} \right\}_{\mathcal{M}_{\mathbb{C}}} \; . \tag{49}$$

The quaternionic matrix which diagonalizes $\mathcal{M}_{\mathbb{C}}$ is

$$\mathcal{S}_{\mathbb{C}} = \texttt{Inverse} \left[\begin{pmatrix} 1 & 1+k \\ \text{-}j & i+j \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 1 & j \\ \frac{1-k}{2} & \text{-}\frac{i+j}{2} \end{pmatrix} \; , \tag{50}$$

and the diagonalized matrix is given by

$$\mathcal{M}_{\mathbb{C}}^{diag} = 2 \begin{pmatrix} \text{-}iR_i & 0 \\ 0 & i \end{pmatrix} . \tag{51}$$

This matrix can be directily obtained from the $M/\mathcal{M}_{\mathbb{C}}$ eigenvalues spectrum by translating, in quaternionic formalism, the matrix

$$M^{diag} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \text{-}2 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & \text{-}2i \end{pmatrix} \; .$$

It is interesting to note that equivalent diagonalized matrices can be obtained from $\mathcal{M}_{\mathbb{C}}^{diag}$ in Eq. (51) by the similarity transformation

$$\mathcal{U}^{\dagger} \, \mathcal{M}_{\mathbb{C}}^{diag} \, \mathcal{U} \; .$$

For example by choosing

$$\mathcal{U} = \begin{pmatrix} \text{-}j & 0 \\ 0 & \frac{1+k}{\sqrt{2}} \end{pmatrix} \; ,$$

one find

$$\mathcal{M}_{\mathbb{C}}^{diag} \; \rightarrow \; 2 \begin{pmatrix} iR_i & 0 \\ 0 & j \end{pmatrix} \; . \tag{52}$$

Appendix B
Two dimensional left quaternionic eigenvalues equations

Let us now examine left quaternionic eigenvalues equations for quaternionic linear operators.

• Hermitian operators

Let

$$\mathcal{H}_{\mathbb{H}} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

be the quaternionic matrix representation associated to an hermitian quaternionic linear operator. We consider its left quaternionic eigenvalue equation

$$\mathcal{H}_{\mathbb{H}} |\widetilde{\psi}> = \tilde{q}\, |\widetilde{\psi}> \quad , \tag{53}$$

where

$$|\widetilde{\psi}> = \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} \in \mathbb{H}^2 \; , \quad \tilde{q} \in \mathbb{H} \; .$$

Eq. (53) can be rewritten by the following quaternionic system

$$\begin{cases} k\, \widetilde{\psi}_2 = \tilde{q}\, \widetilde{\psi}_1 \; , \\ -k\, \widetilde{\psi}_1 = \tilde{q}\, \widetilde{\psi}_2 \; . \end{cases}$$

The solution is

$$\Big\{ \tilde{q} \Big\}_{\mathcal{H}_{\mathbb{H}}} = \Big\{ z + j\beta \Big\}_{\mathcal{H}_{\mathbb{H}}} \; ,$$

where

$$z \in \mathbb{C} \; , \quad \beta \in \mathbb{R} \; , \quad |z|^2 + \beta^2 = 1 \; .$$

The $\mathcal{H}_{\mathbb{H}}$-eigenvectors set is given by

$$\left\{ \begin{pmatrix} \widetilde{\psi}_1 \\ -k(z + j\beta)\widetilde{\psi}_1 \end{pmatrix} \right\}_{\mathcal{H}_{\mathbb{H}}} \; .$$

It is easy to verify that in this case

$$< \widetilde{\psi} | (\tilde{q} - \tilde{q}^\dagger) | \widetilde{\psi} > = 0$$

is verified for quaternionic eigenvalues $\tilde{q} \neq \tilde{q}^\dagger$.

• Anti-hermitian operators

Let

$$\mathcal{A}_{\mathbb{H}} = \begin{pmatrix} j & i \\ i & k \end{pmatrix}$$

be the quaternionic matrix representation associated to an anti-hermitian quaternionic linear operator. Its right complex spectrum is given by

$$\Big\{ \lambda_1 \; , \; \lambda_2 \Big\}_{\mathcal{H}_{\mathbb{H}}} = \Big\{ i\sqrt{2 - \sqrt{2}} \; , \; i\sqrt{2 + \sqrt{2}} \Big\}_{\mathcal{H}_{\mathbb{H}}} \; .$$

We now consider the left quaternionic eigenvalue equation

$$\mathcal{A}_{\mathbb{H}}|\widetilde{\psi}> = \widetilde{q}\,|\widetilde{\psi}> \quad , \tag{54}$$

where

$$|\widetilde{\psi}> = \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} \in \mathbb{H}^2 \ , \quad \tilde{q} \in \mathbb{H} \ .$$

By solving the following quaternionic system

$$\begin{cases} j\,\widetilde{\psi}_1 + i\,\widetilde{\psi}_2 = \tilde{q}\,\widetilde{\psi}_1 \ , \\ i\,\widetilde{\psi}_1 + k\,\widetilde{\psi}_2 = \tilde{q}\,\widetilde{\psi}_2 \ , \end{cases}$$

we find

$$\left\{\tilde{q}_1 \ , \ \tilde{q}_2\right\}_{\mathcal{A}_{\mathbb{H}}} = \left\{\frac{i}{\sqrt{2}} + \frac{j+k}{2} \ , \ \frac{-i}{\sqrt{2}} + \frac{j+k}{2}\right\}_{\mathcal{A}_{\mathbb{H}}} \ ,$$

and

$$\left\{\begin{pmatrix} \widetilde{\psi}_1 \\ (\frac{1}{\sqrt{2}} + \frac{j+k}{2})\widetilde{\psi}_1 \end{pmatrix} \ , \ \begin{pmatrix} \widetilde{\psi}_1 \\ (\frac{-1}{\sqrt{2}} + \frac{j+k}{2})\widetilde{\psi}_1 \end{pmatrix}\right\}_{\mathcal{H}_{\mathbb{H}}} \ .$$

We observe that

$$\left\{|\overline{u}_1\lambda_1 u_1| = \sqrt{2-\sqrt{2}} \ , \ |\overline{u}_2\lambda_2 u_2 = \sqrt{2+\sqrt{2}}\right\}$$

and

$$\left\{|\tilde{q}_1| = 1 \ , \ |\tilde{q}_2| = 1\right\} \ .$$

Thus, left and right eigenvalues cannot associated to the same physical quantity.

• A new possibility

In order to complete our discussion let us discuss for the quaternionic linear operator given in Eq. (43) its left quaternionic eigenvalue equation

$$\mathcal{M}_{\mathbb{H}}|\widetilde{\psi}> = \tilde{q}\,|\widetilde{\psi}> \quad , \tag{55}$$

where

$$|\widetilde{\psi}> = \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} \in \mathbb{H}^2 \ , \quad \tilde{q} \in \mathbb{H} \ .$$

Eq. (55) can be rewritten by the following quaternionic system

$$\begin{cases} i\,\widetilde{\psi}_1 + j\,\widetilde{\psi}_2 = \tilde{q}\,\widetilde{\psi}_1 \ , \\ k\,\widetilde{\psi}_1 + i\,\widetilde{\psi}_2 = \tilde{q}\,\widetilde{\psi}_2 \ . \end{cases}$$

The solution gives for the quaternionic eigenvalues spectrum

$$\left\{\tilde{q}_1 \ , \ \tilde{q}_2\right\}_{\mathcal{M}_{\mathbb{H}}} = \left\{i + \frac{j+k}{\sqrt{2}} \ , \ i - \frac{j+k}{\sqrt{2}}\right\}_{\mathcal{M}_{\mathbb{H}}} \ , \tag{56}$$

and for the eigenvectors set

$$\left\{\begin{pmatrix} 1 \\ \frac{1-i}{\sqrt{2}} \end{pmatrix} \ , \ \begin{pmatrix} 1 \\ \frac{i-1}{\sqrt{2}} \end{pmatrix}\right\}_{\mathcal{M}_{\mathbb{H}}} \ . \tag{57}$$

Let us now consider the following quaternionic linear operator

$$\mathcal{N}_{\mathbb{H}} = \begin{pmatrix} i + \frac{j+k}{\sqrt{2}} & 0 \\ 0 & i - \frac{j+k}{\sqrt{2}} \end{pmatrix} . \tag{58}$$

This operator represents a diagonal operator and has the same left quaternionic eigenvalues spectrum of $\mathcal{M}_{\mathbb{H}}$, notwithstanding such an operator is not equivalent to $\mathcal{M}_{\mathbb{H}}^{diag}$. In fact, the $\mathcal{N}_{\mathbb{H}}$-complex counterpart is characterized by the following eigenvalues spectrum

$$\left\{ i\sqrt{2} \; , \; -i\sqrt{2} \; , \; i\sqrt{2} \; , \; -i\sqrt{2} \right\}_N \; ,$$

different from the eigenvalues spectrum of the M-complex counterpart of $\mathcal{M}_{\mathbb{H}}$. Thus, there is no similarity transformation which relates these two operators in the complex world and consequently by translation there is no a quaternionic matrix which relates $\mathcal{N}_{\mathbb{H}}$ to $\mathcal{M}_{\mathbb{H}}$. So, in the quaternionic world, we can have quaternionic linear operators which have a same left quaternionic eigenvalues spectrum but not related by a similarity transformation.

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