On the gravitational potentials and the Riccati like equations

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Abstract

We present and discuss the generalized Laplace partial differential equation in the de Sitter Universe ($\mathcal{D}$) in several metrics. From Projective Relativity, we show that all angular differential equations have solutions given by the spherical harmonics and all radial differential equations can be written as a Riccati ordinary differential equation. When the radius of $\mathcal{D}$ goes to infinity we reobtain the Minkowskian results. In particular, the radial equations predict the behavior of the gravitational field in $\mathcal{D}$.

We also solve the Laplace equation associated to the Prasad metric. We discuss the so called internal and external spaces that correspond to the symmetry groups $SO_{3,2}$ and $SO_{4,1}$, respectively. Using hyperspherical coordinates we show that both radial differential equations can be led to the Riccati ordinary differential equation. For the Prasad metric with the radius of the universe independent of the parametrization, we associate the solution of the temporal equation with quantum number hypercharge to the internal structure and for the external structure, we associate the energy eigenvalues. Again the radial field equation can be led to the classical case in the limit process of a flat spacetime.

1 Introduction

The de Sitter Universe ($\mathcal{D}$), is a solution of Einstein's equation [1]. In this paper we present and discuss the Laplace differential equation in this universe. Although it can be treated as a four-dimensional Riemannian manifold, $\mathcal{D}$ is not the space where Special Relativity (SR) is valid, because this theory was developed in spaces endowed with Minkowski geometry. We want to adapt SR to cosmology, distinguishing *absolute* spacetime, the effective seat of physical events, from the tangent *relative* spacetime in which phenomena seem to happen to every observer [2].

The correspondence between $\mathcal{D}$ and Minkowski spacetime appears constructing homeomorphisms between these spaces by doing stereographic projections, obtaining the Riemann [3], Börner-Dürr [4] and Beltrami [3] metrics.

From the above metrics, we solve the so called *generalized Laplace differential equation*. We introduce the spherical relativistic coordinates and we conclude after separation of variables that the angular differential equation is the same for all metrics, and admits as solutions the *spherical harmonics*, while the radial differential equations differ; for all of them the classical (Minkowskian) case is obtained when $r \to \infty$, where r is the radius of $\mathcal{D}$. Introducing a suitable change of variables, all radial differential equations can be led to a Riccati equation [5].

For the other side, Arcidiacono [3] proposed a natural extension of translations in a Minkowskian spacetime, which can be thought as rotations in the symmetry group $SO_{4,1}$, restricted to the de Sitter universe $\mathcal{D}$. We can divide $\mathcal{D}$ in internal and external hyperspaces, associating with each one, the symmetry groups $SO_{4,1}$ and $SO_{3,2}$, respectively. After parametrizing these hyperspaces, we again solve the generalized Laplace equation, obtaining the same conclusions we have gotten from the projective metrics.

This paper is organized as follows: in section two we present the Laplace differential equation with Riemann metric, solving the angular differential equation in terms of spherical harmonics and reducing the radial differential equation to a Riccati equation. In section three we discuss the Laplace equation in $\mathcal{D}$ with the Beltrami metric. The radial equation is again led to a Riccati equation. In section four we present and discuss the Börner-Dürr metric. In section five we present the Laplace differential equation for the Prasad metric, by the parametrization of the internal and external hyperspaces associated with $\mathcal{D}$, solving the angular differential equation and reducing the radial differential field equation to a Riccati equation. In section six we discuss the Laplace equation associated with another parametrization of $\mathcal{D}$, getting similar results we have already obtained. Finally, we present our conclusions.

2 The Riemann metric

The stereographic projection of $\mathcal{D}$ on the Euclidean space E_3, which is a tangent hyperplane to $\mathcal{D}$, originates the *Riemann metric* [1]

$$ds^2 = \frac{4r^2}{4r^2+\rho^2}(dx^2+dy^2+dz^2).$$

Let us make a change of variables, introducing spherical coordinates (ρ,θ,φ) in E_3. The Riemann metric transforms into

$$ds^2 = \frac{4r^2}{4r^2+\rho^2}(d\rho^2+\rho^2 d\theta^2+\rho^2 \sin^2\theta d\varphi^2)$$

Substituting the tensor components above in the generalized Laplacian [6] we obtain the Laplace equation:

$$\nabla^2\Phi = \frac{(4r^2+\rho^2)^{3/2}}{4r^2\rho^2}\frac{\partial}{\partial\rho}\left[\frac{\rho^2}{(4r^2+\rho^2)^{1/2}}\frac{\partial\Phi}{\partial\rho}\right] + \frac{(4r^2+\rho^2)}{4r^2\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Phi}{\partial\theta}\right]$$
$$+\frac{(4r^2+\rho^2)}{4r^2\rho^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0. \tag{1}$$

Consider the following separation of variables:

$$\Phi = R(\rho)Y(\theta,\varphi) \equiv RY.$$

It is possible to separate eq.(1) in radial and angular parts, yielding:

$$(4r^2+\rho^2)^{1/2}\frac{d}{d\rho}\left[\frac{\rho^2}{(4r^2+\rho^2)^{1/2}}\frac{dR}{d\rho}\right] = kR \tag{2}$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + kY = 0 \tag{3}$$

where k is a constant.

[1]From now on, the symbol of the intrinsic metric tensor product will be omited; we won't distinguish $ds^2 = g_{ij}dx^i dx^j$ and $ds^2 = g_{ij}dx^i \otimes dx^j$.

In eqs.(2) and (3) the constant factor k is restricted to the values $k = \ell(\ell+1)$, where $\ell = 0, 1, 2 \ldots$ Otherwise, there would be a discontinuity when the azimuthal angle takes the values $\theta = \pm\pi$, since the series solution of the angular azimuthal equation would have values going to infinity at these points. We have chosen integer values for k since for such values, the series degenerates in polynomials which are continuous solutions on the whole real line [7, 8].

With the separation $Y = Y(\theta, \varphi) = \Theta(\theta)\phi(\varphi)$, we can further separate eq.(3), obtaining

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] + \sin^2\theta[\ell(\ell+1)] = -\frac{1}{\phi}\frac{d^2\phi}{d\varphi^2} \tag{4}$$

Each side of eq.(4) has a different independent variable; let the separation constant be m^2, with the condition $-\ell \leq m \leq \ell$, where $m \in \mathcal{Z}$. This choice is made because of the conservation of the rotational symmetry of ϕ: in each direction, ϕ has a unique value. So

$$\phi(\varphi) = Ae^{\pm im\varphi} \tag{5}$$

where A is a constant.

The other equation is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \tag{6}$$

which is the *associated Legendre differential equation*, whose solutions regular at the origin are ${P_l}^m(\cos\theta)$, namely the *associated Legendre functions*.

Introducing the *spherical harmonics* defined by $Y(\theta, \varphi) = {P_l}^m(\cos\theta)e^{\pm im\varphi}$, we obtain the solutions of the angular differential equation.[2]

The radial differential equation can be written as

$$\rho^2\frac{d^2R}{d\rho^2} + \left(2\rho - \frac{\rho^3}{4r^2+\rho^2}\right)\frac{dR}{d\rho} + \ell(\ell+1)R = 0 \tag{7}$$

Considering the following change of variable:

$$\frac{1}{R}\frac{dR}{d\rho} = \eta.$$

[2]In the present case $m \in \mathcal{N}$, what is very useful in Quantum Mechanics, but it is also possible to exist a magnetic quantum number such that $m \leq 0$. Besides, a more precise definition of the spherical harmonics is the one with a normalization factor, given by $Y(\theta, \varphi) = (-1)^m \left[\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}\right]^{1/2} {P_l}^{|m|}(\cos\theta)e^{\pm im\varphi}$.

we can write

$$\rho^2\eta' = -\rho^2\eta^2 + \left(2\rho - \frac{\rho^3}{4r^2+\rho^2}\right)\eta + \ell(\ell+1), \tag{8}$$

which is a Riccati equation.

Besides, in eq.(8) we can carry out the limit

$$\lim_{r\to\infty}\left[\rho^2\frac{d^2R}{d\rho^2} + \left(2\rho - \frac{\rho^3}{4r^2+\rho^2}\right)\frac{dR}{d\rho} + \ell(\ell+1)R\right] = \rho^2\frac{d^2R}{d\rho^2}+2\rho\frac{dR}{d\rho}+\ell(\ell+1)R$$

$$= \frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \ell(\ell+1)R = 0$$

In the particular case where $k=0$, considering that the particle (which is described by a radial wave function) is in its ground state ($\ell=0$), one returns to the classical case, since $\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=0$. In this case, the *scalar field* (or *scalar potential*) has spherical symmetry, and can be written as $\phi=\phi(\rho)$.

3 The Beltrami metric

An observer $\mathcal{O}$ at an arbitrary position in the de Sitter Universe ($\mathcal{D}$) observes every event from its own referential, which is a tangent hyperspace to $\mathcal{D}$. Consider, for example, a light pulse. The luminous pulse follows the intrinsic geodesic of $\mathcal{D}$, but the observer locates it in a direction tangent to S^4.

To construct a faithful representation of events in each one of the infinite tangent hyperspaces we need to establish a relation between the absolute coordinates ξ^α ($\alpha=0,1,\ldots,4$) in $\mathcal{D}$ and the relative coordinates x^μ ($\mu = 0,1,2,3$) in P_{3+1}, the so-called *projective Castelnuovo spacetime*[9].

Among plenty of possibilities, Beltrami chose doing a stereographic projection from the center of $\mathcal{D}$ on a tangent hyperplane to $\mathcal{D}$ at the south pole $(0,0,0,0,-r)$. We can verify the relations

$$\xi^4 = \frac{r}{A} \quad \text{and} \quad \xi^\mu = \frac{x^\mu}{A}, \tag{9}$$

where $A \equiv (1+\frac{\rho^2}{r^2})^{1/2}$, $\rho^2 \equiv x^\mu x_\mu = -x_0^2+x_1^2+x_2^2+x_3^2$ and r is the radius the universe. To a Pythagorean metric in $\mathcal{D}$ there corresponds a projective metric [10] in P_{3+1} associated with the same line element:

$$ds^2 = d\xi^a d\xi_a = A^{-4}[A^2dx^idx_i - \frac{1}{r^2}(x_idx_i)^2], \quad \text{with} \quad a=0,1,2,3. \tag{10}$$

The associated Laplace differential equation is expressed as

$$\nabla^2\phi = \frac{(r^2+\rho^2)^2}{r^4\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\phi}{\partial\rho}\right)+\frac{(r^2+\rho^2)}{r^2\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{(r^2+\rho^2)}{\rho^2r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}=0$$

After separation of variables, the angular differential equation has the same solution as the Riemann metric [6].

The radial differential equation is

$$(r^2+\rho^2)\frac{\rho^2}{r^2}\frac{d^2R}{dr^2}+\frac{2\rho}{r^2}(r^2+\rho^2)\frac{dR}{d\rho}-\ell(\ell+1)R=0 \tag{11}$$

It is important to verify the limit of the equations, when the radius of the de Sitter manifold approaches infinity, which must agree with the relativistic observations on the free (null divergence) Minkowski spacetime. Indeed, when $r\to\infty$ we obtain $\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=0$, leading to the classical field case.

Introducing the same change of variable done in the case of Riemannian metric,

$$\mu=\frac{1}{R}\frac{dR}{d\rho},$$

we can write eq.(11) as

$$\rho^2\mu'=-\mu^2\rho^2-2\rho\mu+\ell(\ell+1)\frac{r^2}{r^2+\rho^2}, \tag{12}$$

which is a Riccati equation.

4 The Börner-Dürr metric

This metric is obtained by the parametrization of the hyperboloid [4]:

$$\xi^a\eta_{ab}\xi^b=-r^2, \qquad \text{with} \qquad \eta_{ab}=\mathrm{diag}(1,-1,-1,-1,-1) \qquad a,b=0,\ldots,4$$

with conformal coordinates $(x^\mu, \mu=0,1,2,3)$. It can be done by a stereographic projection from the north pole $(0,0,0,0,r)$ into a tangent hyperplane on the south pole $(0,0,0,0,-r)$:

$$\xi^\mu=\zeta(x^2)x^\mu, \quad \xi^4=-r\zeta(x^2)\left(1+\frac{x^2}{4r^2}\right)$$

with $\zeta(x^2) = \left(1 - \frac{x^2}{4r^2}\right)^{-1}, \quad x^2 = x^\mu x_\mu = -x_0^2 + x_1^2 + x_2^2 + x_3^2.$

Substituting the metric tensor in the Laplace differential equation we obtain[3]

$$\nabla^2\phi = -\zeta(x^2)\left(\frac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho\phi) + \frac{1}{\rho^2\sin\theta}\left[\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{\sin^2\theta}\right] + \frac{1}{c^2}\partial_{tt}\phi\right) = 0$$

Using the method of separation of variables, we obtain the same angular differential equation, although the former equations are partial differential ones and can be written as follows:

$$\partial_\rho\left(\rho^2\partial_\rho A\right) + \frac{\rho^2}{c^2}\partial_t{}^2 A = -kA \tag{13}$$

where k is a constant.

Now, introducing $A(\rho,t) = R(\rho)T(t)$ and substituting in eq.(13), we obtain two ordinary differential equations as follows:

$$\frac{1}{R}\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{k}{\rho^2} = a^2 \tag{14}$$

and

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = a^2. \tag{15}$$

where a^2 is a constant.

The solution of eq.(15) is an exponential and for the other ordinary differential equation we introduce the same change of variable and we obtain

$$\mu' = -\mu^2 - \frac{2}{\rho}\mu + \frac{k}{\rho^2} + a^2. \tag{16}$$

which is also a Riccati equation.

5 The Prasad metric

The de Sitter Universe ($\mathcal{D}$) can be described as the sum $\wp_1 + \wp_2$ where $\wp_1$ is the external space and $\wp_2$ is the internal one. To each of these spaces there correspond the symmetry groups $SO_{4,1}$ and $SO_{3,2}$ respectively[11].

[3]We have done the change of variables $x_0 \to ict$, where c and t have dimensions of velocity and time, respectively. From now on, we introduce the notation: $\partial_\zeta \equiv \frac{\partial}{\partial_\zeta}$.

5.1 The external space ($\wp_1$)

This hyperspace has constant curvature $1/R^2$ and has a quadratic form associated with it, as follows:

$$R^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2 + (x_5)^2.$$

A possible parametrization [11] for the above quadratic form is

$$\begin{aligned} x_1 &= R\sin\chi\sin\theta\cos\varphi\cosh t \\ x_2 &= R\sin\chi\sin\theta\sin\varphi\cosh t \\ x_3 &= R\sin\chi\cos\theta\cosh t \\ x_4 &= R\sinh t \\ x_5 &= R\cos\chi\cosh t \end{aligned}$$

allowing us to write the line element as

$$ds_+^2 = R^2\cosh^2 t[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)] - R^2 dt^2 \tag{17}$$

For this metric tensor we obtain the Laplace differential equation

$$(R^2\cosh^2 t\sin^2\chi)\nabla^2\phi = \frac{\sin^2\chi}{\cosh t}\partial_t(\cosh^3 t\partial_t\phi) - \partial_t(\sin^2\chi\partial_\chi\phi) +$$

$$+\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{\sin^2\theta}\partial_{\varphi\varphi}\phi = 0. \tag{18}$$

After writing the scalar field ϕ as

$$\phi(\chi, t, \theta, \varphi) = \Gamma(\chi, t)Y(\theta, \varphi)$$

we can separate the Laplace differential equation in two partial differential equations: one of these has the solution known yet (spherical harmonics) while the another is

$$\frac{1}{\cosh t}\partial_t(\cosh^3 t\partial_t\Gamma) - \frac{1}{\sin^2\chi}\partial_\chi(\sin^2\chi\partial_\chi\Gamma) - \frac{\ell(\ell+1)}{\sin^2\chi}\Gamma = 0, \tag{19}$$

with $\ell = 0, 1, 2, 3\ldots$

Relatively to eq.(19), we consider the functional product

$$\Gamma(\chi, t) = \Pi(\chi)\Omega(t)$$

and suppose that the separation constant[4] is α^2. Then, eq.(19) can be separated in two ordinary differential equations:

$$\frac{1}{\sin^2\chi}\frac{d}{d\chi}\left(\sin^2\chi\frac{d\Pi}{d\chi}\right)+\frac{\ell(\ell+1)}{\sin^2\chi}\Pi-\alpha^2\Pi=0 \tag{20}$$

and

$$\frac{1}{\cosh t}\frac{d}{dt}\left(\cosh^3 t\frac{d\Omega}{dt}\right)-\alpha^2\Omega=0. \tag{21}$$

We emphasize eq.(20), namely, the radial one.[5] We can rewrite eq.(20) as

$$\frac{d^2\Pi}{d\chi^2}+2\cot\chi\frac{d\Pi}{d\chi}+\frac{\ell(\ell+1)}{\sin^2\chi}\Pi-\alpha^2\Pi=0. \tag{22}$$

Introducing the variable $j\equiv\cos\chi$, we obtain [12]

$$\frac{d^2\Pi}{dj^2}(1-j^2)-3j\frac{d\Pi}{dj}-\alpha^2\Pi+\frac{\ell(\ell+1)}{1-j^2}\Pi=0 \tag{23}$$

We can rewrite eq.(23) as

$$\mu'=-\mu^2+\frac{1}{1-j^2}\left[3j\mu+\alpha^2-\frac{\ell(\ell+1)}{1-j^2}\right] \tag{24}$$

which is a Riccati equation.

5.2 The internal space ($\wp_2$)

This hyperspace has a negative constant curvature $-1/r^2$ and can be characterized by the symmetry group $SO_{3,2}$. To this group is associated a quadratic form

$$x_1^2+x_2^2+x_3^2-x_4^2-x_6^2=-r^2.$$

Prasad [11] proposed the following parametrization with nonstatic coordinates:

$$\begin{aligned}
x_1 &= r\sinh\chi\sin\theta\cos\varphi\cos t\\
x_2 &= r\sinh\chi\sin\theta\sin\varphi\cos t\\
x_3 &= r\sinh\chi\cos\theta\cos t\\
x_4 &= r\sin t\\
x_6 &= r\cosh\chi\cos t
\end{aligned}$$

[4] $\alpha=0,1,2,3,\ldots$ corresponds to the eingenvalue spectrum of Gegenbauer equation.

[5] The variable χ is defined by $\sin\chi=\dfrac{(x_1)^2+(x_2)^2+(x_3)^2}{R^2}$. Although sin$\chi$ is not the radius of $\mathcal{D}$, it is the so-called *adimensional normalized radius.*

Then, we can write the line element as follows:

$$ds_-^2 = r^2 \cos^2 t[d\chi^2 + \sinh^2 \chi(d\theta^2 + \sin^2 \theta d\varphi^2)] - r^2 dt^2. \quad (25)$$

The generalized Laplace differential equation can be written as

$$r^2 \cos^2 t \sinh^2 \chi \nabla^2 \phi = -\frac{\sin^2 \chi}{\cos t}\partial_t(\cos^3 t \partial_t \phi) + \partial_\chi(\sinh^2 \chi \partial_\chi \phi) +$$

$$+\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{\sin^2\theta}\partial_{\varphi\varphi} = 0 \quad (26)$$

Defining the funcional product

$$\phi(\rho, t, \theta, \varphi) = \Lambda(\rho, t)Y(\theta, \varphi) \equiv \Lambda Y,$$

we can separate eq.(26) in radial-temporal and angular equations:

$$-\frac{1}{\Lambda}\frac{\sinh^2 \chi}{\cos t}\partial_t(\cos^3 t \partial_t \Lambda) + \frac{1}{\Lambda}\partial_\chi(\sinh^2 \chi \partial_\chi \phi) = \ell(\ell + 1) \quad (27)$$

and

$$\frac{1}{Y}\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{Y}\frac{1}{\sin^2\theta}\partial_{\varphi\varphi}\phi = -\ell(\ell+1). \quad (28)$$

where $\ell = 0, 1, 2, \cdots$ It is also worthwhile to mention here that eq.(28) is exactly the angular equation obtained in the precedent case (the internal space $\wp_1$). Once again we separate eq.(27) in radial and temporal equations (we will emphasize the radial equation only).

Taking $\Lambda(\chi, t) = \Xi(\chi)\Psi(t)$ we get two ordinary differential equations

$$\frac{1}{\Psi}\frac{1}{\cos t}\frac{d}{dt}\left(\cos^3 t \frac{d\Psi}{dt}\right) = \beta^2 \quad (29)$$

and

$$\frac{1}{\Xi}\frac{1}{\sinh^2 \chi}\frac{d}{d\chi}\left(\sinh^2 \chi \frac{d\Xi}{d\chi}\right) - \frac{\ell(\ell+1)}{\sinh^2 \chi} = -\beta^2. \quad (30)$$

where β^2 is the separation constant[6] . Introducing $m = \cosh \chi$ and defining $\mu(m) = \frac{1}{\Xi}\frac{d\Xi(m)}{dm}$ we can write eq.(30) as

$$\mu' = -\mu^2 + \frac{1}{1 - m^2}\left[3m\mu + \beta^2 - \frac{\ell(\ell+1)}{1-m^2}\right], \quad (31)$$

which is also a Riccati equation.

[6] $\beta = 0, 1, 2, 3, \ldots$ corresponds to the eingenvalue spectrum of Gegenbauer equation.

6 The Prasad metric with radius of the Universe independent of the parametrization

In the former case (section 5), with the hyperbolic parametrization of $\mathcal{D}$, the classical radial field equations were not obtained in the limit $r \to \infty$. With a new temporal parametrization we are able to obtain the classical case in this limit.

The group that characterizes the external space $\mathcal{D}_+$ is $SO_{4,1}$, while the internal space $\mathcal{D}_-$ is only affected by the symmetry group $SO_{3,2}$. Using cartesian coordinates, we can express[5] the internal and external line elements,

$$ds_+^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2 - dx_4{}^2 + dx_5{}^2,$$

$$ds_-^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2 - dx_4{}^2 - dx_6{}^2.$$

From the transformations proposed by Tolman[13]

$$x_5 \pm x_4 = R_+ \exp[\pm t/R_+](1 - r^2/R_+^2)^{1/2}$$

for $\mathcal{D}_+$ and

$$x_4 \pm i x_6 = R_- \exp[\pm it/R_-](1 + r^2/R_-^2)^{1/2}$$

for $\mathcal{D}_-$, we can substitute these expressions in the expressions for the line elements, obtaining

$$ds_\pm^2 = \frac{dr^2}{1 \mp r^2/R_\pm{}^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 - (1 \mp r^2/R_\pm^2) dt^2$$

The generalized Laplace differential equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(1 \mp \Lambda_\pm r^2)\frac{\partial \psi}{\partial r}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2} - \frac{1}{1 \mp \Lambda_\pm r^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \qquad (32)$$

where $\Lambda_\pm \equiv 1/R_\pm^2$.

Let $\psi(r,t,\theta,\varphi) = Y(\theta,\varphi)\Gamma(r,t)$. Then we can separate eq.(32) as

$$\frac{1}{\Gamma}\frac{\partial}{\partial r}\left[r^2(1 \mp \Lambda_\pm r^2)\frac{\partial \Gamma}{\partial r}\right] - \frac{1}{\Gamma}\frac{r^2}{(1 \mp \Lambda_\pm r^2)}\frac{\partial^2 \Gamma}{\partial t^2} = \ell(\ell+1) \qquad (33)$$

and for the non-angular equation we obtain the same equation that we have gotten formerly.

If we put $\Gamma(r,t) = S(r)T(t)$, we obtain from eq.(33)

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -B.$$

where B is a separation constant. If we are on the internal structure of $(\mathcal{D}_-)$, the temporal equation above has an oscillatory solution

$$T_-(t) = \tau_- \exp(\pm iHt/R_-),$$

associated with the quantum number *hypercharge* (H)[11]. In case we are treating $\mathcal{D}_+$, time will have an exponential character described by the equation associated with the energy eigenvalues (E)

$$T_+(t) = \tau_+ \exp(\pm Et/R_+)$$

For the radial equations we can write

$$\frac{1 \mp \Lambda_\pm r^2}{r^2}\frac{d}{dr}\left[r^2(1 \mp \Lambda_\pm r^2)\frac{dS}{dr}\right] - \frac{\ell(\ell+1)}{r^2}(1 \mp \Lambda_\pm r^2)S = BS. \qquad (34)$$

These equations can be transformed in a classical field equation (for example, the electric or the gravitational fields in $\mathcal{D}$). From the definition $\Lambda_\pm = 1/R_\pm^2$, it is obvious that in the limit where $R \to \infty$, $\Lambda_\pm \to 0$. Using some results of Quantum Mechanics in hyperspherical universes in $\mathcal{D}_+$, with $B = Y^2/R_-^2$, and in $\mathcal{D}_-$,with $B = E^2/R_+^2$, in the limit where $R_\pm \to \infty$ we obtain $B \to 0$. Therefore the radial field equation can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = 0 \qquad (35)$$

in the ground state, where $\ell = 0$.

With the change of variable

$$\frac{1}{R}\frac{dR}{d\rho} = \eta$$

we can transform eq.(34) in two Riccati equations:

$$\eta' = -\eta^2 - \eta\left(\frac{2}{r} \mp \frac{2\Lambda_\pm r}{1 \mp \Lambda_\pm r^2}\right) + \frac{\ell(\ell+1)}{r^2(1 \mp \Lambda_\pm r^2)} + \frac{B}{1 \mp \Lambda_\pm r^2}. \qquad (36)$$

7 Conclusions

For the Riemann, Beltrami and Prasad metrics, the classical case

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left[\rho^2\frac{dR}{d\rho}\right]=0$$

is obtained when we substitute in all radial equations the limit where the manifold radius approaches infinity.

All the three Laplace equations exhibit the same angular equation from the Riemann (section 2), Beltrami (section 3) and Börner-Dürr (section 4) projective metrics. With regard to the radial equations, we have obtained for each metric a second order ordinary differential equation, which we have led into a Riccati differential equation.

We have treated the Prasad metric separately because it comes from a parameterized metric in Projective Relativity. Using these parametrizations, $\mathcal{D}$ is shared in internal and external spaces. The radial differential field equations, obtained from the Laplace differential equation, have different signs, but both are led to similars Riccati equations, different from each other only by the name of the substitution variable. In the first case, with the parametrization, the classical case of a spherically symmetric scalar field cannot be obtained by making the radius of $\mathcal{D}$ approach infinity. This was expected because in cartesian coordinates the radius of $\mathcal{D}$ is independent of the coordinates, while in curvilinear ones the parametrization constrains the radius of $\mathcal{D}$: it is the radial parametric coordinate. Thus the information about the radius is lost, because the Laplace differential equation is homogeneous. In the last treatment, with the parametrization independent of the radius of the Universe, the classical field case is obtained. Finally, by a suitable change of variables, all radial differential field equations can be led to a Riccati equation.

Besides, we have gotten important results on Quantum Mechanics. The temporal equation from the corresponding metrics associates quantum numbers and energy eigenvalues with the internal and external structures of $\mathcal{D}$.

At last, since the Laplace-Beltrami generates a wave partial differential equation (the metric is Lorentzian (3,1)), if we put in all the three Prasad metrics the coordinate $ct = 0$ the wave differential equation is led to a Laplace differential equation.

Acknowledgements

One of us (Roldão da Rocha Jr.) is grateful to FAPESP for financial support during one and a half year of research. We are grateful to Dr. J. Emílio Maiorino for many and useful discussions.

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