Stationary Solution of the Navier-Stokes Equations in a 2d Bounded Domain for Incompressible Flow with Discontinuous Density.

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January 9, 2001

Abstract

We consider the boundary value problem for the stationary Navier-Stokes equations describing an inhomogeneous incompressible fluid in a two dimensional bounded domain. We show the existence of weak solution with prescribed boundary values for the density in L^∞. The solution is obtained as a weak limit of smooth solutions with Hölder continuous density. We also show the existence of a 'trace' for the density.

Key words. stationary Navier-Stokes, incompressible flow, discontinuous density.

AMS Subject Classifications. 35Q30, 76D05

1 Introduction

In this work we are concerned with the motion of an inhomogeneous stationary incompressible fluid in a bounded domain Ω of $\mathbf{R}^2$ (see [1], p. 34),

*Supported by CNPq/Brazil, 300050/92-5.

possibly with holes, i.e. Ω is bounded and simply connected or multiconnected. We assume that the boundary Γ of Ω is smooth, so we can write $\Gamma = \cup_{i=0}^{n}\Gamma_i$, where Γ_i are smooth simple closed curves. The density of mass, velocity, pressure, viscosity (constant) and the density of an external force are denoted, respectively, by ρ, $\mathbf{v} = (v_1, v_2)$, p, ν and $\mathbf{f} = (f_1, f_2)$. Let γ_i denote the part of Γ_i where the fluid is incoming, i.e. $\gamma_i \stackrel{\text{def}}{=} \{x \in \Gamma_i\,;\, \mathbf{v}\cdot\mathbf{n} < 0\}$, where $\mathbf{n}$ denotes the outward unit normal of Γ. We assume that γ_i is connected, possibly empty. Then we formulate the following boundary value problem for Navier-Stokes equations:

$$\left\{\begin{array}{c} \left.\begin{array}{rcl} \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p & = & \nu\Delta\mathbf{v} + \rho\mathbf{f} \\ \multicolumn{3}{c}{\mathrm{div}(\mathbf{v}) = 0, \quad \mathrm{div}(\rho\mathbf{v}) = 0} \end{array}\right\} \quad \text{in} \;\; \Omega \\ \mathbf{v}|\Gamma = \mathbf{v}_0, \quad \rho|\gamma_i = \rho_i, \quad i = 0, 1, \cdots, n. \end{array}\right. \tag{1}$$

The quantities $\mathbf{v}_0$ and ρ_i live in the spaces $L^2(\Gamma)$ and $L^\infty(\gamma_i)$, respectively, with $\mathbf{v}_0$ satisfying

$$\int_{\Gamma_i} \mathbf{v}_0\cdot\mathbf{n}\,ds = 0, \;\; i = 0, 1, \cdots, n. \tag{2}$$

The force density $\mathbf{f}$ is assumed to lie in $L^2(\Omega)$.

N.N. Frolov [2] proved the existence (and regularity) of solution of (1) with smooth density if $\mathbf{v}_0 \in C^1(\Omega)$ and $\rho_i \in C^\alpha(\Omega)$ $(\alpha > 0)$. It is interesting to obtain solution with discontinuous density, since we think this can model a fluid with density stratification (see [3]) and from the mathematical point of view it is important to understand the several solutions (1) have. In this work we prove the existence of solution of (1) with $\rho_i \in L^\infty(\Omega)$ and $\mathbf{v}_0 \in L^2(\Omega)$. In fact, we prove that sequences of Frolov's solutions have a limit which is still a solution of (1). More precisely, by introducing a *stream function* ψ such that

$$\mathbf{v} = \nabla^\perp\psi \stackrel{\text{def}}{=} (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1})$$

($x = (x_1, x_2)$ will denote a generic point in $\overline{\Omega}$), the problem (1) in the case of smooth solution $(\rho, \mathbf{v})$ turns into (see [2])

$$\left\{\begin{array}{ll} \nu\Delta^2\psi = -\mathrm{div}(\omega(\psi)(\nabla^\perp\psi\cdot\nabla)\nabla\psi) + \mathrm{div}(\rho\mathbf{f}^\perp) & \text{in } \Omega, \\ \psi|\Gamma = \psi_0|\Gamma, \qquad \frac{\partial\psi}{\partial\mathbf{n}}|\Gamma = \frac{\partial\psi_0}{\partial\mathbf{n}}|\Gamma, & \end{array}\right. \tag{3}$$

where $\mathbf{f}^\perp = (-f_2, f_1)$, $\mathbf{n}$ is the unit outward vector of Γ, $\tau = \mathbf{n}^\perp$ is a unit tangent vector of Γ, ψ_0 is an appropriate function defined in Ω satisfying

$$\frac{\partial \psi_0}{\partial \tau}|\Gamma = -\mathbf{v}_0 \cdot \mathbf{n} \quad \text{and} \quad \frac{\partial \psi_0}{\partial \mathbf{n}}|\Gamma = \mathbf{v}_0 \cdot \tau, \tag{4}$$

and ω is a scalar function of one variable such that

$$\omega(\psi_0|\gamma_i) = \rho_i, \;\; i = 0, \cdots, n. \tag{5}$$

It is possible to construct ψ_0 such that the intervals Im($\psi_0|\gamma_i$) do not overlap each other. Indeed, choose constants c_i, $i = 0, 1, \cdots, n$, appropriately and define

$$\psi_0(x) \stackrel{\text{def}}{=} -\int_{x_i}^{x} \mathbf{v}_0 \cdot \mathbf{n} ds + c_i \tag{6}$$

for $x \in \Gamma_i$ where x_i is an arbitrary point of Γ_i [2]. The reason why the images Im($\psi_0|\gamma_i$) are intervals of the real line is that $\psi_0|\gamma_i$ is a increasing function, since $(\partial\psi_0/\partial\tau)|\gamma_i = (-\mathbf{v}_0 \cdot \mathbf{n})|\gamma_i > 0$. In what follows, frequently we are going to identify γ_i with Im($\psi_0|\gamma_i$).

Frolov's solution requires ω to be a Hölder continuous function. We take a sequence of smooth solutions of (3), ψ^ϵ, $\rho^\epsilon = \omega^\epsilon(\psi^\epsilon)$, $\epsilon > 0$, where ω^ϵ converges as ϵ goes to zero to some irregular function, say, a Heaviside function, and try to pass to the limit as ϵ goes to zero. We enlighten this idea with the following simpleminded example.

Example 1 (Constant velocity). *If* $\mathbf{v}$ *is constant, say* $\mathbf{v} = (1, 0)$ *for simplicity, then the system (1) becomes*

$$\nabla p = \rho f, \quad \rho_{x_1} = 0, \tag{7}$$

so ρ *is an arbitrary function of* x_2*. Now we are interested in a limit process of (7). The stream function in this case is* $\psi = x_2$*, up to an additive constant. Let's consider the following 'evolution' of (7).*

$$\psi^\epsilon = x_2, \qquad \rho^\epsilon = \omega^\epsilon(\psi^\epsilon) = \omega^\epsilon(x_2),$$

where ω^ϵ *converges (weakly) to some Heaviside function* H*, say* $H = \rho_1\chi_{(-\infty,0)} + \rho_2\chi_{(0,\infty)}$ *for some given constants* ρ_1 *and* ρ_2*. Then* $\psi^\epsilon = \psi = x_2$ *(constant with respect to* ϵ*) converges to* ψ *and* ρ^ϵ *converges weakly to the descontinuous function*

$$\rho(x_1, x_2) = \begin{cases} \rho_1 & \text{if } x_2 < 0 \\ \rho_2 & \text{if } x_2 > 0. \end{cases}$$

It is clear that the pair (ρ, ψ) *is a weak solution of (1).*

Since we deal with nonsmooth density solution, some new issues that not appear in [2] must be considered here. For instance, our density ρ lives in L^∞ only, so we need to prove the existence of trace of ρ on the boundary $\gamma \stackrel{\text{def}}{=} \cup_{i=0}^n \gamma_i$ in some sense and that it equals to the boundary data ρ_i; cf. Definition 1 and Proposition 1 below. The weak formulation of (1) in terms of the stream function ψ, see (3), considered in [2], in which $\rho = \omega(\psi)$, cannot be realized here because this will cause lack of sense when ψ takes values in sets of null measures where ρ has jumps. Thus our formulation of weak solution (1) will be achieved just by integrating by parts the equations in (1) against solenoidal test functions.

The method we use to obtain the main estimates to prove Theorem 1 below appears in [4] and is also used in [2].

Besides this introduction, this paper contains two more sections. In section 2 we give the weak formulation for (1) and prove that the weak solution ρ of $\text{div}(\rho \mathbf{v}) = 0$ (see Definition 1) attains its boundary value in mean in the weak-$*$ sense (see Proposition 1). We note that this is a regularity result type for ρ. In section 3 we state and prove our main result, that is, the existence of a weak solution $(\rho, \mathbf{v})$ of the problem (1) in $L^\infty(\Omega) \times H^1(\Omega)$.

Notations: Throughout this paper we fix the following notations:

Γ	:	$\partial\Omega$
$W^{k,p}(\Omega)$	:	The Sobolev space of order k modelled in $L^p(\Omega)$
$H^k(\Omega)$	:	$W^{k,2}(\Omega)$
$H_0^k(\Omega)$	:	The space of functions in $H^k(\Omega)$ whose derivatives up to order $k-1$ have null trace in Γ
$\lvert \cdot \rvert$	:	Lebesgue measure

2 Weak solutions

We start this section with the definition of weak solution of the boundary value problem for the equation $\text{div}(\rho \mathbf{v}) = 0$.

Definition 1 *Given $\mathbf{v} \in H^1(\Omega)$ with $div(\mathbf{v}) = 0$, we say that $\rho \in L^\infty(\Omega)$ is a weak solution of*

$$\begin{cases} div(\rho \mathbf{v}) &= 0 \quad in \quad \Omega \\ \rho|\gamma_i &= \rho_i \in L^\infty(\gamma_i), \; i = 0, 1, \cdots, n \end{cases} \tag{8}$$

if

$$\int_\Omega \rho\mathbf{v}\cdot\nabla\Phi dx = \int_{\gamma_i}\rho_i(\mathbf{v}_0\cdot\mathbf{n})\Phi ds, \tag{9}$$

for all $\Phi\in C^1(\overline{\Omega})$ *such that* $\phi|(\Gamma/\gamma_i) = 0$, $i = 0,1,\cdots,n$, *where* $\mathbf{v}_0 \stackrel{\text{def}}{=} \mathbf{v}|\Gamma$.

Next we prove that a weak solution of the boundary value problem for the equation (25), accordingly with the above definition, attains the boundary values $\rho_i\in L^\infty(\gamma_i)$ in mean in the weak-$*$ topology of $L^\infty(\gamma_i)$. More precisely, we have the following result:

Proposition 1 *If* ρ *satisfies the above Definition then*

$$\lim_{t\to 0-}\frac{1}{t}\int_0^t \rho(\cdot,\tau)d\tau = \rho_i,$$

in the weak-$*$ topology of $L^\infty(\gamma_i)$, $i = 0,1,\cdots,n$, *for any parametrization* (s,t) *of a tubular neighborhood of* γ_i, *where* s *parametrizes* γ_i *and* t *increases in the direction of* $\mathbf{n}$. *Precisely, we mean that if* $h(s,t) = \gamma_i(s) + t\mathbf{n}(\gamma_i(s))$ *is a local difeomorphism, where* s *belongs to some interval* I *and* $-1 \ll t \ll 1$ *then*

$$\lim_{t\to 0-}\int_I\left(\frac{1}{t}\int_0^t\rho(h(s,\tau))d\tau\right)\varphi(s)ds = \int_I \rho_i(h(s,0))\varphi(s)ds, \tag{10}$$

for any $\varphi\in L^1(I)$.

Proof: 1. An straightforward computation shows that

$$\int_A \rho\mathbf{v}\cdot\nabla_x\Phi dx = \int_t^0\int_{\mathrm{I}}\rho\mathbf{v}\cdot\left((\nabla_{(s,\tau)}\Phi)[\frac{\partial(x_1,x_2)}{\partial(s,t)}]^{-1}|\frac{\partial(x_1,x_2)}{\partial(s,t)}|\right)dsd\tau,$$

where $A \stackrel{\text{def}}{=} \{h(s,\tau); s\in \mathrm{I}, \tau\in(t,0]\}$, $x = (x_1,x_2)$, $\frac{\partial(x_1,x_2)}{\partial(s,t)}$ is the Jacobian matrix of the map h and in the right hand side, $\rho\equiv\rho(s,t)\equiv\rho(h(s,t))$ and $\mathbf{v}\equiv\mathbf{v}(s,t)\equiv\mathbf{v}(h(s,t))$. Notice that

$$[\frac{\partial(x_1,x_2)}{\partial(s,t)}]^{-1}|\frac{\partial(x_1,x_2)}{\partial(s,t)}| = \begin{bmatrix}\frac{\partial x_2}{\partial t} & -\frac{\partial x_1}{\partial t}\\ -\frac{\partial x_2}{\partial s} & \frac{\partial x_1}{\partial s}\end{bmatrix}.$$

Then from (9), with test functions of type $\Phi = \varphi(s)\psi(\tau)$, we have

$$\begin{aligned}&\int_t^0\int_{\mathrm{I}}\rho(v_1\tfrac{\partial x_2}{\partial t} - v_2\tfrac{\partial x_1}{\partial t})\varphi'(s)\psi(\tau)dsd\tau\\ &+\int_t^0\int_{\mathrm{I}}\rho(v_2\tfrac{\partial x_1}{\partial s} - v_1\tfrac{\partial x_2}{\partial s})\varphi(s)\psi'(\tau)dsd\tau\\ =&\int_{\gamma_i}\rho_i(\mathbf{v}_0\cdot\mathbf{n})\varphi\psi ds.\end{aligned} \tag{11}$$

Now taking $\psi(\tau)$ a smooth aproximation of the function $\frac{1}{t}(t-\tau)\chi_{(t,0]}$ and letting $t \to 0-$ we get

$$\lim_{t\to 0-} \frac{1}{t}\int_0^t \int_{\mathrm{I}} \rho(\mathbf{v}\cdot\mathbf{N})\varphi(s)dsd\tau = \int_{\mathrm{I}} \rho_0(\mathbf{v}_0\cdot\mathbf{N}_0)\varphi(s)ds, \tag{12}$$

for all $\varphi \in C^1(\bar{\mathrm{I}})$, where

$$\mathbf{N} \equiv \mathbf{N}(s,t) = (-\frac{\partial x_2}{\partial s}, \frac{\partial x_1}{\partial s})$$

is a normal vector to the curve $\gamma_{i,t} = \mathrm{Im}(\, h(\cdot,t)\,)$, and $\mathbf{N_0} \stackrel{\text{def}}{=} \mathbf{N}(s,0)$.
2. In this step we want to show that we can substitute $\mathbf{v}\cdot\mathbf{N}$ in (12) by $\mathbf{v}_0\cdot\mathbf{N}_0$, i.e.

$$\lim_{t\to 0-} \frac{1}{t}\int_0^t \int_{\mathrm{I}} \rho(\mathbf{v}_0\cdot\mathbf{N}_0)\varphi(s)dsd\tau = \int_{\mathrm{I}} \rho_0(\mathbf{v}_0\cdot\mathbf{N}_0)\varphi(s)ds. \tag{13}$$

Then we write

$$\begin{aligned}
& \tfrac{1}{t}\int_0^t\int_{\mathrm{I}}\rho(\mathbf{v}_0\cdot\mathbf{N}_0)(s)\varphi(s)dsd\tau \\
= \;& \tfrac{1}{t}\int_0^t\int_{\mathrm{I}}\rho(\mathbf{v}\cdot\mathbf{N})(s,t)\varphi(s)dsd\tau \\
& +\tfrac{1}{t}\int_0^t\int_{\mathrm{I}}\rho[(\mathbf{v}_0\cdot\mathbf{N}_0)(s)-(\mathbf{v}\cdot\mathbf{N})(s,t)]\varphi(s)dsd\tau \\
\equiv \;& I + II
\end{aligned}$$

and show that II goes to zero as $t\to 0-$. Indeed,

$$\begin{aligned}
|II| \le ||\rho||_\infty ||\varphi||_\infty \int_t^0\int_{\mathrm{I}} |\tfrac{\partial}{\partial\tau}(\mathbf{v}\cdot\mathbf{N})(s,\tau)|dsd\tau \\
\stackrel{t\to 0-}{\longrightarrow} 0,
\end{aligned}$$

where we have used the following inequality

$$\int_t^0 |f(s,\tau)|d\tau \le |t|\int_t^0 |f(s,\tau)|d\tau$$

for $f = \mathbf{v}_0\cdot\mathbf{N}_0 - \mathbf{v}\cdot\mathbf{N}$.

3. Now, since $\mathbf{v}_0 \in L^2$, it easy to extend (13) for all $\varphi \in L^2(\mathrm{I})$. Besides, by hypothesis, $\mathbf{v}_0\cdot\mathbf{N}_0 > 0$ on I so we can write $\mathrm{I} = \cup_{k=1}^{\infty} I_k$, where $I_k \stackrel{\text{def}}{=} \{s \in \mathrm{I}; \mathbf{v}_0\cdot\mathbf{N}_0 > 1/k\}$. Then taking φ in (13) to be the function

$$\varphi_k \stackrel{\text{def}}{=} \varphi\frac{1}{\mathbf{v}_0\cdot\mathbf{N}_0}\chi_{I_k},$$

we get

$$\lim_{t\to 0-}\frac{1}{t}\int_0^t\int_{I_k}\rho(s,\tau)\varphi ds d\tau=\int_{I_k}\rho_0(s)\varphi ds,$$

for all $k=1,2,\cdots$ and any $\varphi\in L^2(\mathrm{I})$. Let L to be a measurable subset of I. Given an arbitrary $\epsilon>0$, there exists a k such that $|L/L_k|<\epsilon$, $L_k \stackrel{\text{def}}{=} \mathrm{I}\cap I_k$, because $\{I_k\}$ is an increasing sequence of measurable subsets of I whose unioun is I. Then we have

$$\begin{aligned} & |\tfrac{1}{t}\int_t^0\int_L\rho(s,\tau)dsd\tau-\int_L\rho_0(s)ds| \\ \leq\ & |\tfrac{1}{t}\int_t^0\int_{L_k}\rho(s,\tau)dsd\tau-\int_{L_k}\rho_0(s)ds| \\ & +(||\rho||_\infty+||\rho_0||_\infty)|L_k/L|, \end{aligned}$$

so

$$\lim_{t\to 0-}\frac{1}{t}\int_t^0\int_L\rho(s,\tau)dsd\tau=\int_L\rho_0(s)ds$$

for all measurable subset L of I and this is enough to show the weak-$*$ convergence (10). ■

Remark 1 *It is possible to show that the weak-$*$ limit (10) occurs without the mean in t. In fact, for any representative ρ of ρ it turns out that the weak-$*$ limit of $\rho(\cdot,t)$ exists as $t\to 0-$ and is equal to ρ_0. To show this we first notice that from (11) we have that the function*

$$f_\varphi(t)\stackrel{\text{def}}{=}\int_I\rho(\mathbf{v}\cdot\mathbf{N})(s,t)\varphi(s)ds$$

is a BV function, i.e. a function of Bounded Variation, since from (11) we have that

$$|\int f_\varphi(t)\psi'(t)dt|\leq const.||\psi||_\infty$$

for any $\psi\in C^1$. Then there exists the limit $l_\varphi \stackrel{\text{def}}{=} \lim_{t\to 0-} f_\varphi(t)$. Second, we show that this limit is equal to $\int_I\rho_0(s)\varphi(s)ds$ by taking in (11) a sequence of function ψ converging to $\chi_(-\infty,t_0)$ and such that the sequence of its derivatives converges to minus the delta Dirac function centered at t_0, for any Lebesgue point $t_0<0$ of f_φ. Then let $t_0\to 0-$.

We end this section with the weak formulations of (1) and (3). Below, repeated indices mean summation from 1 to 2.

Definition 2 *A pair $(\rho, \mathbf{v}) \in L^\infty(\Omega) \times H^1(\Omega)$ is said to be a weak solution of problem (1) if div$(\mathbf{v}) = 0$ in $H^1(\Omega)$, $(\rho, \mathbf{v})$ satisfies (9), $\mathbf{v} = \mathbf{u} + \mathbf{a}$ where $\mathbf{a} = (a_1, a_2)$ is an appropriate extension of $\mathbf{v}_0$ specified below and $\mathbf{u} = (u_1, u_2) \in H_0^1(\Omega)$, and $(\rho, \mathbf{u})$ satisfies the integral equations*

$$\begin{aligned} \nu \int_\Omega \nabla(u_j + a_j) \cdot \nabla \Phi_j dx \quad & - \int_\Omega \rho(u_j + a_j)(\mathbf{u} + \mathbf{a}) \cdot \tfrac{\partial \mathbf{\Phi}}{\partial x_j} dx \\ & = \int_\Omega \rho \mathbf{f} \cdot \mathbf{\Phi} dx, \end{aligned} \tag{14}$$

for all $\mathbf{\Phi} = (\Phi_1, \Phi_2) \in C_0^\infty(\Omega; \mathbf{R}^2)$ such that div$(\mathbf{\Phi}) = 0$. Cf. [4] and [2].

Remark 2 (Extension of $\mathbf{v}_0$): Let ψ_0 be a function in $H^2(\Omega)$ satisfying the boundary conditions (4),(6), e.g. we can take ψ_0 as the solution of the biharmonic equation in Ω with the boundary conditions (4),(6). We take the extension $\mathbf{a}$ of $\mathbf{v}_0$ as follows (cf. [4, pp. 27, 120] and [2]): $\mathbf{a} \equiv \mathbf{a}_\delta = \nabla^\perp(\psi_0 \zeta)$ in Ω, $\zeta \equiv \zeta_\delta$, $0 < \delta \ll 1$, where ζ_δ is a smooth function (a twice differentiable cutoff function) such that $\zeta_\delta = 1$ near Γ and zero at all points of Ω with distance from Γ bigger than δ; besides,

$$|\zeta_\delta(x)| \le c, \qquad |\nabla \zeta_\delta(x)| \le c/\delta, \tag{15}$$

where c is a constant independent of x and δ. The equations (4) are equivalent to

$$(\nabla^\perp \psi_0)|\Gamma = \mathbf{v}_0. \tag{16}$$

Remark 3 *For smooth solutions, we write $\mathbf{u} = \nabla^\perp \varphi$ where $\varphi \in H_0^2(\Omega)$, and $\rho = \omega(\psi)$, where ω was introduced in (3). Besides, the solenoidal test functions $\mathbf{\Phi}$ can be written as $\nabla^\perp \theta$ for $\theta \in C_0^\infty(\Omega)$. Then Definition 2 turns into the following Definition for weak solution of (3).*

Definition 3 *A function $\psi \in H^2(\Omega)$ is said to be a weak solution of problem (3) if $\psi = \varphi + \psi_{0,\delta}$ where $\psi_{0,\delta} = \psi_0 \zeta_\delta$ and $\varphi \in H_0^2(\Omega)$, and φ satisfies the integral equation*

$$\begin{aligned} & \nu \int_\Omega \Delta(\varphi + \psi_{0,\delta}) \Delta \theta dx \\ & - \int_\Omega \omega(\varphi + \psi_{0,\delta})(\nabla^\perp \varphi + \nabla^\perp \psi_{0,\delta})_j (\nabla^\perp \varphi + \nabla^\perp \psi_{0,\delta}) \cdot \frac{\partial(\nabla^\perp \theta)}{\partial x_j} dx \\ & = \int_\Omega \omega(\varphi + \psi_{0,\delta}) \mathbf{f} \cdot \nabla^\perp \theta dx, \end{aligned} \tag{17}$$

for all $\theta \in C_0^\infty(\Omega)$

3 The main results

The goal of this section is to prove the following result:

Theorem 1 *If* $\rho_i \in L^\infty(\gamma_i)$, $i = 0, 1, \cdots, n$ *and* $\mathbf{v}_0 \in L^2(\Omega)$ *then the problem (1) has a weak solution* $(\rho, \mathbf{v})$ *in* $L^\infty(\Omega) \times H^1(\Omega)$.

To prove this result we will use

Theorem 2 *If a sequence* $\{(\rho^\epsilon, \mathbf{v}^\epsilon)\}$ *of solutions of (1) converges to* $(\rho, \mathbf{v})$ *in* $(L^\infty(\Omega), weak - *) \times (H^1(\Omega), weak)$ *then* $(\rho, \mathbf{v})$ *is also a solution of (1).*

Theorem 2 is essentially a corollary of

Proposition 2 *The operator*

$$(\rho, \mathbf{v}) \longmapsto \frac{\partial}{\partial x_j} (\rho v_j \mathbf{v}) \tag{18}$$

is sequencially continuous from $(L^\infty(\Omega), weak - *) \times (H^1(\Omega), weak)$ *into* $(H^{-1}(\Omega), weak - *)$.

We will prove first this Proposition, then we will prove Theorems 2 and 1.

Proof of Proposition 2: First we notice that the map (18) is well defined, i.e. $\frac{\partial}{\partial x_j}(\rho v_j \mathbf{v}) \in H^{-1}(\Omega)$ for any $(\rho, \mathbf{v}) \in L^\infty(\Omega) \times H^1(\Omega)$. Indeed, for any $\mathbf{\Phi} \in H_0^1(\Omega)$ we have

$$\begin{aligned} |\langle \tfrac{\partial}{\partial x_j}(\rho v_j \mathbf{v}), \mathbf{\Phi}\rangle| &\stackrel{\text{def}}{=} |-\int_\Omega \rho v_j \mathbf{v} \cdot \tfrac{\partial \mathbf{\Phi}}{\partial x_j} dx| \\ &\leq ||\rho||_{L^\infty} ||\mathbf{v}||_{L^4}^2 ||\mathbf{\Phi}||_{H_0^1} \\ &\leq \mathrm{c} ||\rho||_{L^\infty} ||\mathbf{v}||_{H^1}^2 ||\mathbf{\Phi}||_{H_0^1}, \end{aligned}$$

where c is the constant of the Sobolev imbedding $H^1(\Omega) \to L^4(\Omega)$. It remains to prove that if $\{(\rho^k, \mathbf{v}^k)\}$, $k = 1, 2, \cdots$, is any sequence in $L^\infty(\Omega) \times H^1(\Omega)$ such that ρ^k converges to ρ in $(L^\infty(\Omega), \text{weak} - *)$ and $\mathbf{v}^k$ converges to $\mathbf{v}$ in $(H^1(\Omega), \text{weak})$ then there exists a subsequence $k_1 < k_2 < \cdots$ such that

$$\lim_{i \to \infty} \int_\Omega \rho^{k_i} v_j^{k_i} \mathbf{v}^{k_i} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx = \int_\Omega \rho v_j \mathbf{v} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx. \tag{19}$$

(This is intuitive, since, because of the imbedding $H^1(\Omega) \to L^4(\Omega)$ being compact, the left hand side of the above equation is a *weak-strong-strong* triple.) We write

$$\int_\Omega \rho^{k_i} v_j^{k_i} \mathbf{v}^{k_i} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx - \int_\Omega \rho v_j \mathbf{v} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx \equiv I + II + III,$$

where $I \stackrel{\text{def}}{=} \int_\Omega (\rho^{k_i} - \rho) v_j \mathbf{v} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx$, $II \stackrel{\text{def}}{=} \int_\Omega \rho^{k_i} (v_j^{k_i} - v_j) \mathbf{v} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx$ and $III \stackrel{\text{def}}{=} \int_\Omega \rho^{k_i} v_j^{k_i} (\mathbf{v}^{k_i} - \mathbf{v}) \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx$, and we show that $\lim_{i\to\infty} I, II, III = 0$. Indeed,

$$I = \langle \rho^{k_i} - \rho, v_j \mathbf{v} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} dx \rangle$$

and $v_j \mathbf{v} \cdot \frac{\partial \mathbf{\Phi}}{\partial x_j} \in L^1(\Omega)$, so $\lim_{i\to\infty} I = 0$, since ρ^k converges to ρ in $(L^\infty(\Omega), \text{weak} - *)$;

$$|II| \le \sup_k \{||\rho^k||_{L^\infty}\} ||v_j^{k_i} - v_j||_{L^4} ||\mathbf{v}||_{L^4} ||\frac{\partial \mathbf{\Phi}}{\partial x_j}||_{L^2}$$

and $||v_j^{k_i} - v_j||_{L^4} \stackrel{i\to\infty}{\longrightarrow} 0$ for some subsequence $k_1 < k_2 < \cdots$ because of the compact imbedding $H^1(\Omega) \to L^4(\Omega)$, so $\lim_{i\to\infty} II = 0$; analogously,

$$|III| \le \sup_k \{||\rho^k||_{L^\infty}\} ||\mathbf{v}||_{L^4} ||\mathbf{v}^{k_i} - \mathbf{v}||_{L^4} ||\mathbf{\Phi}||_{H^1}$$

goes to zero as i goes to ∞. ■

Proof of Theorem 2: We need to check all the equations in Definition 2. Regarding equation (14), the nonlinear term is dealt by Proposition 2 and the linear terms do not offer great difficult to deal with. The equation $\text{div}(\mathbf{v}) = 0$ in $H^1(\Omega)$, i.e. $\int_\Omega \mathbf{v} \cdot \nabla \Phi dx = 0$ for all $\Phi \in C_0^\infty(\Omega)$, is straightforward to verify. The remainder equation (9) can also be easily verified as follows:

$$\begin{aligned} &\int_\Omega \rho^\epsilon \mathbf{v}^\epsilon \cdot \nabla \Phi dx - \int_\Omega \rho \mathbf{v} \cdot \nabla \Phi dx \\ &\le \langle (\rho^\epsilon - \rho), \mathbf{v} \cdot \nabla \Phi \rangle + \int_\Omega \rho^\epsilon (\mathbf{v}^\epsilon - \mathbf{v}) \cdot \nabla \Phi dx \\ &\equiv I + II; \end{aligned}$$

I goes to zero because $\mathbf{v} \cdot \nabla \Phi \in L^1(\Omega)$ and II goes to zero passing to some subsequence because of the compact imbedding $H^1(\Omega) \to L^2(\Omega)$. ■

Remark 4 *Theorem 2 also holds if we consider boundary conditions depending on* ϵ, ρ_i^ϵ, $\mathbf{v}_0^\epsilon$, *such that* ρ_i^ϵ *converges to* ρ_i *in weak-$*$ topology of* $L^\infty(\gamma_i)$ *for* $i = 0, 1, \cdots, n$ *and* $\mathbf{v}_0^\epsilon$ *converges to* $\mathbf{v}_0$ *weakly in* $L^2(\Omega)$.

Proof of Theorem 1: Let $\{\psi_0^\epsilon\}$ and $\{\rho_i^\epsilon\}$, $\epsilon > 0$, be sequences of smooth functions such that $\{\psi_0^\epsilon\}$ converges to ψ_0 in $H^2(\Omega)$, where ψ_0 is defined in Remark 2, and $\{\rho_i^\epsilon\}$ converges to ρ_i in the weak-$*$ topology of $L^\infty(\gamma_i)$ for $i = 0, 1, \cdots, n$, as ϵ goes to zero. Let $\{\omega^\epsilon\}$, $\epsilon > 0$, be a bounded sequence in $L^\infty(\mathbf{R})$ of Hölder continuous functions such that $\omega^\epsilon(\psi_0^\epsilon|\gamma_i) = \rho_i^\epsilon$, $i = 0, 1, \cdots, n$. Let also ψ^ϵ be a smooth solution of (3) with ω^ϵ in place of ω and ψ_0^ϵ in place of ψ_0. Then, we are going to show that the sequence $\{(\rho^\epsilon, \mathbf{v}^\epsilon)\} \stackrel{\text{def}}{=} \{(\omega^\epsilon(\psi^\epsilon), \nabla^\perp\psi^\epsilon)\}$ has a subsequence that converges to a weak solution $(\rho, \mathbf{v})$ of (1) in $(L^\infty(\Omega), \text{weak} - *) \times (H^1(\Omega), \text{weak})$.

We first notice that since ω^ϵ is uniformly bounded in $L^\infty(\mathbf{R})$, so it is ρ^ϵ. The rest of the proof consists in proving that $\{\nabla^\perp\psi^\epsilon\}$ is a bounded sequence in $H^2(\Omega)$, so $\{\mathbf{v}^\epsilon\} = \{\nabla^\perp\psi^\epsilon\}$ is a bounded sequence in $H^1(\Omega)$. That will be enough to proof Theorem 1 by Banach-Alaouglu's theorem and Theorem 2.

To estimate $\{\mathbf{v}^\epsilon\}$ we use the technique in [4] and used in [2]. The plan is to take $\theta = \varphi^\epsilon$ $(= \psi^\epsilon - \psi_{0,\delta}^\epsilon)$ in (17), but there is a difficulty, that is to deal with the term

$$\int_\Omega u_j^\epsilon \nabla^\perp\psi_{0,\delta}^\epsilon \cdot \frac{\partial \mathbf{u}^\epsilon}{\partial x_j} dx, \qquad \mathbf{u}^\epsilon \stackrel{\text{def}}{=} \nabla^\perp\varphi^\epsilon, \quad \mathbf{u} = (u_1, u_2),$$

because this is a quadratic term in $\mathbf{u}^\epsilon$. Here it comes the subtle idea in [4] of using the arbitrariness of $0 < \delta \ll 1$ in (17). Let us take $\theta = \varphi^{\epsilon,\eta} \stackrel{\text{def}}{=} \varphi^\epsilon + \psi_{0,\delta}^\epsilon - \psi_{0,\eta}^\epsilon$ in (17) where $0 < \eta \leq \delta$. We remark that $\varphi^{\epsilon,\eta}$ is in fact a solution of (17) when we replace $\psi_{0,\delta}^\epsilon$ by $\psi_{0,\eta}^\epsilon$. Notice also that this $\theta \in H_0^2(\Omega)$ for all $0 < \eta \leq \delta$, so it is allowed to use it in (17). Then we get

$$\begin{aligned} \nu||\varphi^{\epsilon,\eta}||_2^2 = -\nu\int_\Omega \Delta\psi_{0,\eta}^\epsilon \Delta\varphi^{\epsilon,\eta} dx \quad &- \int_\Omega \rho^\epsilon v_j^\epsilon (\nabla^\perp\psi_{0,\eta}^\epsilon) \cdot \tfrac{\partial}{\partial x_j}(\nabla^\perp\varphi^{\epsilon,\eta}) dx \\ &- \int_\Omega \rho^\epsilon \mathbf{f} \cdot (\nabla^\perp\varphi^{\epsilon,\eta})\, dx, \end{aligned} \tag{20}$$

where $||\cdot||_2$ stands for the norm $||\Delta\cdot||_{L^2(\Omega)}$ in $H_0^2(\Omega)$ and we used that

$$\int_\Omega \rho^\epsilon v_j^\epsilon \mathbf{w} \cdot \frac{\partial}{\partial x_j}\mathbf{w} dx = 0, \qquad \mathbf{w} = \nabla^\perp\varphi^{\epsilon,\eta},$$

since $\mathrm{div}(\rho^\epsilon \mathbf{v}^\epsilon) = 0$. Indeed,

$$\int_\Omega \rho^\epsilon v_j^\epsilon \mathbf{w} \cdot \tfrac{\partial \mathbf{w}}{\partial x_j} dx = \tfrac{1}{2}\int_\Omega \rho^\epsilon v_j^\epsilon \tfrac{\partial}{\partial x_j} |\mathbf{w}|^2 dx$$
$$\tfrac{1}{2}\int_\Omega (\rho^\epsilon \mathbf{v}^\epsilon) \cdot \nabla(|\mathbf{w}|^2) dx = -\tfrac{1}{2}\int_\Omega (\mathrm{div}(\rho^\epsilon \mathbf{v}^\epsilon))|\mathbf{w}|^2 dx = 0.$$

Let $N_{\epsilon,\eta} \stackrel{\text{def}}{=} ||\varphi^{\epsilon,\eta}||_2$. From (20) and Hölder inequalities we obtain

$$\begin{aligned} \nu N_{\epsilon,\eta}^2 \leq \;& \nu ||\psi_{0,\eta}^\epsilon||_2 N_{\epsilon,\eta} + c_\infty \left(||\mathbf{f}||_{L^2} + ||\psi_{0,\eta}^\epsilon||_{W^{1,4}}^2\right) N_{\epsilon,\eta} \\ & + c_\infty \int_\Omega |(\nabla^\perp \varphi^{\epsilon,\eta})_j (\nabla^\perp \psi_{0,\eta}^\epsilon) \cdot \tfrac{\partial}{\partial x_j} (\nabla^\perp \varphi^{\epsilon,\eta})| dx \\ \equiv \;& K N_{\epsilon,\eta} + ||\omega||_{L^\infty} I_{\epsilon,\eta}, \end{aligned} \tag{21}$$

where c_∞ is a constant that bounds $||\rho^\epsilon||_{L^\infty}$, $K \stackrel{\text{def}}{=} \nu ||\psi_{0,\eta}^\epsilon||_2 + c_\infty(||\mathbf{f}||_{L^2} + ||\psi_{0,\eta}^\epsilon||_{W^{1,4}}^2)$ and $I_{\epsilon,\eta}$ is the integral occuring in (21). Notice that

$$||\psi_{0,\eta}^\epsilon||_{W^{1,4}} \leq c_1 ||\psi_{0,\eta}^\epsilon||_{H^2} = ||\psi_0^\epsilon \zeta_\eta||_{H^2} \leq c(\eta), \tag{22}$$

where c_1 is the constant of the Sobolev imbedding $H^2(\Omega) \to W^{1,4}(\Omega)$ and $c(\eta)$ is a constant independent of ϵ. Regarding $I_{\epsilon,\eta}$, defining

$$\Omega_\eta \stackrel{\text{def}}{=} \{x \in \Omega \,;\, \mathrm{dist}(x, \Gamma) < \eta\}$$

and using (15) we have the following estimate:

$$\begin{aligned} I_{\epsilon,\eta} \leq \;& \frac{c}{\eta} \int_{\Omega_\eta} |(\nabla^\perp \varphi^{\epsilon,\eta})_j \psi_0^\epsilon| \, |\tfrac{\partial}{\partial x_j} (\nabla^\perp \varphi^{\epsilon,\eta})| \, dx \\ & + c \int_{\Omega_\eta} |(\nabla^\perp \varphi^{\epsilon,\eta})_j| \, |\nabla^\perp \psi_0^\epsilon| \, |\tfrac{\partial}{\partial x_j} (\nabla^\perp \varphi^{\epsilon,\eta})| \, dx \\ \leq \;& \frac{c_1}{\eta} ||\nabla^\perp \varphi^{\epsilon,\eta}||_{L^2(\Omega_\eta)} \, ||\nabla^\perp \varphi^{\epsilon,\eta}||_{H^1(\Omega_\eta)} \\ & + c ||\nabla^\perp \varphi^{\epsilon,\eta}||_{L^4(\Omega_\eta)} \, ||\psi_0^\epsilon||_{W^{1,4}(\Omega_\eta)} \, ||\nabla^\perp \varphi^{\epsilon,\eta}||_{H^1(\Omega_\eta)}, \end{aligned} \tag{23}$$

where c_1 is a constants that bounds $c||\psi_0^\epsilon||_{L^\infty} \leq \mathrm{const.} ||\psi_0^\epsilon||_{H^2}$. Now, let us suppose that $\{\varphi^\epsilon\} \equiv \{\varphi^{\epsilon,\delta}\}$ is not a bounded sequence in $H_0^2(\Omega)$. Then there exists a subsequence $\{\varphi^{\epsilon_k}\}$ such that $N_k \stackrel{\text{def}}{=} ||\varphi^{\epsilon_k}||_2$ goes to infinity as $k \to \infty$. However,

$$\mathbf{z}^k \stackrel{\text{def}}{=} \frac{\mathbf{u}^{\epsilon_k}}{N_k}, \qquad \mathbf{u}^{\epsilon_k} \stackrel{\text{def}}{=} \nabla^\perp \varphi^{\epsilon_k}$$

is a bounded sequence in $H_0^1(\Omega)$, then is precompact in $L^4(\Omega)$. Thus, up to passing to some subsequence, there exists a $\mathbf{z} \in H_0^1(\Omega)$ such that $\{\mathbf{z}^k\}$

converges to $\mathbf{z}$: weakly in $H_0^1(\Omega)$ and strongly in $L^4(\Omega)$. We claim that the same holds for

$$\mathbf{z}^{k,\eta} \stackrel{\text{def}}{=} \frac{\mathbf{u}^{\epsilon_k,\eta}}{N_{k,\eta}}, \qquad \mathbf{u}^{\epsilon_k,\eta} \stackrel{\text{def}}{=} \nabla^\perp \varphi^{\epsilon_k,\eta},$$

i.e. $\{\mathbf{z}^{k,\eta}\}$ also converges to $\mathbf{z}$ (independent of η) as $k \to \infty$: weakly in $H_0^1(\Omega)$ and strongly in $L^4(\Omega)$. *Proof:*

$$\begin{aligned} ||\mathbf{z}^{k,\eta} - \mathbf{z}^k||_{H^1} &= ||(\tfrac{\mathbf{u}^{\epsilon_k}}{N_{k,\eta}} - \tfrac{\mathbf{u}^{\epsilon_k}}{N_k}) + \tfrac{\nabla^\perp(\psi_{0,\delta}^{\epsilon_k} - \psi_{0,\eta}^{\epsilon_k})}{N_{k,\eta}}||_{H^1} \\ &\leq |\tfrac{1}{N_{k,\eta}} - \tfrac{1}{N_k}|\ ||\mathbf{u}^{\epsilon_k}||_{H^1} + \tfrac{1}{N_{k,\eta}}||\psi_{0,\delta}^{\epsilon_k} - \psi_{0,\eta}^{\epsilon_k}||_{H^2} \\ &= \tfrac{|N_{k,\eta} - N_k|}{N_{k,\eta}}||\mathbf{z}^k||_{H^1} + \tfrac{1}{N_{k,\eta}}||\psi_{0,\delta}^{\epsilon_k} - \psi_{0,\eta}^{\epsilon_k}||_{H^2} \end{aligned}$$

and

$$\begin{aligned} |N_{k,\eta} - N_k| &= |\ ||\varphi^{\epsilon_k,\eta}||_2 - ||\varphi^{\epsilon_k}||_2\ | \\ &\leq ||\varphi^{\epsilon_k,\eta} - \varphi^{\epsilon_k}||_2 = ||\psi_{0,\eta}^{\epsilon_k} - \psi_\delta^{\epsilon_k}||_2 \\ &= ||\psi_0^{\epsilon_k}(\zeta_\eta - \zeta_\delta)||_2 \leq c(\eta,\delta), \end{aligned}$$

where $c(\eta,\delta)$ is a constant independent of ϵ. Besides, $\lim_{k\to\infty} N_{k,\eta} = \lim_{k\to\infty} N_k = \infty$. Thus $\{\mathbf{z}^{\epsilon_k,\eta} - \mathbf{z}^k\}$ converges strongly in $H_0^1(\Omega)$ to zero as $k \to \infty$.$\blacksquare$

Next, fix η for a while in (21), divide (21) by $N^2_{\epsilon_k,\eta}$ and let k goes to infinity. Then, using (22), (23) and the above claim, we obtain

$$\nu \leq c_2 \frac{c_1}{\eta}\, ||\mathbf{z}||_{L^2(\Omega_\eta)} + c_2 c ||\mathbf{z}||_{L^4(\Omega_\eta)}\, ||\psi_0||_{W^{1,4}(\Omega_\eta)}, \tag{24}$$

where c_2 is a constant that bounds $c_\infty||\mathbf{z}^{\epsilon_k,\eta}||_{H^1}$. Now, by a Poincaré type inequality, we have that

$$\frac{1}{\eta}\, ||\mathbf{z}||_{L^2(\Omega_\eta)} \leq c_3 ||\mathbf{z}||_{H^1(\Omega_\eta)},$$

where c_3 is a constant independent of η. Then we get a contradiction by letting η goes to zero in (24). $\blacksquare$

We finish this paper with the following remark.

Remark 5 For smooth solution, the equation

$$\operatorname{div}(\rho \mathbf{v}) = 0 \tag{25}$$

is equivalent to $\nabla\rho\cdot\mathbf{v} = 0$, since the fluid is incompressible ($\operatorname{div}(\mathbf{v}) = 0$), so ρ is constant along the integral curves of $\mathbf{v}$. For nonsmooth ρ, the equation (25) is seen in the weak sense and we can easily check the following by Green's theorem: If $\mathbf{v}$ is continuous and ρ has a jump along a curve $\mathcal{S}$ and is smooth on both sides of $\mathcal{S}$ then

$$\int_{\mathcal{S}} \Phi[\rho]\mathbf{v} \cdot d\mathbf{s} = 0,$$

for all test function Φ, where $[\rho]$ is the jump of ρ along $\mathcal{S}$, so

$$\begin{array}{c} [\rho](\mathbf{v} \cdot \mathbf{n}) = 0 \\ \textit{(shock condition)} \end{array} \tag{26}$$

where $\mathbf{n}$ is a unit normal to $\mathcal{S}$. (The first equation in (1) yields a shock condition for the pressure p; see §1 in [5].) Since $[\rho] \neq 0$, the shock condition (26) implies that $\mathbf{v}$ is tangent to $\mathcal{S}$, so $\mathcal{S}$ is in fact a integral curve of $\mathbf{v}$. In conclusion, jumps on ρ can only occur on integral curves of $\mathbf{v}$. With respect to the question whether ρ is constant along integral curves of $\mathbf{v}$, we should mention that this is true even for irregular divergent free vector fields $\mathbf{v}$, i.e. $\mathbf{v} \in H^1(\Omega)$ with $\mathrm{div}(\mathbf{v}) = 0$, treated by DiPerna–Lions in [6] (see also [7]). Accordingly, the only solution of the initial value problem for transport equation

$$\begin{cases} u_t - \mathbf{v} \cdot \nabla_x u = 0 \quad \text{in} \;\; \mathbf{R} \times \Omega \\ u(\cdot, 0) = \rho \end{cases} \tag{27}$$

in the space $L^\infty(\mathbf{R}; L^1(\Omega))$ is given by $u(x,t) = \rho(X(x,t))$, where $X \in C(\mathbf{R}; L^1(\Omega))$ is the flux of $\mathbf{v}$, which satisfies $X(x,\cdot) \in C^1(\Omega)$ and

$$\begin{cases} \frac{\partial X(x,t)}{\partial t} = \mathbf{v}(X(x,t)) \\ X(x,0) = x \end{cases} \tag{28}$$

for a.e. $x \in \Omega$ [6, 7]. Now, our solution ρ of (25) is obviously a stationary solution of (27), so $\rho(X(x,t)) \equiv \rho(x)$ for a.e. $x \in \Omega$, by uniqueness. We remark that although ρ lives only in $L^\infty(\Omega)$ the composition $\rho \circ X(\cdot, t)$ makes sense for all t because $X(\cdot, t)$ is Lebesgue measure invariant for all t, since $\mathrm{div}(\mathbf{v}) = 0$ [6, 7]. Now, for regular vector fields, say $\mathbf{v} \in W^{1,p}(\Omega)$ with $p > 2$, one can show that there is a neighborhood U in $\overline{\Omega}$ of each γ_i such that all $x \in U$ has a backward exit time of Ω, i.e. there exists a $t_x < 0$ such that $X(x, -t_x) \in \Gamma$ and $X(x,-t) \in \Omega$ if $t_x < t \leq 0$ [8]. Then the boundary datum ρ_i is achieved along the integral curves of $\mathbf{v}$, i.e.

$$\lim_{t \to t_x +} \rho(X(x,-t)) = \rho_i(X(x,-t_x)) \tag{29}$$

for all $x \in U$. We do not know if trajectories of irregular vector fields have an exit time. Another open question, as far as we know, is if a smooth boundary datum $\mathbf{v}_0$, say $\mathbf{v}_0 \in C^\infty$, with $\rho_i \in L^\infty$, yields a smooth (or continuous) velocity $\mathbf{v}$.

Acknowledgment: I thank M.R. Medar for introducing me the problem (1) and acquainting me with the references [2], [8] and [1]. I would like also to thank M.C. Lopes, H.N. Lopes and J.L. Boldrini for the discussions and suggestions on this work.

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