Quaternionic Groups in Physics: A Panoramic Review

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Abstract

Due to the non-commutative nature of quaternions we introduce the concept of left and right action for quaternionic numbers. This gives the opportunity to manipulate appropriately the $\mathbb{H}$-field. The standard problems arising in the definitions of transpose, determinant and trace for quaternionic matrices are overcome. We investigate the possibility to formulate a *new approach* to Quaternionic Group Theory. Our aim is to highlight the possibility of looking at new quaternionic groups by the use of left and right operators as fundamental step toward a clear and complete discussion of Unification Theories in Physics.

1 Introduction

Complex numbers have played a dual role in Physics, first as a technical tool in resolving differential equation (classical optics) or via the theory of analytic functions for performing real integrations, summing series, etc.; secondly in a more essential way in the development of Quantum Mechanics and later Field Theory. With quaternions, for the first type of application, i.e. as a means to simplify calculations, we can quote the original work of Hamilton [1], but this only because of the late development of vector algebra by Gibbs and Heaviside [2]. Even Maxwell used quaternions as a tool in his calculations, e.g. in the *Treatise of Electricity and Magnetism* [3] where we find the ∇-operator expressed by the three quaternionic imaginary units.

Notwithstanding the Hamilton's conviction that quaternions would soon play a role comparable to, if not greater than, that of complex numbers the use of quaternions in Physics was very limited [4]. Nevertheless, in the last decades, we find a renewed interest in the application of non-commutative fields in Mathematics and Physics. In Physics, we quote quaternionic versions of Gauge Theories [5]-[8], Quantum Mechanics and Fields [9]-[14], Special Relativity [15]. In Mathematics, we find applications of quaternions for Tensor Products [16, 17], Group Representations [18].

In this paper we aim to give a *new* panoramic review of quaternionic groups. We use the adjective "new" since the elements of our matrices will not be simple quaternions but *left and right operators*, originally introduced with the name of "barred operators" [19].

In Physics, particularly Quantum Mechanics, we are accustomed to distinguishing between "states" and "operators". Even when the operators are represented by numerical matrices, the squared form of operators distinguishes them from the column structure of the spinors states. Only for one-component fields and operators is there potential confusion. In extending Quantum Mechanics defined over the complex field to quaternions it has almost always been assumed that matrix operators contain elements which are "numbers" indistinguishable from those of the state vectors. *This is an unjustified limitation.* In fact, (non-commutative) hyper-complex theories require left/right operators.

This paper is organized as follows: In section 2, we introduce the quaternionic algebras. In section 3, we show that the non-commutative nature of the quaternionic field suggests the use of left/right operators. In section 4, we find the appropriate definitions of transpose, trace and determinant for quaternionic matrices. Such a section contains the *new* classification of quaternionic groups. In section 5, we present some applications of left/right operators in Physics. Our conclusions are drawn in the final section.

2 Quaternionic States

Complex numbers can be constructed from the real numbers by introducing a quantity i whose square is -1:

$$c = r_1 + ir_2 \qquad (r_{1,2} \in \mathbb{R}) \ .$$

Likewise, we can construct the quaternions from the complex numbers in exactly the same way by introducing another quantity j whose square is -1,

$$q = c_1 + jc_2 \qquad (c_{1,2} \in \mathbb{C}) \ ,$$

and which anti-commutes with i

$$ij = -ji = k \ .$$

In introducing the quaternionic algebra, let us follow the conceptual approach of Hamilton. In 1843, the Irish mathematician attempted to generalize the complex field in order to describe the rotations in the three-dimensional space. He began by looking for numbers of the form

$$x + iy + jz \ ,$$

with $i^2 = j^2 = -1$. Hamilton's hope was to do for three-dimensional space what complex numbers do for the plane. Influenced by the existence of a complex number norm

$$c^* c = (\mathrm{Re}\, c)^2 + (\mathrm{Im}\, c)^2 \ ,$$

when he looked at its generalization

$$(x - iy - jz)(x + iy + jz) = x^2 + y^2 + z^2 - (ij + ji)yz \ ,$$

to obtain a real number, he had to adopt the anti-commutative law of multiplication for the imaginary units. Nevertheless, with only two imaginary units we have no chance of constructing a new numerical field, because assuming

$$ij = \alpha + i\beta + j\gamma \qquad (\alpha, \beta, \gamma \in \mathbb{R}) \ ,$$

and using the multiplication associativity, $-i(ij) = -i^2 j = j$, we find

$$j = \beta - i\alpha - ij\gamma = i\alpha - \beta - (\alpha + i\beta + j\gamma)\, \gamma \ ,$$

which implies

$$\alpha = \beta = 0 \qquad \text{and} \qquad \gamma^2 = -1 \ .$$

Thus, we must introduce a third imaginary unit $k \neq i, j$, with

$$k = ij = -ji \ .$$

The $\mathbb{H}$-field is therefore characterized by three imaginary units i, j, k which satisfy the following multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1 \ . \tag{1}$$

Numbers of the form

$$q = x_0 + ix + jy + kz \qquad (\ x_0, x, y, z \in \mathbb{R}\ ) \ , \tag{2}$$

are called (real) *quaternions*. They are added, subtracted and multiplied according to the usual laws of arithmetic, except for the commutative law of multiplication.

Similarly to rotations in a plane that can be concisely expressed by complex number, a rotation about an axis passing through the origin and parallel to a given unitary vector $\hat{u} \equiv (u_x, u_y, u_z)$ by an angle α can be obtained taking the following *quaternionic* transformation

$$\exp\left(\tfrac{\alpha}{2}\, \vec{h} \cdot \vec{u}\right)\, \vec{h} \cdot \vec{r}\, \exp\left(-\tfrac{\alpha}{2}\, \vec{h} \cdot \vec{u}\right) \ ,$$

where

$$\vec{h} \equiv (i, j, k) \qquad \text{and} \qquad \vec{r} \equiv (x, y, z) \ .$$

In section 5, we shall see how the quaternionic number q in Eq. (2), with the identification $x_0 \equiv ct$, can be used to formulate a one-dimensional version of the Lorentz group [15]. We obtain the natural generalization of Hamilton's idea

complex/plane $\rightarrow$ pure imaginary quaternions/space $\rightarrow$ quaternions/space-time ,

completing the unification of algebra and geometry.

Let us now consider the (full) conjugate of q

$$q^\dagger = x_0 - ix - jy - kz \ . \tag{3}$$

We observe that $q^\dagger q$ and $qq^\dagger$ are both equal to the real number

$$N(q) = x_0^2 + x^2 + y^2 + z^2 \ ,$$

which is called the norm of q. When $q \neq 0$, we can define

$$q^{-1} = q^{\dagger}/N(q) \ ,$$

so quaternions form a zero-division ring.

An important difference between quaternionic and complex numbers is related to the definition of the conjugation operation. Whereas with complex numbers we can define only one type of conjugation

$$i \rightarrow -i \ ,$$

working with quaternionic numbers we can introduce different conjugation operations. Indeed, with three imaginary units we have the possibility to define besides the standard conjugation (3), the six new operations

$$\begin{array}{lcl} (i,j,k) & \rightarrow & (-i,+j,+k) \ , \ (+i,-j,+k) \ , \ (+i,+j,-k) \ ; \\ (i,j,k) & \rightarrow & (+i,-j,-k) \ , \ (-i,+j,-k) \ , \ (-i,-j,+k) \ . \end{array}$$

These last six conjugations can be concisely represented by q and $q^{\dagger}$ as follows

$$\begin{array}{lcl} q & \rightarrow & -iq^{\dagger}i \ , \ -jq^{\dagger}j \ , \ -kq^{\dagger}k \ , \\ q & \rightarrow & -iqi \ \ , \ -jqj \ \ , \ -kqk \ \ . \end{array}$$

It could seem that the only independent conjugation be represented by $q^{\dagger}$. Nevertheless, $q^{\dagger}$ can also be expressed in terms of q, in fact

$$q^{\dagger} = -\tfrac{1}{2}\,(q + iqi + jqj + kqk) \ . \qquad (4)$$

We conclude this section by introducing a compact notation to represent quaternionic states. Let

$$h^{\mu} \equiv \left(1\,,\,-\vec{h}\right) \qquad \text{and} \qquad h_{\mu} \equiv \left(1\,,\,\vec{h}\right) \qquad (5)$$

denote the $\mathbb{H}$-field generators and

$$x^{\mu} \equiv (\,x_0\,,\,\vec{x}\,) \qquad \text{and} \qquad x_{\mu} \equiv (\,x_0\,,\,-\vec{x}\,) \qquad (6)$$

be contravariant and covariant real quadrivectors,

$$g^{\mu\nu} = (+,-,-,-) \ .$$

Quaternionic states will be written in terms of (5,6) as follows

$$q = x^{\mu}h_{\mu} = x_{\mu}h^{\mu} \equiv x_0 + \vec{h}\cdot\vec{x} \ ,$$

and consequently the (full) quaternionic conjugate, $q^{\dagger}$ will read

$$q^{\dagger} = x^{\mu}h^{\mu} = x_{\mu}h_{\mu} \equiv x_0 - \vec{h}\cdot\vec{x} \ .$$

From now on, Greek letters will be run from 0 to 3.

3 Quaternionic Left/Right Operators

Due to the non-commutative nature of quaternions we must distinguish between $q\vec{h}$ and $\vec{h}q$. Thus, it is appropriate to consider left and right-actions for our imaginary units i, j and k. Let us define the operators

$$L_\mu \equiv \left(\mathbb{1}\,,\,\vec{L}\right) \in \mathbb{H}^L \;, \qquad \vec{L} = (L_i, L_j, L_k) \;, \tag{7}$$

and

$$R_\mu \equiv \left(\mathbb{1}\,,\,\vec{R}\right) \in \mathbb{H}^R \;, \qquad \vec{R} = (R_i, R_j, R_k) \;, \tag{8}$$

which act on quaternionic states in the following way

$$L_\mu : \; \mathbb{H} \to \mathbb{H} \;, \qquad L_\mu q = h_\mu q \in \mathbb{H} \;, \tag{9}$$

and

$$R_\mu : \; \mathbb{H} \to \mathbb{H} \;, \qquad R_\mu q = q\, h_\mu \in \mathbb{H} \;. \tag{10}$$

The algebra of left/right generators can be concisely expressed by

$$L_i^2 = L_j^2 = L_k^2 = L_i L_j L_k = R_i^2 = R_j^2 = R_k^2 = R_k R_j R_i = -\mathbb{1} \;,$$

and by the commutation relations

$$[\, L_{i,j,k} \,,\, R_{i,j,k} \,] = 0 \;.$$

In this section we will discuss three different types of operators. Operators $\mathbb{H}$-linear, $\mathbb{C}$-linear, $\mathbb{R}$-linear from the right. For simplicity of notation we introduce

$$\mathcal{O}_{\mathbb{X}} : \; \mathbb{H} \to \mathbb{H} \;,$$

to represent quaternionic operators right-linear on the $\mathbb{X}$-field. Operators which act only from the left,

$$\mathcal{O}_{\mathbb{H}} = a^\mu L_\mu \in \mathbb{H}^L \;, \qquad a^\mu \in \mathbb{R}^4 \;,$$

are obviously $\mathbb{H}$-linear from the right

$$L_\mu(q\lambda) = (L_\mu q)\,\lambda \;, \qquad \lambda \in \mathbb{H} \;,$$

and $\mathbb{R}$-linear from the left

$$L_\mu(\rho q) = \rho\,(L_\mu q) \;, \qquad \rho \in \mathbb{R} \;.$$

Let us now consider the sixteen generators

$$M_{\mu\nu} \equiv L_\mu \otimes R_\nu \;.$$

Due to left and right actions of the imaginary units i, j, k the corresponding operator

$$\mathcal{O}_{\mathbb{R}} = a^{\mu\nu} M_{\mu\nu} \equiv a^{\mu\nu} L_\mu \otimes R_\nu \in \mathbb{H}^L \otimes \mathbb{H}^R \;, \qquad a^{\mu\nu} \in \mathbb{R}^{16} \;,$$

are restricted to be $\mathbb{R}$-linear

$$M_{\mu\nu}(q\rho) = (M_{\mu\nu} q)\,\rho = \rho\,(M_{\mu\nu} q) \;, \qquad \rho \in \mathbb{R} \;.$$

Finally, considering the right action to the only i-complex imaginary unit

$$M_{\mu n} \equiv L_\mu \otimes R_n \ , \qquad {\scriptstyle n\,=\,1,\,2} \ ,$$

we obtain right $\mathbb{C}$-linear operators

$$\mathcal{O}_{\mathbb{C}} = a^{\mu n} M_{\mu n} \equiv a^{\mu n} L_\mu \otimes R_n \in \mathbb{H}^L \otimes \mathbb{C}^R \subset \mathbb{H}^L \otimes \mathbb{H}^R \ , \qquad a^{\mu n} \in \mathbb{R}^8 \ ,$$

in fact

$$M_{\mu n}(q\zeta) = (M_{\mu n} q)\, \zeta \ , \qquad \zeta \in \mathbb{C} \ .$$

The classification of right $\mathbb{H}/\mathbb{C}/\mathbb{R}$-linear can be summarized by

$$\mathcal{O}_{\mathbb{H}} \subset \mathcal{O}_{\mathbb{C}} \subset \mathcal{O}_{\mathbb{R}} \ ,$$

note that all these operators are obviously $\mathbb{R}$-linear from the left.

Let us now analyze the product of two right $\mathbb{R}$-linear operators

$$\mathcal{O}^a_{\mathbb{R}} = a^{\mu\nu} M_{\mu\nu} \ , \quad \mathcal{O}^b_{\mathbb{R}} = b^{\tau\sigma} M_{\tau\sigma} \ ,$$

in terms of left/right quaternionic generators is given by

$$\mathcal{O}^a_{\mathbb{R}}\, \mathcal{O}^b_{\mathbb{R}} = a^{\mu\nu} b^{\tau\sigma} L_\mu L_\tau \otimes R_\sigma R_\nu \ .$$

From such a relation we can immediately obtain the product of right $\mathbb{C}$-linear and $\mathbb{R}$-linear operators. The *full* conjugation operation for left/right operators is defined by a simultaneous change in the sign of left/right quaternionic imaginary units, i.e.

$$L^\dagger_\mu = L^\mu = g^{\mu\nu} L_\nu \qquad \text{and} \qquad R^\dagger_\mu = R^\mu = g^{\mu\nu} R_\nu \ .$$

Thus,

$$\mathcal{O}^\dagger_{\mathbb{R}} = a^{\mu\nu} L^\dagger_\mu \otimes R^\dagger_\nu = a^{\mu\nu} L^\mu \otimes R^\nu \quad \in \mathbb{H}^L \otimes \mathbb{H}^R \ . \tag{11}$$

For operators product conjugations we have

$$\begin{aligned} \left(\mathcal{O}^a_{\mathbb{R}}\, \mathcal{O}^b_{\mathbb{R}}\right)^\dagger &= a^{\mu\nu} b^{\tau\sigma} \left(L_\mu L_\tau \otimes R_\sigma R_\nu\right)^\dagger \\ &= a^{\mu\nu} b^{\tau\sigma} \left(L_\mu L_\tau\right)^\dagger \otimes \left(R_\sigma R_\nu\right)^\dagger \\ &= a^{\mu\nu} b^{\tau\sigma} L^\dagger_\tau L^\dagger_\mu \otimes R^\dagger_\nu R^\dagger_\sigma \\ &= \mathcal{O}^{b\,\dagger}_{\mathbb{R}}\, \mathcal{O}^{a\,\dagger}_{\mathbb{R}} \ . \end{aligned}$$

In section 4, dealing with quaternionic matrices we shall distinguish between (right) $\mathbb{R}$-linear quaternionic groups,

$$\mathsf{GL}\,(N, \mathbb{H}^L \otimes \mathbb{H}^R) \ ,$$

and $\mathbb{C}$-linear quaternionic groups,

$$\mathsf{GL}\,(N, \mathbb{H}^L \otimes \mathbb{C}^R) \ .$$

For a clear and complete discussion of standard quaternionic groups,

$$\mathsf{GL}\,(N, \mathbb{H}^L) \ ,$$

the reader is referred to Gilmore's book [20]. The use of left/right operators give new opportunities in Quaternionic Group Theory. Let us observe as follows: The so-called *symplectic* complex representation of a quaternion (state) q

$$q = c_1 + jc_2 \qquad (c_{1,2} \in \mathbb{C}) \ ,$$

by a complex column matrix, is

$$q \ \leftrightarrow \ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \ . \tag{12}$$

The operator representation of L_i, L_j and L_k consistent with the above identification

$$L_i \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \ , \qquad L_j \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \ , \qquad L_k \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \ , \tag{13}$$

has been known since the discovery of quaternions. It permits any quaternionic number or matrix to be translated into a complex matrix, *but not necessarily vice-versa.* Eight real numbers are required to define the most general 2×2 complex matrix but only four are needed to define the most general quaternion. In fact since every (non-zero) quaternion has an inverse, only a subclass of invertible 2×2 complex matrices are identifiable with quaternions. $\mathbb{C}$-linear quaternionic operators complete the translation [19]. The right quaternionic imaginary unit

$$R_i \ \leftrightarrow \ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \ ,$$

adds four additional degrees of freedom, obtained by matrix multiplication of the corresponding matrices,

$$R_i \ , \ L_iR_i \ , \ L_jR_i \ , \ L_kR_i \ ,$$

and so we have a set of rules for translating from any 2×2 complex matrices to $\mathbb{C}$-linear operators, $\mathcal{O}_{\mathbb{R}}$. This opens new possibilities for quaternionic numbers, see for example the one-dimensional version of the Glashow group [7]. Obviously, this translation does not apply to odd-dimensional complex matrices [21].

4 Quaternionic Groups

Every set of basis vectors in V_N a vector space, can be related to every other coordinate system by an $N \times N$ non singular matrix. The $N \times N$ matrix groups involved in changing bases in the vector spaces $\mathbb{R}_N$, $\mathbb{C}_N$ and $\mathbb{H}^L_N$ are called *general linear groups* of $N \times N$ matrices over the reals, complex and quaternions

$$\begin{array}{ccccc}
\mathsf{GL}\,(N,\mathbb{R}) & \rightarrow & \mathsf{GL}\,(N,\mathbb{C}) & \rightarrow & \mathsf{GL}\,(N,\mathbb{H}^L) \\
 & & & & \downarrow \\
 & & \mathsf{GL}\,(2N,\mathbb{C}) & \leftrightarrow & \mathsf{GL}\,(N,\mathbb{H}^L \otimes \mathbb{C}^R) \\
 & & & & \downarrow \\
\mathsf{GL}\,(4N,\mathbb{R}) & & \leftrightarrow & & \mathsf{GL}\,(N,\mathbb{H}^L \otimes \mathbb{H}^R) \ .
\end{array}$$

Before discussing the groups $\mathsf{GL}(N, \mathbb{H}^L \otimes \mathbb{H}^R)$ and $\mathsf{GL}(N, \mathbb{H}^L \otimes \mathbb{C}^R)$, we introduce a new definition of transpose for quaternionic matrices which will allow us to overcome the difficulties due to the non-commutative nature of the quaternionic field (our definition, applying to standard quaternions, will be extended to complex and real linear quaternions).

4.1 Quaternionic Transpose Definition

The customary convention of defining the transpose M^t of the matrix M is

$$(M^t)_{rs} = M_{sr} \ . \tag{14}$$

In general, however, for two quaternionic matrices M and N one has

$$(MN)^t \neq N^t M^t \ ,$$

whereas this statement hold as an equality for complex matrices. For example, the usual definition (14) implies

$$\left[\begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right]^t = (q_1 p_1 + q_2 p_2)^t = q_1 p_1 + q_2 p_2 \ , \tag{15}$$

and

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^t \begin{pmatrix} q_1 & q_2 \end{pmatrix}^t = p_1^t q_1^t + p_2^t q_2^t = p_1 q_1 + p_2 q_2 \ , \tag{16}$$

which are equal only if we use a commutative states. How can we define orthogonal quaternionic states and orthogonal groups? By looking at the previous example, we see that the problem arises in the different position of factors $q_{1,2}$ and $p_{1,2}$. The solution is very simple once seen. It is possible to give a quaternionic transpose which reverses the order of factors. We have three (equivalent) possibilities to define q^t, namely

$$x_0 - ix_1 + jx_2 + kx_3 \ , \quad x_0 + ix_1 - jx_2 + kx_3 \ , \quad x_0 + ix_1 + jx_2 - kx_3 \ . \tag{17}$$

We choose

$$q^t = x_0 + ix_1 - jx_2 + kx_3 \ , \tag{18}$$

which *goes back* to the usual definition for complex numbers, $c^t = c \in \mathbb{C}(1, i)$. In this way, the transpose of a product of two quaternions q and p is the product of the transpose quaternions in reverse order

$$(qp)^t = p^t q^t \ .$$

The proof is straightforward if we recognize the following relation between transpose q^t and conjugate $q^\dagger$,

$$q^t = -jq^\dagger j \ .$$

What happens for left/right quaternionic operators? Observing that for quaternionic states

$$i^t = i \ , \quad j^t = -j \ , \quad k^t = k \ ,$$

the natural generalization for left/right quaternionic operators is

$$L_i^t = L_i \ , \quad L_j^t = -L_j \ , \quad L_k^t = L_k \quad \Rightarrow \quad L_\mu^t = -L_j L_\mu^\dagger L_j \ ,$$

and

$$R_i^t = R_i \ , \quad R_j^t = -R_j \ , \quad R_k^t = R_k \quad \Rightarrow \quad R_\mu^t = -R_j R_\mu^\dagger R_j \ .$$

One-dimensional $\mathbb{R}$-linear transpose operators read

$$\begin{aligned} \mathcal{O}_{\mathbb{R}}^t &= a^{\mu\nu} L_\mu^t \otimes R_\nu^t \\ &= L_j R_j \mathcal{O}_{\mathbb{R}}^\dagger R_j L_j \quad \in \mathbb{H}^L \otimes \mathbb{H}^R \ . \end{aligned} \tag{19}$$

Thus, in the quaternionic world the transpose M^t of the matrix $M \in \mathsf{GL}\,(N, \mathbb{H}^L \otimes \mathbb{H}^R)$, defined by

$$(M^t)_{rs} = {M_{sr}}^t \quad \in \mathbb{H}^L \otimes \mathbb{H}^R \ ,$$

can be written as

$$M^t = L_j R_j M^\dagger R_j L_j = R_j L_j M^\dagger L_j R_j \ , \tag{20}$$

where

$$(M^\dagger)_{rs} = {M_{sr}}^\dagger \quad \in \mathbb{H}^L \otimes \mathbb{H}^R \ .$$

With this new definition of quaternionic transpose, the relation

$$\begin{aligned} (MN)^t &= L_j R_j (MN)^\dagger R_j L_j = L_j R_j N^\dagger M^\dagger R_j L_j \\ &= L_j R_j N^\dagger R_j L_j L_j R_j M^\dagger R_j L_j \\ &= N^t M^t \end{aligned}$$

also holds for non-commutative numbers. Finally, for $\mathbb{C}$-linear operators, Eq. (19) reduces to

$$\begin{aligned} \mathcal{O}_{\mathbb{C}}^t &= a^{\mu n} L_\mu^t \otimes R_n \\ &= -L_j \mathcal{O}_{\mathbb{C}}^\dagger L_j \ . \end{aligned} \tag{21}$$

The fundamental property of reverse ring the order of factors for the transpose of quaternionic products is again preserved.

4.2 $\mathbb{X}$-Mappings

In discussing the classification of the classical (matrix) groups, it is necessary to introduce one additional concept: the *metric*. Our matrix element are $\mathcal{O}_{\mathbb{X}}$-operators, and so it is appropriate to adopting the metric function, $\mathcal{M}_{\mathbb{X}}$, mapping of a pair of vectors into a number field $\mathbb{X}$

$$\mathcal{M}_{\mathbb{X}} : \mathbb{H}_N \times \mathbb{H}_N \to \mathbb{X} \ , \qquad \mathcal{M}_{\mathbb{X}}(\Psi, \Phi) = (\Psi, \Phi)_{\mathbb{X}} \ ,$$

with $\Psi, \Phi \in \mathbb{H}_N$ and $(\Psi, \Phi)_{\mathbb{X}} \in \mathbb{X}$. Let us now recall the following theorem: *The subset of transformations of basis in V_N which preserves the mathematical structure of a metric forms a subgroup of general linear groups.*

	bilinear symmetric		*orthogonal*
Groups preserving	bilinear antisymmetric	metrics are called	*symplectic*
	sesquilinear symmetric		*unitary* .

The previous theorem is valid for all real and complex metric-preserving matrix groups. It is also valid for quaternionic groups that preserve sesquilinear metrics, since two quaternions obey $(q_1 q_2)^\dagger = q_2^\dagger q_1^\dagger$. It is not true for quaternionic matrices and bilinear metrics, since two quaternions do not generally commute. Nevertheless, it is still possible to associate subgroups of $\mathsf{GL}\,(N,\mathbb{H}^L)$ with groups that preserve bilinear metrics. In the literature this is done in the following way. "*Each quaternion in* $\mathsf{GL}\,(N,\mathbb{H}^L)$ *is replaced by the corresponding* 2×2 *complex matrix using the translation rules (13). The subset of matrices in this complex* $2N\times 2N$ *matrix representation of* $\mathsf{GL}\,(N,\mathbb{H}^L)$ *that leaves invariant a bilinear metric forms a group, since the theorem is valid for bilinear metrics on complex linear vector spaces. We can associate an* $N\times N$ *quaternion-valued matrix with each* $2N\times 2N$ *complex-valued matrix in the resulting groups that preserve bilinear metrics in the space* $\mathbb{C}_{2N\times 2N}$, *which is a representation for the space* $\mathbb{H}^L_{N\times N}$" - *Gilmore* [20].

Once we write our complex matrix, we can trivially obtain the generators of complex orthogonal groups in a standard manner and then we can translate back into quaternionic language. But this is *surely a laborious procedure*. Defining an appropriate transpose for quaternionic numbers (18), we can overcome the just-cited difficulty. Besides, using the symplectic representation (12), the most general transformation (on quaternionic states) will be necessarily represented by $\mathbb{C}$-linear quaternionic operators, $\mathcal{O}_{\mathbb{C}}$, and for the invariant metric we have to require a "complex" projection, $\mathcal{M}_{\mathbb{C}} : \mathbb{H}\times\mathbb{H}\to\mathbb{C}$,

$$\mathcal{M}_{\mathbb{C}}(q^t q) = (q^t q)_{\mathbb{C}} = [(c_1 - jc_2^*)(c_1 + jc_2)]_{\mathbb{C}} = c_1^2 + c_2^2 \ .$$

We wish to emphasize that the introduction of the imaginary unit R_i in complex linear quaternionic operators

$$(R_i)^\dagger = -R_i \ ,$$

necessarily implies a complex inner product. The "new" imaginary units R_i represents an anti-hermitian operator, and so it must verify

$$\begin{array}{ccc} \int\ (R_i\psi)^\dagger\varphi & = & -\int\ \psi^\dagger R_i\varphi \\ \downarrow & & \downarrow \\ \int\ (\psi i)^\dagger\varphi & = & -\int\ \psi^\dagger\varphi i \ . \end{array}$$

The previous relation is true only if we adopt a *complex projection*

$$\int_{\mathbb{C}} \equiv \frac{1 - L_i R_i}{2} \int \ ,$$

for the inner products

$$\int_{\mathbb{C}} (R_i\psi)^\dagger\varphi = -i\int_{\mathbb{C}} \psi^\dagger\varphi = -\int_{\mathbb{C}} \psi^\dagger\varphi i = -\int\ \psi^\dagger R_i\varphi \ .$$

Obviously, for $\mathbb{R}$-linear operators

$$(\vec{R})^\dagger = -\vec{R} \ ,$$

implies real inner products,

$$\mathcal{M}_{\mathbb{R}} : \mathbb{H}\times\mathbb{H}\to\mathbb{R} \ .$$

4.3 One-Dimensional Quaternionic Groups

The generators of the unitary and orthogonal groups satisfy the following constraints

Groups :	Generators :
Unitary	$A + A^\dagger = 0$,
Orthogonal	$A + A^t = 0$.

For one-dimensional quaternionic groups, we find

Groups	Generators
$\mathsf{U}(1,\mathbb{H}^L)$	$\vec{L}$
$\mathsf{U}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\vec{L}\,,\;R_i$
$\mathsf{U}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$	$\vec{L}\,,\;\vec{R}$
$\mathsf{O}(1,\mathbb{H}^L)$	L_j
$\mathsf{O}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$	$L_j\,,\;L_jR_i$
$\mathsf{O}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$	$L_j\,,\;L_jR_i\,,\;L_jR_k\,,\;R_j\,,\;L_iR_j\,,\;L_kR_j$

At this point, we make a number of observations:

1. - The difference between orthogonal and unitary groups is manifest for complex linear quaternionic groups because of the different numbers of generators.

2. - Orthogonal and unitary real linear quaternionic groups have the same number of generators.

3. - The real groups $\mathsf{U}(N_+,N_-)$ and $\mathsf{O}(N_+,N_-)$ are identical (there is no difference between bilinear and sesquilinear metrics in a real vector space) and this suggest a possible link between $\mathsf{U}(N,\mathbb{H}^L\otimes\mathbb{H}^R)$ and $\mathsf{O}(N,\mathbb{H}^L\otimes\mathbb{H}^R)$.

4. - For real linear quaternionic groups, the invariant metric requires a "real" projection (note that $\vec{R}$ represent anti-hermitian operators only for real inner products). Let us show the "real" invariant metric for $\mathsf{U}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$ and $\mathsf{O}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$,

$$
\begin{aligned}
(x^\dagger y)_{\mathbb{R}} &= [(x_0-ix_1-jx_2-kx_3)(y_0+iy_1+jy_2+ky_3)]_{\mathbb{R}} = x_0y_0+x_1y_1+x_2y_2+x_3y_3\ ,\\
(x^t y)_{\mathbb{R}} &= [(x_0+ix_1-jx_2+kx_3)(y_0+iy_1+jy_2+ky_3)]_{\mathbb{R}} = x_0y_0-x_1y_1+x_2y_2-x_3y_3\ .
\end{aligned}
$$

We can immediately recognize the invariant metric of $\mathsf{O}(4)$ and $\mathsf{O}(2,2)$. To complete the analogy between one-dimensional real linear quaternionic operators and 4-dimensional real matrices, we observe that besides the subgroups related to the $\dagger$-conjugation (where the sign of all three imaginary units is changed) and the t-conjugation (where only $j\to -j$), we can define a new subgroup which leaves invariant the following real metric

$$(x^\dagger g y)_{\mathbb{R}}\ ,$$

where

$$g = -\tfrac{1}{2}\,L_\mu R_\mu = -\tfrac{1}{2}\left(\mathbb{1}+\vec{L}\cdot\vec{R}\right)\ .$$

Explicitly,

$$(x^\dagger g y)_{\mathbb{R}} = [(x_0-ix_1-jx_2-kx_3)(y_0-iy_1-jy_2-ky_3)]_{\mathbb{R}} = x_0y_0-x_1y_1-x_2y_2-x_3y_3\ .$$

The new subgroup,

$$\tilde{\mathsf{O}}(1,\mathbb{H}^L\otimes\mathbb{H}^R)\ ,\qquad gA+A^\dagger g=0\ ,$$

represents the one-dimensional quaternionic counterpart of the Lorentz group $\mathsf{O}(1,3)$ [15].

The classical groups which occupy a central place in group representation theory and have many applications in various branches of Mathematics and Physics are the unitary, special unitary, orthogonal, and symplectic groups. In order to define special groups, we must define an appropriate trace for our matrices. In fact, for non-commutative numbers the trace of the product of two numbers is not the trace of the product with reversed factors. With complex linear quaternions we have the possibility to give a new definition of "complex" trace (Tr) by

$$Tr\ \mathcal{O}^a_{\mathbb{C}} = Tr\ a^{\mu n} L_\mu \otimes R_n = a^{00} + i a^{01}\ . \tag{22}$$

Such a definition implies that for any two complex linear quaternionic operators $\mathcal{O}^a_{\mathbb{C}}$ and $\mathcal{O}^b_{\mathbb{C}}$

$$Tr\ (\mathcal{O}^a_{\mathbb{C}} \mathcal{O}^b_{\mathbb{C}}) = Tr\ (\mathcal{O}^b_{\mathbb{C}} \mathcal{O}^a_{\mathbb{C}})\ .$$

For real linear quaternions we need to use the standard definition of "real" trace (tr)

$$tr\ \mathcal{O}^a_{\mathbb{R}} = a^{00}\ , \tag{23}$$

since the previous "complex" definition (22) gives

$$Tr\ (\mathcal{O}^a_{\mathbb{R}} \mathcal{O}^b_{\mathbb{R}}) \neq Tr\ (\mathcal{O}^b_{\mathbb{R}} \mathcal{O}^a_{\mathbb{R}})\ .$$

For example, for

$$\mathcal{O}^a_{\mathbb{R}} = R_j \quad \text{and} \quad \mathcal{O}^b_{\mathbb{R}} = R_k\ ,$$

we find

$$\begin{aligned} Tr\ (\mathcal{O}^a_{\mathbb{R}} \mathcal{O}^b_{\mathbb{R}}) &= Tr\ (R_j R_k) = -Tr\ (R_i) = -1\ , \\ Tr\ (\mathcal{O}^b_{\mathbb{R}} \mathcal{O}^a_{\mathbb{R}}) &= Tr\ (R_k R_j) = +Tr\ (R_i) = +1\ . \end{aligned}$$

4.4 N-Dimensional Quaternionic Groups

We recall that the generators of the unitary, special unitary, orthogonal groups must satisfy the following conditions [22]

$$\begin{array}{lll} \mathsf{U}(N) & A + A^\dagger = 0\ , & \\ \mathsf{SU}(N) & A + A^\dagger = 0\ , & Tr\ A = 0\ , \\ \mathsf{O}(N) & A + A^t = 0\ . & \end{array}$$

These conditions also apply for quaternionic groups. For complex symplectic groups we find

$$\mathsf{Sp}(2N) \qquad \mathcal{J} A + A^t \mathcal{J} = 0\ ,$$

where

$$\mathcal{J} = \begin{pmatrix} \mathbf{0}_{N\times N} & \mathbf{1}_{N\times N} \\ -\mathbf{1}_{N\times N} & \mathbf{0}_{N\times N} \end{pmatrix}\ .$$

Working with quaternionic numbers, we can construct a group preserving a non-singular antisymmetric metric, for N odd as well as N even. Thus for quaternionic symplectic groups we have

$$\mathsf{Sp}(N) \qquad \mathcal{J} A + A^t \mathcal{J} = 0\ ,$$

with

$$\mathcal{J}_{2N\times 2N} = \begin{pmatrix} \mathbf{0}_{N\times N} & \mathbf{1}_{N\times N} \\ -\mathbf{1}_{N\times N} & \mathbf{0}_{N\times N} \end{pmatrix}\ , \qquad \mathcal{J}_{(2N+1)\times(2N+1)} = \begin{pmatrix} \mathbf{0}_{N\times N} & \mathbf{0}_{N\times 1} & \mathbf{1}_{N\times N} \\ \mathbf{0}_{1\times N} & L_j & \mathbf{0}_{1\times N} \\ -\mathbf{1}_{N\times N} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times N} \end{pmatrix}\ .$$

The generators of one-dimensional groups with complex and real linear quaternions are given in the following table

One-dimensional quaternionic groups

Groups	Generators		
$\mathsf{U}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\mathcal{A}_{\mathbb{C}}+\mathcal{A}_{\mathbb{C}}{}^{\dagger}=0$	:	$\vec{L}\,,\;R_i$
$\mathsf{SU}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\mathcal{A}_{\mathbb{C}}+\mathcal{A}_{\mathbb{C}}{}^{\dagger}=0,\;\;Tr\;\mathcal{A}_{\mathbb{C}}=0$	:	$\vec{L}$
$\mathsf{O}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\mathcal{A}_{\mathbb{C}}+\mathcal{A}_{\mathbb{C}}{}^{t}=0$	:	$L_j\,,\;L_jR_i$
$\mathsf{Sp}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$	$L_j\mathcal{A}_{\mathbb{C}}+\mathcal{A}_{\mathbb{C}}{}^{t}L_j=0$	:	$\vec{L}\,,\;\vec{L}R_i$
$\mathsf{U}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$	$\mathcal{A}_{\mathbb{R}}+\mathcal{A}_{\mathbb{R}}{}^{\dagger}=0$	:	$\vec{L}\,,\;\vec{R}$
$\mathsf{O}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$	$\mathcal{A}_{\mathbb{R}}+\mathcal{A}_{\mathbb{R}}{}^{t}=0$	:	$L_j\,,\;L_jR_i\,,\;L_jR_k\,,\;R_j\,,\;L_iR_j\,,\;L_kR_j$
$\tilde{\mathsf{O}}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$	$g\mathcal{A}_{\mathbb{R}}+\mathcal{A}_{\mathbb{R}}{}^{\dagger}g=0$	:	$\vec{L}-\vec{R}\,,\;\vec{L}\times\vec{R}$
$\mathsf{Sp}(1,\mathbb{H}^L\otimes\mathbb{H}^R)$	$L_j\mathcal{A}_{\mathbb{R}}+\mathcal{A}_{\mathbb{R}}{}^{t}L_j=0$	:	$\vec{L}\,,\;\vec{L}R_i\,,\;\vec{L}R_k\,,\;R_j$

We conclude our classification of quaternionic groups giving the general formulas for counting the generators of generic N-dimensional groups as function of N.

N-dimensional quaternionic groups

$\mathsf{U}(N,\mathbb{H}^L)$	$\leftrightarrow$	$\mathsf{USp}(2N,\mathbb{C})$	$N(2N+1)$
$\mathsf{U}(N,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\leftrightarrow$	$\mathsf{U}(2N,\mathbb{C})$	$4N^2$
$\mathsf{U}(N,\mathbb{H}_L\otimes\mathbb{H}^R)$	$\leftrightarrow$	$\mathsf{O}(4N)$	$2N(4N-1)$
$\mathsf{SU}(N,\mathbb{H}^L)$	$\equiv$	$\mathsf{U}(N,\mathbb{H}^L)$	
$\mathsf{SU}(N,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\leftrightarrow$	$\mathsf{SU}(2N,\mathbb{C})$	$4N^2-1$
$\mathsf{SU}(N,\mathbb{H}^L\otimes\mathbb{H}^R)$	$\equiv$	$\mathsf{U}(N,\mathbb{H}^L\otimes\mathbb{H}^R)$	
$\mathsf{O}(N,\mathbb{H}^L)$	$\leftrightarrow$	$\mathsf{SO}^*(2N,\mathbb{C})$	$N(2N-1)$
$\mathsf{O}(N,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\leftrightarrow$	$\mathsf{O}(2N,\mathbb{C})$	$2N(2N-1)$
$\mathsf{O}(N,\mathbb{H}^L\otimes\mathbb{H}^R)$	$\leftrightarrow$	$\mathsf{O}(2N_+,2N_-)$	$2N(4N-1)$
$\tilde{\mathsf{O}}(N,\mathbb{H}^L\otimes\mathbb{H}^R)$	$\leftrightarrow$	$\mathsf{O}(3N_+,N_-)$	$2N(4N-1)$
$\mathsf{Sp}(N,\mathbb{H}^L)$	$\leftrightarrow$	$\mathsf{USp}(2N,\mathbb{C})$	$N(2N+1)$
$\mathsf{Sp}(N,\mathbb{H}^L\otimes\mathbb{C}^R)$	$\leftrightarrow$	$\mathsf{Sp}(2N,\mathbb{C})$	$2N(2N+1)$
$\mathsf{Sp}(N,\mathbb{H}^L\otimes\mathbb{H}^R)$	$\leftrightarrow$	$\mathsf{Sp}(4N,\mathbb{R})$	$2N(4N+1)$

5 Physical Applications

In the last years the left/right action of the quaternionic numbers, expressed by left/right operators, $\mathcal{O}_{\mathbb{R}}$, $\mathcal{O}_{\mathbb{C}}$, $\mathcal{O}_{\mathbb{H}}$, has been very useful in overcoming difficulties owing to the non-commutativity of quaternions. Among the successful applications of left/right operators we mention the one-dimensional quaternionic formulation of Lorentz boosts.

5.1 Special Relativity

The Lorentz group, $\mathsf{O}(3,1)$, is characterized by six parameters, three for rotations and three for boosts. Corresponding to these six parameters there are six generators. The anti-hermitian generators associated to spatial rotations and the hermitian boost generators satisfy the following commutation relations

$$\begin{aligned}
&\mathcal{A}_x=[\mathcal{A}_y,\mathcal{A}_z]\;, &&\mathcal{A}_x=[\mathcal{B}_z,\mathcal{B}_y]\;, &&\mathcal{B}_x=[\mathcal{A}_y,\mathcal{B}_z]=[\mathcal{B}_y,\mathcal{A}_z]\;,\\
&\mathcal{A}_y=[\mathcal{A}_z,\mathcal{A}_x]\;, &&\mathcal{A}_y=[\mathcal{B}_x,\mathcal{B}_z]\;, &&\mathcal{B}_y=[\mathcal{A}_z,\mathcal{B}_x]=[\mathcal{B}_z,\mathcal{A}_x]\;,\\
&\mathcal{A}_z=[\mathcal{A}_x,\mathcal{A}_y]\;, &&\mathcal{A}_z=[\mathcal{B}_y,\mathcal{B}_x]\;, &&\mathcal{B}_z=[\mathcal{A}_x,\mathcal{B}_y]=[\mathcal{B}_x,\mathcal{A}_y]\;,
\end{aligned}\tag{24}$$

The idea of combining left and right imaginary units gives the possibility to obtain a one-dimensional quaternionic representation for boost generators by $\mathbb{R}$-linear quaternionic operators [15]

$$\text{Boost generators:} \qquad \tfrac{1}{2}\,\vec{L}\times\vec{R}\ ,$$

$$\text{Rotation generators:} \quad \tfrac{1}{2}\,\left(\vec{L}-\vec{R}\right)\ .$$

The four real quantities which identify the space-time point (ct, x, y, z) are represented by the quaternion

$$q = ct + ix + jy + kz\ .$$

The one-dimensional group $\tilde{\mathsf{O}}(1, \mathbb{H}^L \otimes \mathbb{H}^R)$ represents the quaternionic counterpart of the four-dimensional Lorentz group $\mathsf{O}(3,1)$. Infinitesimal rotations about the x-axis and boosts (ct, x) are respectively given by

$$\begin{aligned}
\tilde{q}_r &= \left[1 + \tfrac{\theta_x}{2}\,(L_i - R_i)\right]\, q \\
&= ct + ix + j(y - \theta_x z) + k(z + \theta_x y)\ , \qquad (25)\\
\tilde{q}_b &= \left[1 + \tfrac{\varphi_x}{2}\,(L_j R_k - L_k R_j)\right]\, q \\
&= ct + \varphi_x x + i(x + \varphi_x ct) + jy + kz\ . \qquad (26)
\end{aligned}$$

In analogy to the connection between the rotation group $\mathsf{O}(3)$ and the special unitary group $\mathsf{SU}(2)$, there is a natural correspondence between the Lorentz group $\mathsf{O}(3,1)$ and the special linear group $\mathsf{SL}(2,\mathbb{C})$. The use of left/right quaternionic imaginary units gives the possibility to extend such connections to "quaternionic" group,

$$\begin{array}{lclcl}
\mathsf{O}(3) & \sim & \mathsf{SU}(2) & \leftrightarrow & \mathsf{U}(1, \mathbb{H}^L)\ , \\
\mathsf{O}(3,1) & \sim & \mathsf{SL}(2,\mathbb{C}) & \leftrightarrow & \mathsf{SL}(1, \mathbb{H}^L \otimes \mathbb{C}^R)\ .
\end{array}$$

In fact, combining left quaternionic imaginary units, $\vec{L}$, with right "complex" imaginary unit, R_i, we obtain the following one-dimensional representation for rotation and boost generators

$$\mathcal{A}_x = L_i/2\ , \quad \mathcal{A}_y = L_j/2\ , \quad \mathcal{A}_z = L_k/2 \qquad \in \mathbb{H}^L\ , \qquad (27)$$

and

$$\mathcal{B}_x = L_i R_i/2\ , \quad \mathcal{B}_y = L_j R_i/2\ , \quad \mathcal{B}_z = L_k R_i/2 \qquad \in \mathbb{H}^L \otimes \mathbb{C}^R\ . \qquad (28)$$

Now, the four real quantities which identify the space-time point are represented by the symplectic decomposition of quaternionic spinor states

$$q = \xi + j\eta\ , \qquad \xi, \eta \in \mathbb{C}(1, i)\ , \qquad (29)$$

where

$$q(1+i)q^{\dagger} \equiv ct + ix + jy + kz\ .$$

With this identifications, $\mathsf{O}(3,1)$-transformations on

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}\ ,$$

are equivalent to one-dimensional transformations on quaternionic spinors (29). A detailed discussion is found in ref. [23].

New possibilities, coming out from the use of $\mathbb{C}$-linear quaternionic operators, also appear in Quantum Mechanics and Field Theory, e.g. they allow an appropriate definition of the momentum operator [12], quaternionic version of standard relativistic equations [12, 13], Lagrangian formalism [25], electroweak model [7] and grand unification theories [8].

5.2 Dirac Equation

In the complex world, the Dirac equation read indifferently as

$$i\partial_t\psi = H\psi \quad \text{or} \quad \partial_t\psi i = H\psi \ .$$

In the quaternionic world there is a clear difference in choosing a left or right position for our complex imaginary unit i. In fact, by requiring norm conservation

$$\partial_t \int d^3x\, \psi^\dagger \psi = 0 \ ,$$

we find that a left position of the imaginary unit i in the quaternionic Dirac equation,

$$L_i\partial_t\psi \equiv i\partial_t\psi = H\psi \ ,$$

gives

$$\partial_t \int d^3x\, \psi^\dagger \psi = \int d^3x\, \psi^\dagger [H, i]\psi \ ,$$

in general $\neq 0$ for quaternionic Hamiltonians. A right position of the imaginary unit i,

$$R_i\partial_t\psi \equiv \partial_t\psi i = H\psi \ ,$$

ensures the norm conservation. From covariance, by treating time and space in the same way, we obtain the following "quaternionic" momentum operator

$$p^\mu \leftrightarrow R_i\partial^\mu \quad \Rightarrow \quad p^\mu\psi \leftrightarrow R_i\partial^\mu\psi \equiv \partial^\mu\psi i \ . \tag{30}$$

Finally, the quaternionic Dirac equation reads

$$R_i\gamma^\mu\partial_\mu\psi \equiv \gamma^\mu\partial_\mu\psi i = m\psi \ , \quad [\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu} \ , \tag{31}$$

with

$$\gamma_\mu \in \mathcal{M}_2(\mathbb{H}^L \otimes \mathbb{C}^R) \quad \text{and} \quad \psi \in \mathbb{H}_2 \ .$$

Another fundamental ingredient in the formulation of quaternionic relativistic quantum mechanics is represented by the adoption of a *complex geometry* [24], necessary in order to guarantee that $R_i\vec{\partial}$ be an hermitian operator

$$\int d^3x\, \varphi^\dagger R_i\vec{\partial}\psi = \int d^3x\, (R_i\vec{\partial}\varphi)^\dagger\psi \ .$$

The previous relation implies

$$\int d^3x\, \varphi^\dagger \vec{\partial}\psi i = -i \int d^3x\, \vec{\partial}\varphi^\dagger\psi \ \text{(after integration by parts)} \ = i \int d^3x\, \varphi^\dagger\vec{\partial}\psi \ .$$

The different position of the imaginary unit i forces the use of a *complex projection* [24] for inner products

$$\int d^3x \ \rightarrow \ \int_{\mathbb{C}} d^3x \ . \tag{32}$$

5.3 Lagrangian Formalism

The use of the variational principle within quaternionic quantum mechanics is non-trivial because of the non commutative nature of quaternions. In this subsection, we write the Dirac Lagrangian density corresponding to the two-component Dirac equation. This Lagrangian is *complex projected* as anticipated in previous articles [25]. The traditional form for the Dirac Lagrangian density is

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \; . \tag{33}$$

The position of the imaginary unit i is purely conventional in (33) but with a quaternionic number field we must recognize that the ∂_{μ} operator is more precisely part of the first quantized momentum operator $R_i\partial_{\mu}$ and that hence to ensure $\mathcal{L}$ be an hermitian quantity we must taken a complex projection of the kinetic term

$$\begin{aligned}\mathcal{L}_{kin} &= \left(\bar{\psi}\gamma^{\mu}R_i\partial_{\mu}\psi\right)_{\mathbb{C}} \\ &\equiv \left(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi i\right)_{\mathbb{C}} \\ &= \left(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\right)_{\mathbb{C}} i \; .\end{aligned}$$

The requirement of hermiticity however says nothing about the Dirac mass term in eq. (33). It is here that appeal to the variational principle must be made. A variation $\delta\psi$ in ψ cannot brought to the extreme right because of the imaginary unit i. The only consistent procedure is to generalize the variational rule that says that ψ and $\bar{\psi}$ must be varied *independently*. We thus apply independent variations to ψ ($\delta\psi$) and ψi ($\delta(\psi i)$). Similarly for $\delta\bar{\psi}$ and $\delta(i\bar{\psi})$. Now, to obtain the desired Dirac equation for ψ and its adjoint equation for $\bar{\psi}$, we must modify the mass term into

$$\mathcal{L}_{mass} = -m\left(\bar{\psi}\psi\right)_{\mathbb{C}} \; .$$

The final result for $\mathcal{L}$ is

$$\mathcal{L} = \left(\bar{\psi}\gamma^{\mu}R_i\partial_{\mu}\psi - m\bar{\psi}\psi\right)_{\mathbb{C}} \; . \tag{34}$$

5.4 Electroweak Models

Let us now eaxamine the fermion/quark sector of Salam-Weinberg model [26]. The first family is represented by

$$\begin{pmatrix}\nu \\ e\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}u \\ d\end{pmatrix} \; .$$

In the standard representation [7, 12]

$$\Psi_L = \begin{pmatrix}\nu_L & u_L \\ e_L & d_L\end{pmatrix} \; , \quad \Psi_R = \begin{pmatrix}\nu_R & u_R \\ e_R & d_R\end{pmatrix} \qquad \in \mathcal{M}_4(\mathbb{H}) \; . \tag{35}$$

The massless fermion electroweak Lagrangian

$$\mathcal{L}_{fermion} = \left(\bar{\Psi}_L\gamma^{\mu}R_i\partial_{\mu}\Psi_L + \bar{\Psi}_R\gamma^{\mu}R_i\partial_{\mu}\Psi_R\right)_{\mathbb{C}} \; , \tag{36}$$

where

$$\gamma_{\mu} \in \mathcal{M}_4(\mathbb{C}^R) \; ,$$

is global invariant under the quaternionic Glashow gauge group [27]

$$\mathsf{U}(1, \mathbb{H}^L)_L \otimes \mathsf{U}(1, \mathbb{C}^R)_Y \; . \tag{37}$$

In the chiral representation [23] $\nu_{L/R}$, $e_{L/R}$, $u_{L/R}$ and $d_{L/R}$ are one-dimensional quaternionic spinors and so can be accommodate in

$$\Psi_L = \begin{pmatrix} \nu_L & u_L \\ e_L & d_L \end{pmatrix} \quad , \quad \Psi_R = \begin{pmatrix} \nu_R & u_R \\ e_R & d_R \end{pmatrix} \quad \in \mathcal{M}_2(\mathbb{H}) \ . \tag{38}$$

The Lagrangian for the masseless fermion sector

$$\mathcal{L}_{fermion} = \left(\bar{\Psi}_L \gamma^\mu R_i \partial_\mu \Psi_L + \bar{\Psi}_R \gamma^\mu R_i \partial_\mu \Psi_R\right)_{\mathbb{C}} \ , \tag{39}$$

is now global invariant under the "right complex" gauge group

$$\mathsf{SU}(2, \mathbb{C}^R)_L \otimes \mathsf{U}(1, \mathbb{C}^R)_Y \ . \tag{40}$$

6 Conclusions

The more exciting possibility that quaternionic or octonionic equations will eventually play a significant role in Mathematics and Physics is synonymous, for some physicist, with the advent of a revolution in Physics comparable to that of Quantum Mechanics.

For example, Adler suggested [28] that the color degree of freedom postulated in the Harari-Shupe model [29, 30] (where we can think of quarks and leptons as composites of other more fundamental fermions, preons) could be sought in a non-commutative extension of the complex field. Surely a stimulating idea. Nevertheless, we think that it would be very strange if standard Quantum Mechanics did not permit a quaternionic or octonionic description other than in the trivial sense that complex numbers are contained within the quaternions or octonions.

In the last few years much progress has been achieved in manipulating such fields. We quote the quaternionic version of electroweak theory [7], where the Glashow group is expressed by the one-dimensional quaternionic group $\mathsf{U}(1, \mathbb{H}^L) \otimes \mathsf{U}(1, \mathbb{C}^R)$, quaternionic GUTs [8] and Special Relativity, where the Lorentz group is represented by $\hat{\mathsf{O}}(1, \mathbb{H}^L \otimes \mathbb{H}^R)$. We also recall new possibilities related to the use of octonions in Quantum Mechanics [31], in particular in writing a one-dimensional octonionic Dirac equation. The link between octonionic and quaternionic versions of standard Quantum Physics is represented by the use of a complex geometry [24].

In this paper we observed that beyond the study of matrix groups with "simple" quaternionic elements, $\mathcal{O}_{\mathbb{H}}$, one can consider more general groups with matrix elements of the form $\mathcal{O}_{\mathbb{R}}$ and $\mathcal{O}_{\mathbb{C}}$. To the best of our knowledge these more general matrix groups have not been studied in the literature. We overcome the problems arising in the definitions of transpose, determinant and trace for quaternionic matrices. For octonionic fields [32] we must admit a more complicated situation, yet our discussion can be also proposed for non-associative numbers.

Finally, we hope that this paper emphasizes the possibility of using hyper-complex numbers in Mathematics and Physics and could represent an important step towards a complete and clear discussion on Hyper-complex Group and Field Theories.

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Appendix A

We give the translation rules between quaternionic left/right $\mathbb{R}$-linear operators and 4×4 real matrices:

$$L_i \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; , \quad L_j \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \; , \quad L_k = L_i L_j \; ,$$

$$R_i \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \; , \quad R_j \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \; , \quad R_k = R_j R_i \; .$$

From these identifications we can obtain the *full* translation. For example, the matrix counterpart of the operator $L_j R_k$ is soon achieved by

$$L_j R_k \quad \leftrightarrow \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right] =$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \; .$$

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