Reflexive Spaces of Linear Operators

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Abstract

In this note we improve a result of Ruckle[9] and Holub[3], which characterizes when the space $L(E;F)$ is reflexive.

Let E and F be Banach spaces, let $L(E;F)$ denote the Banach space of all continuous linear operators from E into F, and let $K(E;F)$ denote the subspace of all compact members of $L(E;F)$. Let $A(E;F)$ denote the subspace of all *approximable operators*, that is $A(E;F)$ is the norm-closure of $E' \otimes F$, the subspace of all finite rank operators.

Let τ_c denote the topology of compact convergence on $L(E;F)$, that is the locally convex topology defined by the seminorms of the form $\|T\| = sup_{x \in K} \|Tx\|$, where K varies among the compact subsets of E.

We recall that E is said to have the *approximation property* if the identity operator on E lies in the τ_c-closure of $E' \otimes E$. Likewise E is said to have the *compact approximation property* if the identity operator on E lies in the τ_c-closure of $K(E;E)$. Then we have the following theorem.

Theorem. *Let E and F be reflexive Banach spaces. Consider the following conditions:*
(a) $L(E;F) = A(E;F)$.
(b) $L(E;F) = K(E;F)$.
(c) For each $T \in L(E;F)$ there exists $x \in E$ with $\|x\| = 1$ such that $\|Tx\| = \|T\|$.
(d) $L(E;F)$ is reflexive.
(e) $L(E;F)' = (L(E;F), \tau_c)'$.
Then the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ always hold. If either E or F has the compact approximation property, then $(e) \Rightarrow (b)$. And if either E or F has the approximation property, then $(e) \Rightarrow (a)$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious, and the implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$ have been elegantly proved by Holub[3].

$(d) \Rightarrow (e)$: We have the canonical isometric isomorphisms

$$L(E;F) = Bil(E,F') = (E\tilde{\otimes}_\pi F')'$$

(see [1, p.27]). And $L(E;F)$ being reflexive, we conclude that

$$L(E;F)' = E\tilde{\otimes}_\pi F'.$$

Hence every $\Phi \in L(E;F)'$ admits a representation of the form

$$\Phi(T) = \sum_{n=1}^{\infty} y'_n(Tx_n)$$

for every $T \in L(E;F)$, where

$$(x_n)_{n=1}^{\infty} \subset E, \qquad (y'_n)_{n=1}^{\infty} \subset F', \qquad \sum_{n=1}^{\infty} \|x_n\| \|y'_n\| < \infty.$$

Thus $L(E;F)' \subset (L(E;F),\tau_c)'$, by [1, p. 62] or [6, p. 31]. Since the other inclusion is obvious, (e) follows.

$(e) \Rightarrow (b)$ if either E or F has the compact approximation property: If either E or F has the compact approximation property, then one can readily see that $K(E;F)$ is a dense subspace of $(L(E;F),\tau_c)$. This in tandem with (e) implies that

$$L(E;F) = \overline{K(E;F)}^{\tau_c} = \overline{K(E;F)}^{\|\cdot\|} = K(E;F).$$

If either E or F has the approximation property, then a similar argument shows that $(e) \Rightarrow (a)$.

Remarks. (1) Ruckle[9] proved that $(b) \Leftrightarrow (d)$ when both E and F have the approximation property. Holub[3] proved that the implications $(b) \Rightarrow (c) \Rightarrow (d)$ always hold, and that $(d) \Rightarrow (b)$ when either E or F has the approximation property. See also [4] for another proof of the implication $(b) \Rightarrow (d)$, and [1, p. 196] or [2, p. 247] for another proof of the equivalence $(b) \Leftrightarrow (d)$ when either E or F has the approximation property.

(2) A refinement of the proof of the implication $(d) \Rightarrow (e)$ shows that conditions (d) and (e) are actually equivalent. But I do not know whether any of the other implications in the theorem may be reversed in general.

(3) Every ℓ_p with $p \neq 2$ contains a closed subspace without the compact approximation property. See [6, p. 94] for the case $p > 2$ and [7, p. 107] for the case $p < 2$. Willis[10] has given an example of a reflexive Banach space which has the compact approximation property, but fails to have the approximation property.

Example. Let E and F be closed subspaces of ℓ_p and ℓ_q, respectively, where $1 < p, q < 2$. Let $R : E \hookrightarrow \ell_p$ and $S : F \hookrightarrow \ell_q$ denote the inclusion mappings. Let $\tilde{T} = S \circ T \circ R'$ for each $T \in L(E'; F)$. Then one can readily see that the mapping

$$T \in L(E'; F) \to \tilde{T} \in L(\ell_{p'}; \ell_q)$$

is an isometric embedding, where $(1/p) + (1/p') = 1$. Moreover T is compact if and only if $\tilde{T}$ is compact. Since $p' > q$, it follows from a result of Pitt[8] that $L(\ell_{p'}; \ell_q) = K(\ell_{p'}; \ell_q)$ (see [5, p. 208] for a direct proof). Hence we see that $L(E'; F)$ is reflexive and $L(E'; F) = K(E'; F)$. Furthermore, taking into account Remark (3) and the fact that *a reflexive Banach space has the compact approximation property if and only if its dual has the same property* (a standard argument shows this), we may even assume that neither E' nor F has the compact approximation property.

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