# Quaternionic Lorentz Group
# and
# Chiral Representation


Stefano De Leo

*Departamento de Matemática Aplicada, UNICAMP*
CP 6065, 13081-970 Campinas (SP) Brasil
deleo@ime.unicamp.br
*Dipartimento di Fisica, Universitá degli Studi di Lecce*
via Arnesano, CP 193, 73100 Lecce, Italia
deleos@le.infn.it


December, 1998


**Abstract**

Analogous to the correspondence between unitary quaternions, $\mathsf{U}(1,\mathbb{H}^L)$, and rotation group, $\mathsf{O}(3)$, there is a correspondence between special linear quaternions, $\mathsf{SL}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$, and Lorentz group, $\mathsf{O}(3,1)$. The transformation laws of quaternionic one-dimensional spinors allow a quaternionic derivation of Dirac equation. The quaternionic chiral representation for gamma-matrices plays a fundamental role in quaternionic electroweak models.


## 1 Introduction

We discuss a one-dimensional quaternionic version of the complex group $\mathsf{SL}(2,\mathbb{C})$. The transformation properties of quaternionic spinors under such a group give the possibility to write a quaternionic Dirac equation for two-dimensional spinors $\psi$, which, in the case of massless particles, decouples into two one-dimensional equations, quaternionic counterpart of Weyl equations. This new approach to quaternionic Dirac equation supplies a quaternionic chiral representation for gamma-matrices which plays a fundamental role in quaternionic gauge models for electroweak interactions [1, 2].

Two ingredients are necessary to formulate quaternionic versions of the group $\mathsf{SL}(2,\mathbb{C})$, Dirac and Weyl equations: The introduction of right $\mathbb{C}$-linear operators [3, 4] and the adoption of a complex geometry [5], namely complex projection for inner products [6].

The paper is structured as follows: In the following section, we introduce the quaternionic field [7], $\mathbb{H}$, and develop the relevant material concerning the correspondence between unitary quaternions and spatial rotations [8]. Section 3 provides a detailed exposition of the quaternionic Lorentz group [9] and contains a brief discussion on left/right operators. Section 4 is intended to motivate our study of quaternionic Lorentz spinors. In such a section, we formulate the quaternionic Dirac equation in the chiral representation. In section

5, we present a brief discussion on the quaternionic Dirac-Rotelli equation [10]. There, we justify the adoption of a complex geometry and summarize the main motivations of using the quaternionic chiral representation to formulate an electroweak theory.

# 2 Unitary Quaternionic Group and Spatial Rotations

Spatial rotations can be represented by

$$\widetilde{V} = \mathcal{R} V \ ,$$

where the vector column

$$\widetilde{V} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \ ,$$

defines the rotated vector and $\mathcal{R}$ denotes an orthogonal $3 \times 3$ real matrix

$$\mathcal{R}\mathcal{R}^t = \mathbb{1} \ .$$

In terms of infinitesimal generators, the previous equation reads

$$\mathcal{A} = -\mathcal{A}^t. \tag{1}$$

The rotation group, $\mathsf{O}(3)$, is a non-abelian Lie group, characterized by three parameters. Corresponding to these three parameters there are three anti-hermitian generators defined by Eq. (1). Explicitly,

$$\mathcal{A}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \ , \quad \mathcal{A}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ , \quad \mathcal{A}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

Such generators satisfy the following commutation relations

$$\mathcal{A}_x = [\mathcal{A}_y, \mathcal{A}_z] \ , \quad \mathcal{A}_y = [\mathcal{A}_z, \mathcal{A}_x] \ , \quad \mathcal{A}_z = [\mathcal{A}_x, \mathcal{A}_y] \ . \tag{2}$$

It is possible to represent spatial rotations by quaternions? Which is the quaternionic counterpart of $\mathsf{SU}(2)$? Can quaternions do for the space what complex do for the plane? Before to answering to such questions, let us recall some algebraic properties of the quaternionic field, $\mathbb{H}$.

A quaternion can be defined by four real entries or by two complex quantities, symplectic decomposition [11, 12],

$$q = a_0 + ia_1 + ja_2 + ka_3 = \xi + j\eta \ , \qquad a_{0,1,2,3} \in \mathbb{R} \ , \quad \xi, \eta \in \mathbb{C}(1, i) \ , \tag{3}$$

and three, associative but non commutative, imaginary quaternionic units $i$, $j$, $k$ which satisfy

$$i^2 = j^2 = k^2 = ijk = -1 \ . \tag{4}$$

The *full* conjugate, denoted by $q^\dagger$, is given by

$$q^\dagger = a_0 - ia_1 - ja_2 - ka_3 = \xi^* - j\eta \ .$$

Let us now introduce the quaternionic group $\mathsf{U}(1,\mathbb{H}^L)$, characterized by unitary operators

$$\mathbf{u}\mathbf{u}^\dagger = \mathbf{u}^\dagger\mathbf{u} = 1 \qquad \mathbf{u} \in \mathbb{H}^L \ .$$

A generic element of $\mathsf{U}(1,\mathbb{H}^L)$ is

$$\mathbf{u} = \exp\left[\left(i\theta_x + j\theta_y + k\theta_z\right)/2\right] \ . \tag{5}$$

What means $\mathbb{H}^L$? Working with quaternions, we must admit *left* and *right* action for imaginary units, and so the most general transformation on quaternionic states will be represented by operators $\in \mathbb{H}^L \otimes \mathbb{H}^R$. In this section we shall use only left operators and so it is not necessary to use different notations to differentiate left and right quaternionic imaginary units. This will become necessary in the next section, where the motivations in using left/right operators appear clear.

One-dimensional quaternionic spinors transform under $\mathsf{U}(1,\mathbb{H}^L)$,

$$q \to \mathbf{u}\, q \ . \tag{6}$$

Using quaternionic spinors, we can construct the anti-hermitian quaternionic state $qiq^\dagger$

$$qi \to \mathbf{u}\, qi \ , \quad q^\dagger \to q^\dagger \mathbf{u}^\dagger \ , \quad qiq^\dagger \to \mathbf{u}\, qiq^\dagger \mathbf{u}^\dagger \ . \tag{7}$$

Once identified the anti-hermitian quaternion $qiq^\dagger$ with the three spatial coordinates

$$qiq^\dagger \equiv ix + jy + kz \quad \Rightarrow \quad x \equiv |\xi|^2 - |\eta|^2 \ , \quad y - iz \equiv 2i\eta\xi^* \ , \tag{8}$$

we can prove that an $\mathsf{O}(3)$-transformation on

$$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ ,$$

is equivalent to a $\mathsf{U}(1,\mathbb{H}^L)$-transformation on the quaternionic spinor

$$q = \xi + j\eta \ .$$

The $\mathbf{u}$-transformation (5) is characterized by three real parameters, just like the number of real parameters in spatial rotations. Let us find the explicit relation between these two sets of parameters. In taking the quaternionic spinor transformation

$$\tilde{\xi} + j\tilde{\eta} = \exp\left[\left(i\theta_x + j\theta_y + k\theta_z\right)/2\right](\xi + j\eta) = [A + jB]\,(\xi + j\eta) \ ,$$

we obtain

$$\begin{aligned} \tilde{x} &= |\tilde{\xi}|^2 - |\tilde{\eta}|^2 = \left(|A|^2 - |B|^2\right)x - i\left(AB - A^*B^*\right)y + \left(AB + A^*B^*\right)z \ , \\ \tilde{y} - i\tilde{z} &= 2i\tilde{\eta}\tilde{\xi}^* = A^{*\,2}\,(y - iz) + B^2\,(y + iz) + 2iA^*B\,x \ . \end{aligned} \tag{9}$$

Setting $\mathbf{u} = e^{\frac{i}{2}\theta_x}$, from Eqs. (9) we have

$$\tilde{x} = x \ , \quad \tilde{y} - i\tilde{z} = e^{-i\theta_x}(y - iz) \quad \Rightarrow \quad \tilde{x} = x \ , \quad \tilde{y} = y\cos\theta_x - z\sin\theta_x \ , \quad \tilde{z} = y\sin\theta_x + z\cos\theta_x \ ,$$

which represents a rotation about the $x$-axis through an angle $\theta_x$. Hence, we gain the following identification

$$e^{\frac{i}{2}\theta_x} \quad \leftrightarrow \quad \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{array} \right) \ . \tag{10}$$

In a similar way, putting $\mathbf{u} = e^{\frac{j}{2}\theta_y}$ and $\mathbf{u} = e^{\frac{k}{2}\theta_z}$ we find

$$e^{\frac{j}{2}\theta_y} \quad \leftrightarrow \quad \left( \begin{array}{ccc} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{array} \right) \ , \tag{11}$$

and

$$e^{\frac{k}{2}\theta_z} \quad \leftrightarrow \quad \left( \begin{array}{ccc} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{array} \right) \ . \tag{12}$$

In this way, we establish a correspondence between the one-dimensional unitary quaternionic group, $\mathsf{U}(1, \mathbb{H}^L)$, and the three-dimensional orthogonal real group, $\mathsf{O}(3)$. The above mentioned groups are algebraically isomorphic. It is immediate to check that the quaternionic unitary generators

$$\mathcal{A}_x = i/2 \ , \quad \mathcal{A}_y = j/2 \ , \quad \mathcal{A}_z = k/2 \ , \tag{13}$$

satisfy the commutation relations (2). We observe that the factor $\frac{1}{2}$ is responsible for the global topological distinction between $\mathsf{U}(1, \mathbb{H}^L)$ and $\mathsf{O}(3)$. Increasing the angles $\theta_x$, $\theta_y$, $\theta_z$ by $2\pi$

$$\mathbf{u} \to -\mathbf{u} \ , \quad \mathcal{R} \to \mathcal{R} \ ,$$

so the elements $\mathbf{u}$ and $-\mathbf{u}$ in $\mathsf{U}(1, \mathbb{H}^L)$ both correspond to the rotation $\mathcal{R}$ in $\mathsf{O}(3)$. Thus, there is a two-to-one mapping of the elements of $\mathsf{U}(1, \mathbb{H}^L)$ onto those of $\mathsf{O}(3)$. Finally, the quaternionic (one-dimensional) unitary group, $\mathsf{U}(1, \mathbb{H}^L)$, represents the quaternionic "translation" of the complex (two-dimensional) special unitary group, $\mathsf{SU}(2)$.

## 3 Quaternions and Lorentz Group

The Lorentz group, $\mathsf{O}(3,1)$, is characterized by six parameters, three for rotations and three for boosts. Corresponding to these six parameters there are six generators. The anti-hermitian generators associated to spatial rotations are

$$\mathcal{A}_x = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right) \ , \quad \mathcal{A}_y = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right) \ , \quad \mathcal{A}_z = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \ ,$$

and the hermitian boosts generators

$$\mathcal{B}_x = \left( \begin{array}{cccc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \ , \quad \mathcal{B}_y = \left( \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \ , \quad \mathcal{B}_z = \left( \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right) \ .$$

Such generators satisfy the following commutation relations

$$\begin{array}{lll} \mathcal{A}_x = [\mathcal{A}_y, \mathcal{A}_z] \ , & \mathcal{A}_x = [\mathcal{B}_z, \mathcal{B}_y] \ , & \mathcal{B}_x = [\mathcal{A}_y, \mathcal{B}_z] = [\mathcal{B}_y, \mathcal{A}_z] \ , \\ \mathcal{A}_y = [\mathcal{A}_z, \mathcal{A}_x] \ , & \mathcal{A}_y = [\mathcal{B}_x, \mathcal{B}_z] \ , & \mathcal{B}_y = [\mathcal{A}_z, \mathcal{B}_x] = [\mathcal{B}_z, \mathcal{A}_x] \ , \\ \mathcal{A}_z = [\mathcal{A}_x, \mathcal{A}_y] \ , & \mathcal{A}_z = [\mathcal{B}_y, \mathcal{B}_x] \ , & \mathcal{B}_z = [\mathcal{A}_x, \mathcal{B}_y] = [\mathcal{B}_x, \mathcal{A}_y] \ , \end{array} \tag{14}$$

Observe that by changing the sign of our boost generators we do not modify the previous commutation relations. This will give the possibility to introduce two *inequivalent* quaternionic representations of the Lorentz group.

We showed in the previous section a possible quaternionic one-dimensional representations (13) for the anti-hermitian rotations operators of $\mathcal{O}(3)$. The aim of this section is to extend such a correspondence to include boost generators. To do it, let us introduce *left/right operators*. Such operators will represent the left/right actions of the imaginary quaternionic units and , in the sequel, will be denoted by

$$\vec{L} \equiv (L_i, L_j, L_k) \quad \text{ and } \quad \vec{R} \equiv (R_i, R_j, R_k) \ , \tag{15}$$

with

$$\vec{L} \ : \ \mathbb{H} \to \mathbb{H} \ , \quad \vec{L}\, q \equiv \vec{h}\, q \quad \text{ and } \quad \vec{R} \ : \ \mathbb{H} \to \mathbb{H} \ , \quad \vec{R}\, q \equiv q\, \vec{h} \ , \qquad \vec{h} \equiv (i, j, k) \ .$$

The algebra of left and right operators can be concisely expressed by

$$L_i^2 = L_j^2 = L_k^2 = L_i L_j L_k = R_i^2 = R_j^2 = R_k^2 = R_k R_j R_i = -\mathbb{1} \ ,$$

and

$$[\, L_{i,j,k} \ , \ R_{i,j,k} \,] = 0 \ .$$

The idea of combining left and right imaginary units gives the possibility to obtain a one dimensional representation for boost generators by $\mathbb{C}$-linear quaternionic operators

$$\mathcal{O}_{\mathbb{C}}(qz) = (\mathcal{O}_{\mathbb{C}} q) z \ , \qquad z \in \mathbb{C}(1, i) \ .$$

Explicitly,

$$\mathcal{A}_x = L_i/2 \ , \quad \mathcal{A}_y = L_j/2 \ , \quad \mathcal{A}_z = L_k/2 \qquad \in \mathbb{H}^L \ , \tag{16}$$

and

$$\mathcal{B}_x = \pm L_i R_i/2 \ , \quad \mathcal{B}_y = \pm L_j R_i/2 \ , \quad \mathcal{B}_z = \pm L_k R_i/2 \qquad \in \mathbb{H}^L \otimes \mathbb{C}^R \ . \tag{17}$$

These two inequivalent quaternionic representations for the Lorentz generators imply two different transformation laws for quaternionic spinors

$$\mathbf{s}_+ = \exp\left[\vec{L} \cdot \left(\vec{\theta} + R_i\ \vec{\varphi}\right)/2\right] \quad \text{ and } \quad \mathbf{s}_- = \exp\left[\vec{L} \cdot \left(\vec{\theta} - R_i\ \vec{\varphi}\right)/2\right] \ , \tag{18}$$

where

$$\vec{\theta} \equiv (\theta_x, \theta_y, \theta_z) \ , \quad \vec{\varphi} \equiv (\varphi_x, \varphi_y, \varphi_z) \ .$$

For consistence, we introduce two one-dimensional quaternionic spinors, $q_\pm$, which transform in the following way

$$q_\pm \to \mathbf{s}_\pm\, q_\pm \ , \qquad q_\pm = \xi_\pm + j \eta_\pm \ , \quad \xi_\pm, \eta_\pm \in \mathbb{C} \ . \tag{19}$$

Due to the right $\mathbb{C}$-linearity of boost generators, the spinor $q_\pm i$ transforms like $q_\pm$

$$q_\pm i \to \mathbf{s}_\pm \, q_\pm i \ .$$

Taking the quaternion $q_\pm(1+i)q_\pm^\dagger$ defined by the four space-time coordinates

$$\begin{gathered} q_\pm(1+i)q_\pm^\dagger \equiv ct + ix + jy + kz \quad \Rightarrow \\ ct \equiv |\xi_\pm|^2 + |\eta_\pm|^2 \ , \quad x \equiv |\xi_\pm|^2 - |\eta_\pm|^2 \ , \quad y - iz \equiv 2i\eta_\pm \xi_\pm^* \ , \end{gathered} \tag{20}$$

we will show that $\mathsf{O}(3,1)$-transformations on

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \ ,$$

are equivalent to $\mathbf{s}_\pm$-transformations on the quaternionic spinors

$$q_\pm = \xi_\pm + j\eta_\pm \ .$$

We follow the proof presented in the previous section. Putting $\mathbf{s}_\pm = e^{L_i \frac{\theta_x}{2}}$,

$$q_\pm \to e^{\frac{i}{2}\theta_x} q_\pm \ , \quad q_\pm^\dagger \to q_\pm^\dagger e^{-\frac{i}{2}\theta_x} \ ,$$

we obtain

$$q_\pm(1+i)q_\pm^\dagger \to e^{\frac{i}{2}\theta_x} \left[ q_\pm(1+i)q_\pm^\dagger \right] e^{-\frac{i}{2}\theta_x} = q_\pm q_\pm^\dagger + e^{\frac{i}{2}\theta_x} q_\pm i q_\pm^\dagger e^{-\frac{i}{2}\theta_x} \ .$$

Consequently,

$$c\tilde{t} + i\tilde{x} + j\tilde{y} + k\tilde{z} = ct + e^{\frac{i}{2}\theta_x} \left( ix + jy + kz \right) e^{-\frac{i}{2}\theta_x} \ ,$$

which implies

$$e^{L_i \frac{\theta_x}{2}} \quad \leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & -\sin\theta_x \\ 0 & 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \ . \tag{21}$$

In a similar way, choosing $\mathbf{s}_\pm = e^{L_j \frac{\theta_y}{2}}$ and $\mathbf{s}_\pm = e^{L_k \frac{\theta_z}{2}}$, we find

$$e^{L_j \frac{\theta_y}{2}} \quad \leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & \sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \ , \tag{22}$$

and

$$e^{L_k \frac{\theta_z}{2}} \quad \leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{23}$$

Our next objective is to evaluating the correspondence between one-dimensional quaternionic boosts and $4\times 4$ real matrix representing space/time transformations. Let us consider the first generator of quaternionic boosts in Eq. (17), $\mathbf{s}_\pm = e^{\pm L_i R_i \frac{\varphi_x}{2}}$,

$$q_\pm \to e^{\pm L_i R_i \frac{\varphi_x}{2}} q_\pm \equiv q_\pm \cosh\tfrac{\varphi_x}{2} \pm i q_\pm i \sinh\tfrac{\varphi_x}{2} \ .$$

The transformation law associated with such a quaternionic generator gives for the quaternionic space-time vector

$$q_\pm(1+i)q_\pm^\dagger \to \left(q_\pm \cosh\tfrac{\varphi_x}{2} \pm i q_\pm i \sinh\tfrac{\varphi_x}{2}\right)(1+i)\left(q_\pm^\dagger \cosh\tfrac{\varphi_x}{2} \pm i q_\pm^\dagger i \sinh\tfrac{\varphi_x}{2}\right) \ ,$$

which implies

$$\tilde{q}_\pm(1+i)\tilde{q}_\pm^\dagger = q_\pm(1+i)q_\pm^\dagger \cosh^2\tfrac{\varphi_x}{2} - i q_\pm(1+i)q_\pm^\dagger i \sinh^2\tfrac{\varphi_x}{2} \pm \{i, q_\pm(i-1)q_\pm^\dagger i\} \sinh\tfrac{\varphi_x}{2}\cosh\tfrac{\varphi_x}{2} \ .$$

Observing that

$$q_\pm q_\pm^\dagger \equiv ct \ , \quad q_\pm i q_\pm^\dagger \equiv ix + jy + kz \ , \quad \{i, q_\pm(i-1)q_\pm^\dagger i\} = -2(x+ict) \ ,$$

the previous equation can be directly written in terms of space-time coordinates as follows

$$\begin{aligned} c\tilde{t} + i\tilde{x} + j\tilde{y} + k\tilde{z} &= (ct+ix)\left(\cosh^2\tfrac{\varphi_x}{2} + \sinh^2\tfrac{\varphi_x}{2}\right) \mp 2\,(ct+ix)\sinh\tfrac{\varphi_x}{2}\cosh\tfrac{\varphi_x}{2} + jy + kz \\ &= ct\cosh\varphi_x \mp x\sinh\varphi_x + i\,(x\cosh\varphi_x \mp ct\sinh\varphi_x) + jy + kz \ . \end{aligned}$$

Thus, we obtain the following identification between our quaternionic transformations, $\mathbf{s}_\pm = e^{\pm L_i R_i \frac{\varphi_x}{2}}$, and Lorentz boosts which mix time $t$ and spatial coordinate $x$,

$$e^{\pm L_i R_i \frac{\varphi_x}{2}} \quad \leftrightarrow \quad \begin{pmatrix} \cosh\varphi_x & \mp\sinh\varphi_x & 0 & 0 \\ \mp\sinh\varphi_x & \cosh\varphi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{24}$$

Finally, by setting $\mathbf{s}_\pm = e^{\pm L_j R_i \frac{\varphi_y}{2}}$ and $\mathbf{s}_\pm = e^{\pm L_k R_i \frac{\varphi_z}{2}}$, we obtain

$$e^{\pm L_j R_i \frac{\varphi_y}{2}} \quad \leftrightarrow \quad \begin{pmatrix} \cosh\varphi_y & 0 & \mp\sinh\varphi_y & 0 \\ 0 & 1 & 0 & 0 \\ \mp\sinh\varphi_y & 0 & \cosh\varphi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{25}$$

and

$$e^{\pm L_k R_i \frac{\varphi_z}{2}} \quad \leftrightarrow \quad \begin{pmatrix} \cosh\varphi_z & 0 & 0 & \mp\sinh\varphi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mp\sinh\varphi_z & 0 & 0 & \cosh\varphi_z \end{pmatrix} \ . \tag{26}$$

In the previous section, we observed that unitary quaternions, $\mathsf{U}(1,\mathbb{H}^L)$, represent the quaternionic counterpart of complex special unitary groups, $\mathsf{SU}(2)$. In this section, we proved that (16,17) represent the quaternionic counterpart of the generators of complex special linear groups, $\mathsf{SL}(2,\mathbb{C})$. The corresponding quaternionic group will be thus indicated by $\mathsf{SL}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$. For a completed and detailed review of quaternionic group theory, we refer the reader to [4, 13, 14]. The situation can be summarized by

$$\begin{array}{ccccc} \mathsf{U}(1,\mathbb{H}^L) & \leftrightarrow & \mathsf{SU}(2) & \sim & \mathsf{O}(3) \\ \mathsf{SL}(1,\mathbb{H}^L\otimes\mathbb{C}^R) & \leftrightarrow & \mathsf{SL}(2,\mathbb{C}) & \sim & \mathsf{O}(3,1) \end{array}$$

## 4 Quaternionic Dirac Equation and Chiral Representation

As remarked in section 3, corresponding to the two possible signs of $\vec{\mathcal{B}}$ in Eq. (17), we have

$$q_+ \;\to\; \mathbf{s}_+\, q_+ = \exp\left[\vec{L}\cdot\left(\vec{\theta}+R_i\;\vec{\varphi}\right)/2\right]\; q_+ \;,$$

and

$$q_- \;\to\; \mathbf{s}_-\, q_- = \exp\left[\vec{L}\cdot\left(\vec{\theta}-R_i\;\vec{\varphi}\right)/2\right]\; q_- \;.$$

These *inequivalent* representations of the Lorentz group are related to *two* different types of one-component quaternionic spinors, $q_\pm$, which represent the quaternionic counterpart of the standard complex *dotted* and *undotted* spinors. Such spinors correspond to the representations

$$\left(\tfrac{1}{2},0\right) \quad \text{and} \quad \left(0,\tfrac{1}{2}\right)$$

of the Lorentz group. In fact, by defining

$$\vec{\mathcal{A}}_1=\tfrac{1}{2}\left(\vec{\mathcal{A}}-\vec{\mathcal{B}}\,R_i\right) \quad \text{and} \quad \vec{\mathcal{A}}_2=\tfrac{1}{2}\left(\vec{\mathcal{A}}+\vec{\mathcal{B}}\,R_i\right) \;, \tag{27}$$

from Eqs. (14), we obtain

$$\mathcal{A}_{m,x}=\left[\,\mathcal{A}_{m,y}\,,\,\mathcal{A}_{m,z}\,\right]\,,\quad \mathcal{A}_{m,y}=\left[\,\mathcal{A}_{m,z}\,,\,\mathcal{A}_{m,x}\,\right]\,,\quad \mathcal{A}_{m,z}=\left[\,\mathcal{A}_{m,x}\,,\,\mathcal{A}_{m,y}\,\right]\,,\qquad {\scriptstyle m\,=\,1,\,2}\;,$$

and

$$\left[\,\vec{\mathcal{A}}_1\,,\,\vec{\mathcal{A}}_2\,\right]=0\;.$$

The quaternionic Lorentz group is thus essentially

$$\mathsf{U}_1(1,\mathbb{H}^L)\otimes\mathsf{U}_2(1,\mathbb{H}^L)\;,$$

and states transform by two angular momenta

$$(j_1,j_2)\;\;.$$

The first one corresponding to $\vec{\mathcal{A}}_1$ and the second to $\vec{\mathcal{A}}_2$. As special case,

$$\begin{array}{llcllc}
\vec{\mathcal{A}}=\vec{L}/2\;, & \vec{\mathcal{B}}=+\vec{L}R_i/2 & \Rightarrow & \vec{\mathcal{A}}_1=\vec{L}/2\;, & \vec{\mathcal{A}}_2=0\;, & \left(\tfrac{1}{2},0\right)\;,\\
\vec{\mathcal{A}}=\vec{L}/2\;, & \vec{\mathcal{B}}=-\vec{L}R_i/2 & \Rightarrow & \vec{\mathcal{A}}_2=\vec{L}/2\;, & \vec{\mathcal{A}}_1=0\;, & \left(0,\tfrac{1}{2}\right)\;.
\end{array}$$

Under parity, $\mathcal{P}$, velocity changes sign, hence the boost generators $\vec{\mathcal{B}}$ change sign

$$\mathcal{P}\text{-transformation}\;:\quad \vec{\mathcal{B}}\;\to\;-\vec{\mathcal{B}}\;.$$

The rotation generators $\vec{\mathcal{A}}$, behaving like axial vectors, $\vec{r}\times\vec{p}$, does not change sign

$$\mathcal{P}\text{-transformation}\;:\quad \vec{\mathcal{A}}\;\to\;\vec{\mathcal{A}}\;.$$

So, we have

$$\mathcal{P}\text{-transformation}\;:\quad (j_1,0)\leftrightarrow(0,j_2)\;,\quad q_+\leftrightarrow q_-\;.$$

By introducing parity, it is no longer sufficient to consider one-dimensional quaternionic spinors $q_+$ and $q_-$ separately, but we need to define two-dimensional quaternionic spinors

$$\psi=\left(\begin{array}{c}q_+\\ q_-\end{array}\right)\;. \tag{28}$$

Finally, under Lorentz transformations

$$\psi \rightarrow \begin{pmatrix} \mathbf{s}_+ & 0 \\ 0 & \mathbf{s}_- \end{pmatrix} \begin{pmatrix} q_+ \\ q_- \end{pmatrix} ,$$

and under parity

$$\psi \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_+ \\ q_- \end{pmatrix} .$$

The two-dimensional spinor $\psi$ is an *irreducible* representation of the Lorentz group *extended* by parity. Let us now consider Lorentz boosts, $\vec{\theta} = 0\,,\ \vec{\varphi} \neq 0$,

$$q_+ \rightarrow \exp\left(\vec{L}R_i \cdot \vec{\varphi}/2\right)\, q_+ = \exp\left(L_{\mathbf{n}}R_i\varphi/2\right)\, q_+ \ ,$$

where

$$L_{\mathbf{n}} \equiv \vec{L}\cdot\vec{n} = n_1 L_i + n_2 L_j + n_3 L_k \ , \quad L_{\mathbf{n}}^2 = -\mathbb{1} \ , \quad \varphi = \sqrt{\varphi_x^2 + \varphi_y^2 + \varphi_z^2} \ .$$

Consequently,

$$q_+ \rightarrow \left(\cosh\tfrac{\varphi}{2} + L_{\mathbf{n}}R_i \sinh\tfrac{\varphi}{2}\right) q_+ \ .$$

The $q_-$ transformation is soon obtained from the previous one by changing $\varphi \rightarrow -\varphi$,

$$q_- \rightarrow \left(\cosh\tfrac{\varphi}{2} - L_{\mathbf{n}}R_i \sinh\tfrac{\varphi}{2}\right) q_- \ .$$

By supposing original spinors refering to particles at rest, $q_{\pm(0)}$, and the transformed spinors, $q_{\pm(\vec{p})}$ to a particles with momentum $\vec{p}$, and observing that

$$\cosh\varphi = \gamma = E/m \ , \quad \sinh\varphi = \beta\gamma = p/m \ , \qquad (c=1) \ ,$$

which implies

$$\cosh\tfrac{\varphi}{2} = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}} = \left(\frac{E+m}{2m}\right)^{\frac{1}{2}} \ , \quad \sinh\tfrac{\varphi}{2} = \left(\frac{\gamma-1}{2}\right)^{\frac{1}{2}} = \left(\frac{E-m}{2m}\right)^{\frac{1}{2}} \ ,$$

we find

$$\begin{aligned} q_{+(\vec{p})} &= \left[\left(\frac{E+m}{2m}\right)^{\frac{1}{2}} + L_{\mathbf{n}}R_i\left(\frac{E-m}{2m}\right)^{\frac{1}{2}}\right] q_{+(0)} \\ &= \left(E + m + \vec{L}R_i\cdot\vec{p}\right)\left[2m\left(E+m\right)\right]^{-\frac{1}{2}}\, q_{+(0)} \ , \end{aligned}$$

and

$$\begin{aligned} q_{-(\vec{p})} &= \left[\left(\frac{E+m}{2m}\right)^{\frac{1}{2}} - L_{\mathbf{n}}R_i\left(\frac{E-m}{2m}\right)^{\frac{1}{2}}\right] q_{-(0)} \\ &= \left(E + m - \vec{L}R_i\cdot\vec{p}\right)\left[2m\left(E+m\right)\right]^{-\frac{1}{2}}\, q_{-(0)} \ . \end{aligned}$$

At rest, $q_{+(0)} = q_{-(0)}$, thus with a bit of algebra we obtain

$$m\, q_{+(\vec{p})} = \left(E + \vec{L}R_i\cdot\vec{p}\right)\, q_{-(\vec{p})} \quad \text{and} \quad m\, q_{-(\vec{p})} = \left(E - \vec{L}R_i\cdot\vec{p}\right)\, q_{+(\vec{p})} \ .$$

In matrix form, these equations read

$$\left(\begin{array}{cc} -m & E+\vec{L}R_i\cdot\vec{p} \\ E-\vec{L}R_i\cdot\vec{p} & -m \end{array}\right)\left(\begin{array}{c} q_{+(\vec{p})} \\ q_{-(\vec{p})} \end{array}\right) . \tag{29}$$

By adopting the following representation for gamma matrices

$$\gamma^0 \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) , \quad \vec{\gamma} \equiv \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) \vec{L}R_i , \tag{30}$$

Eq. (29) becomes

$$(\gamma^\mu p_\mu - m)\psi_{(p)} = 0 .$$

This is the quaternionic Dirac equation for massive spin $\frac{1}{2}$ particle in the *chiral* representation. For massless particles, such an equation decouples into two equations, each characterized by a one-component quaternionic spinor

$$\left(E+\vec{L}R_i\cdot\vec{p}\right) q_{-(\vec{p})} = 0 ,$$

and

$$\left(E-\vec{L}R_i\cdot\vec{p}\right) q_{+}(\vec{p}) = 0 ,$$

which represent the quaternionic Weyl equations. Since, for a massless particle, $E = p$, these equations also read

$$\vec{L}R_i\cdot\hat{p}\, q_{-(\vec{p})} = -q_{-(\vec{p})} , \quad \vec{L}R_i\cdot\hat{p}\, q_{+(\vec{p})} = q_{+(\vec{p})} . \tag{31}$$

The operator

$$\vec{L}R_i\cdot\hat{p}$$

measures the component of the spin in the direction of momentum and defines the *quaternionic helicity operator.*

# 5 Chiral and Dirac Representations

The derivation of the quaternionic Dirac equation given above differs from the original one formulated by Rotelli [10]. The equation presented in this paper is obtained directly from transformation properties of quaternionic Lorentz spinors, whereas the Rotelli equation follows the Dirac original approach. The difference becomes clear by observing the representations for gamma matrices. In the Dirac-Rotelli equation we find:

$$\gamma^0 \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) , \quad \vec{\gamma} \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \vec{L} .$$

In the Dirac-De Leo equation we obtain the following set of gamma-matrices

$$\gamma^0 \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) , \quad \vec{\gamma} \equiv \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) \vec{L}R_i .$$

Apparently, the Dirac-Rotelli equation contains left operators, $\gamma^\mu \in \mathcal{M}_2(\mathbb{H}^L)$, whereas our Dirac equation is based on a right $\mathbb{C}$-linear quaternionic algebra, $\gamma^\mu \in \mathcal{M}_2(\mathbb{H}^L \otimes \mathbb{C}^R)$. Nevertheless, the use of right imaginary unit, $R_i$, is only hidden in the Rotelli approach.

In the complex world, the Dirac equation reads indifferently as

$$i\partial_t\psi = H\psi \quad \text{or} \quad \partial_t\psi i = H\psi \ .$$

In the quaternionic world there is a clear difference in choosing a left or right position for our complex imaginary unit $i$. In fact, by requiring norm conservation

$$\partial_t \int d^3x\, \psi^\dagger\psi = 0 \ ,$$

we find that a left position of the imaginary unit $i$ in the quaternionic Dirac equation,

$$L_i\partial_t\psi \equiv i\partial_t\psi = H\psi \ ,$$

gives

$$\partial_t \int d^3x\, \psi^\dagger\psi = \int d^3x\, \psi^\dagger[H, i]\psi \ ,$$

in general $\neq 0$ for quaternionic Hamiltonians. A right position of the imaginary unit $i$,

$$R_i\partial_t\psi \equiv \partial_t\psi i = H\psi \ ,$$

ensures the norm conservation. From covariance, by treating time and space in the same way, we obtain the following "quaternionic" momentum operator

$$p^\mu \leftrightarrow R_i\partial^\mu \quad \Rightarrow \quad p^\mu\psi \leftrightarrow R_i\partial^\mu\psi \equiv \partial^\mu\psi i \ . \tag{32}$$

Finally, the quaternionic Dirac equation reads

$$R_i\gamma^\mu\partial_\mu\psi \equiv \gamma^\mu\partial_\mu\psi i = m\psi \ , \quad [\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu} \ . \tag{33}$$

Another fundamental ingredient in the formulation of quaternionic relativistic quantum mechanics is represented by the adoption of a *complex geometry* [5], necessary in order to guarantee that $R_i\vec{\partial}$ be an hermitian operator

$$\int d^3x\, \varphi^\dagger R_i\vec{\partial}\psi = \int d^3x\, (R_i\vec{\partial}\varphi)^\dagger\psi \ .$$

The previous relation implies

$$\int d^3x\, \varphi^\dagger\vec{\partial}\psi i = -i\int d^3x\, \vec{\partial}\varphi^\dagger\psi \ \text{(after integration by parts)} \ = i\int d^3x\, \varphi^\dagger\vec{\partial}\psi \ .$$

The different position of the imaginary unit $i$ forces the use of a *complex projection* [6] for inner products

$$\int d^3x \ \rightarrow \ \int_{\mathbb{C}} d^3x \ . \tag{34}$$

By introducing the following matrices

$$\Upsilon^\mu \equiv R_i\gamma^\mu \quad \in \mathbb{H}^L \otimes \mathbb{C}^R \ ,$$

the quaternionic Dirac equation (33) becomes

$$\Upsilon^\mu\partial_\mu\psi = m\psi \ , \quad [\Upsilon^\mu, \Upsilon^\nu] = -2g^{\mu\nu} \ . \tag{35}$$

The Rotelli representation modifies in

$$\Upsilon^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_i \ , \quad \vec{\Upsilon} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{L} R_i \ ,$$

and the chiral representation in

$$\Upsilon^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_i \ , \quad \vec{\Upsilon} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{L} \ .$$

These $\Upsilon$-matrices give the possibility to construct the following quaternionic algebra

$$\Upsilon^0 \ , \quad \Upsilon^{123} = \Upsilon^1 \Upsilon^2 \Upsilon^3 \ , \quad \Upsilon^5 = \Upsilon^0 \Upsilon^{123} \ , \tag{36}$$

in fact,

$$(\Upsilon^0)^2 = (\Upsilon^{123})^2 = (\Upsilon^5)^2 = \Upsilon^0 \Upsilon^{123} \Upsilon^5 = -\mathbb{1} \ .$$

Standard and chiral representations diagonalize respectively $\Upsilon^0$ and $\Upsilon^5$. The *new* (Maiorana) representation

$$\Upsilon^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_i \ , \quad \vec{\Upsilon} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{L} R_i \ .$$

diagonalizes $\Upsilon^{123}$. By simple algebraic manipulations, it is immediate to show that these three representations, corresponding to three possible choices in the diagonalization of $\Upsilon^0$ , $\Upsilon^{123}$, $\Upsilon^5$, generate the "complex" group

$$\mathsf{SU}(2, \mathbb{C}^R) \ .$$

It is not our purpose to study here quaternionic gauge theories. We would like spend only some considerations on quaternionic fermion states. The first family in the Salam-Weinberg model [15] is represented by

$$\begin{pmatrix} \nu \\ e \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u \\ d \end{pmatrix} \ .$$

In the chiral representation leptons $\nu_{L/R}$, $e_{L/R}$ and quarks $u_{L/R}$, $d_{L/R}$ are given in terms of one-dimensional quaternionic spinors and so they can be accommodate in the following $2 \times 2$ quaternionic matrices

$$\Psi_L = \begin{pmatrix} \nu_L & u_L \\ e_L & d_L \end{pmatrix} \ , \quad \Psi_R = \begin{pmatrix} \nu_R & u_R \\ e_R & d_R \end{pmatrix} \ . \tag{37}$$

The massless fermion electroweak Lagrangian [16]

$$\mathcal{L}_F = \left( \bar{\Psi}_L \Upsilon^\mu \partial_\mu \Psi_L + \bar{\Psi}_R \Upsilon^\mu \partial_\mu \Psi_R \right)_{\mathbb{C}} \ , \tag{38}$$

is global invariant under the Glashow gauge group [17]

$$\mathsf{SU}(2, \mathbb{C}^R)_L \otimes \mathsf{U}(1, \mathbb{C}^R)_Y \ . \tag{39}$$

## 6 Conclusions

This work was intended as an attempt to motivate the use of quaternions in physics [18], in particular relativistic quantum mechanics [19, 20] and gauge theories [1, 2, 6].

By following the Rider approach [21, 22], we obtained a two-dimensional quaternionic Dirac equation by using the transformation properties of quaternionic spinors under

$$\mathsf{SL}(1, \mathbb{H}^L \otimes \mathbb{C}^R) \ ,$$

quaternionic counterpart of $\mathsf{SL}(2, \mathbb{C})$. This gives as result the quaternionic version of chiral representation which plays an important role in the fermion sector of electroweak Lagrangian. The quaternionic gamma-matrices in the chiral representation also suggest that the true space for quaternionic $2 \times 2$ gamma-matrices be a left-quaternionic/right-complex space, namely

$$\gamma^\mu \in \mathcal{M}_2(\mathbb{H}^L \otimes \mathbb{C}^R) \ ,$$

In the special case of Dirac-Rotelli representation, $\gamma^\mu \in \mathcal{M}_2(\mathbb{H}^L) \subset \mathcal{M}_2(\mathbb{H}^L \otimes \mathbb{C}^R)$, the right $\mathbb{C}$-linear algebra is hidden in the definition of momentum operator $p^\mu = R_i \partial^\mu$.

It is interesting to observe that all possible representations for gamma matrices are generated by *three special representations*, diagonalization of $\Upsilon^0$ , $\Upsilon^{123}$, $\Upsilon^5$, which have the properties to generate the group $\mathsf{SU}(2, \mathbb{C}^R)$ and consequently the Glashow group [17] of Salam-Weinberg model [15]. In a forthcoming paper [23] will be discussed in detail a quaternionic electroweak theory based on the "complex" group

$$g_2 \, \mathsf{SU}(2, \mathbb{C}^R)_L \otimes g_Y \mathsf{U}(1, \mathbb{C}^R)_Y \ .$$

Formulations of left/right symmetric models [24, 25] require the simple generalization

$$g_{2,L} \, \mathsf{SU}(2, \mathbb{C}^R)_L \otimes g_{2,R} \, \mathsf{SU}(2, \mathbb{C}^R)_R \otimes g_1 \mathsf{U}(1, \mathbb{C}^R) \ .$$

Grand unification models and super-symmetric theories [26] could require the choice of *effective* quaternionic gauge groups [27].

## References


[1] Y. Nagamoto, J. Math. Phys. **33**, 4020 (1992).
A. Galpenin and E. Ivanov, Ann. Phys. **230**, 201 (1994).
B. de Wit and A. Van Proeyen, Int. J. Mod. Phys. D **3**, 31 (1994).

[2] S. De Leo an P. Rotelli, Phys. Rev. D **45**, 575 (1992); Int. J. Mod. Phys. A **10**, 4359 (1995); J. Phys. G **22**, 1137 (1996).

[3] S. De Leo and P. Rotelli, Prog. Theor. Phys. **92**, 917 (1994); **96**, 247 (1996).

[4] S. De Leo and G. Ducati, *Quaternionic Groups in Physics: A Panoramic Review* (submitted to Int. J. Theor. Phys.),

[5] J. Rembieliński, J. Phys. A **11**, 2323 (1978).

[6] L. P. Horwitz and L. C. Biedenharn, Ann. Phys. **157**, 432 (1984).

[7] W. R. Hamilton, *Elements of Quaternions* (Chelsea Publishing Co., N.Y., 1969).

[8] S. De Leo and P. Rotelli, Nuovo Cim. **B110**, 33 (1995).

[9] S. De Leo, J. Math. Phys. **37**, 2955 (1996).

[10] P. Rotelli, Mod. Phys. Lett. A **4**, 933 (1989).

[11] F. Gürsey, *Symmetries in Physics (1600-1980)*: Proceedings of the $1^{st}$ International Meeting on the History of Scientific Ideas, Seminari d' Historia de les Ciences, Barcellona, Spain, 557 (1983).

[12] F. Gürsey and C. H. Tze, *On the Role of Division, Fordan and Related Algebras in Particle Physics* (World Scientific, Singapore, 1996).

[13] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).

[14] S. Adler, J. Math. Phys. **37** 2352 (1996).

[15] A. Salam, Weak and electromagnetic interactions, in *Elementary Particle Theory; Nobel Symposium No. 8*, ed. N. Svartholm, Almqvist and Wiksell, Stockholm (1969).
S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).

[16] S. De Leo and P. Rotelli, Mod. Phys. Lett. **A11**, 357 (1996).

[17] S. L. Glashow, Nucl. Phys. **22**, 579 (1961).

[18] D. Finkelstein, J. M. Jauch, S. Schiminovich and D. Speiser D, J. Math. Phys. **3**, 207 (1962); **4**, 788 (1963).
D. Finkelstein, J. M. Jauch and D. Speiser, J. Math. Phys. **4**, 136 (1963).

[19] S. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford UP, New York, 1995).

[20] D. Finkelstein, J. M. Jauch and D. Speiser, *Notes on Quaternion Quantum Mechanics*, in Logico-Algebraic Approach to Quantum Mechanics II, Hooker (Reidel, Dordrecht 1979), 367-421.

[21] L. Ryder, *Quantum Field Theory* (Cambridge University, Cambridge 1988).

[22] F. H. Gaioli and E. T. Garcia Alvarez, Am. J. Phys. **63**, 177 (1995).

[23] S. De Leo, *Quaternionic Standard Model* (in preparation).

[24] R. N. Mohapatra and J. C. Pati, Phys. Rev. **D11**, 566 (1975).

[25] F. Cuypers, *Au-delà du Modèle Standard*, PSI Report 97-03, May 1997.

[26] G. Altarelli, *The Standard Electroweak Theory and Beyond*, hep-ph/9811456, November 1998.

[27] S. De Leo, Int. J. Theor. Phys. **35**, 1821 (1996).