Partitions with "$\left\lfloor \frac{N+1}{2} \right\rfloor$ copies of N"

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Abstract: *A general theorem on partitions is given which, in two special cases, presents very interesting combinatorial interpretations.*

1. Introduction

In this paper we shall prove a result in partitions wich involve the concept of "$f(N)$ copies of N". Associate to this we have another concept: "multiset" M (for MacMahon). To exemplify these concepts we present bellow the 6 partitions of 3 with "N copies of N":

$$3_1, 3_2, 3_3$$
$$2_1 + 1_1, 2_2 + 1_1$$
$$1_1 + 1_1 + 1_1$$

where the multiset in this case is

$$M_0 = \{1_1, 2_1, 2_2, 3_1, 3_2, 3_3, 4_1, 4_2, 4_3, 4_4, ...\}.$$

Some results in the theory of partitions have been obtained in this context. Among them we have, due to MacMahon, that

$$\sum_{n=0}^{\infty} \pi(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

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where $\pi(n)$ denote the number of plane partitions.

Also Agarwal and Andrews presented in [2] the following theorem:

"Let $A_0(k,n)$ denote the number of partitions of n with N copies of N such that if the weighted difference of any pair of summands m_i, r_j is nonpositive, then it is even and satisfies

$$m_i - r_j \geq -2\min\{i-1, j-1, k-3\}.$$

Let $B_\ell(k,n)$ denote the number of partitions of n into parts $\equiv 0, \pm 2(k-\ell)$ (mod $4k+2$). Then

$$A_0(k,n) = B_0(k,n)$$

for all $n \geq 0$ and $k \geq 2$".

and in [1] Agarwal proved that

"For $k \geq 3$, let $c_k(n)$ denote the number of partitions of n with "N copies of N" such that each pair of summands m_i, r_j satisfies $|m_i - r_j| > i + j + k$. Then

$$\sum_{n=0}^{\infty} c_k(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n[1+\frac{(k+3)(n-1)}{2}]}}{(q;q)_n (q;q^2)_n} \text{''} \ .$$

For our theorem we shall work into the following multisets

$$M_\ell = \{k_r \in M_0 \mid k \equiv r(\text{mod}2) \text{ and } r \geq \ell\}$$

where $\ell = 1, 2, 3, \ldots$.

We finish this section by observing that from the definition of M_ℓ we have

$$M_1 \underset{\neq}{\supset} M_2 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} M_\ell \underset{\neq}{\supset} M_{\ell+1} \underset{\neq}{\supset} \ldots$$

that

$$M_1 = \{1_1, 2_2, 3_1, 3_3, 4_2, 4_4, 5_1, 5_3, 5_5, \ldots\},$$

$$M_2 = \{2_2, 3_3, 4_2, 4_4, 5_3, 5_5, 6_2, 6_4, 6_6, ...\}$$

and that, in general, M_ℓ is a multiset with "$\left\lfloor \frac{N+2-\ell}{2} \right\rfloor$ copies of N".

2. The Theorem

Theorem 2.1. Let $A_\ell(n)$ be the number of partitions of n with parts in M_ℓ such that the difference between any two consecutive parts a_i and b_j satisfy

$$|a_i - b_j| \geq i + j. \tag{2.1}$$

Then

$$\sum_{n=0}^{\infty} A_\ell(n) q^n = \sum_{n=0}^{\infty} \frac{q^{\ell n^2}}{(q;q)_{2n}}. \tag{2.2}$$

Proof. Let $A_\ell(m,n)$ denote the number of partitions enumerated by $A_\ell(n)$ with the added restriction that there are exactly m parts. Next we show that $A_\ell(m,n)$ satisfies the following recurrence relation:

$$\begin{aligned} A_\ell(m,n) &= A_\ell(m, n-2m) + A_\ell(m-1, n-2m\ell+\ell) \\ &+ A_\ell(m, n-2m+1) - A_\ell(m, n-4m+1). \end{aligned} \tag{2.3}$$

To prove (2.3) we split the partitions enumerated by $A_\ell(m,n)$ into three classes: (a) those in which k_k is not a part; (b) those in which ℓ_ℓ is a part and (c) those in which k_k is a part for $k > \ell$.

If in those from class (a) we subtract "2" from each part without changing the subscripts we are left with a partition of $n - 2m$ in exactly m parts and these are the ones enumerated by $A_\ell(m, n-2m)$. From those in class (b) if we drop the part "ℓ_ℓ" and subtract 2ℓ from each of the remaining parts (keeping the subscript) we are left with a partition of $n - 2m\ell + \ell$ in exactly $m - 1$ parts and these are the ones enumerated by $A_\ell(m-1, n-2m\ell+\ell)$.

Finally from those in class (c) we subtract "2" from each part different from k_k and replace k_k by $(k-1)_{k-1}$. In doing this, we are left with a partition of $n-2m+1$ in exactly m parts which are enumerated by $A_\ell(m, n-2m+1)$. At this point we have to observe that by this transformation we get only those partitions of $n-2m+1$ into m parts which contain a part of the form "r_r" with $r \geq \ell$. For this reason the partitions of class (c) can be put in an one to one correspondence with those that are enumerated by $A_\ell(m, n-2m+1) - A_\ell(m, n-4m+1)$ and this finished the proof of (2.3).

The transformations just described are possible by the following reasons:
(i) the elements in class (a) for not having parts of the form "k_k" all of its parts are, then, of the form r_s where $r - s \geq 2$.
(ii) the parts of a partition in class (b) that are distinct from ℓ_ℓ are greater than or equal to 3ℓ because of (2.1), i.e, for $r_j + \ell_\ell, r - \ell \geq j + 2\ell$.
(iii) due to the transformation done in class (a) the partitions of $n-2m+1$ with m parts without having part of form "r_r" is enumerated by $A_\ell(m, n-4m+1)$.

Now we define

$$F_\ell(z;q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_\ell(m,n) z^m q^n$$

and using (2.3) we get:

$$\begin{aligned}
F_\ell(z;q) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (A_\ell(m, n-2m) + \\
&\quad A_\ell(m-1, n-2m\ell+\ell) + A_\ell(m, n-2m+1) \\
&\quad -A_\ell(m, n-4m+1)) z^m q^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_\ell(m, n-2m) z^m q^n + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_\ell(m-1, n-2m\ell+\ell) z^m q^n \\
&+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_\ell(m, n-2m+1) z^m q^n - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_\ell(m, n-4m+1) z^m q^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_\ell(m, n-2m)(zq^2)^m q^{n-2m} +
\end{aligned}$$

$$zq^{\ell}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{\ell}(m-1,n-2m\ell+\ell)(zq^{2\ell})^{m-1}q^{n-2m\ell+\ell}+$$
$$\frac{1}{q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{\ell}(m,n-2m+1)(zq^{2})^{m}q^{n-2m+1}-$$
$$\frac{1}{q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{\ell}(m,n-4m+1)(zq^{4})^{m}q^{n-4m+1}$$
$$= F_{\ell}(zq^{2};q)+zq^{\ell}F_{\ell}(zq^{2\ell};q)+\frac{1}{q}F_{\ell}(zq^{2};q)-\frac{1}{q}F_{\ell}(zq^{4};q) \qquad (2.4)$$

Assuming $F_{\ell}(z;q)=\sum_{n=0}^{\infty}\gamma(q,n)z^{n}$ and using (2.4) we may compare coefficients of z^n obtaining:

$$\gamma(q,n) = q^{2n}\gamma(q,n)+q^{2\ell n-\ell}\gamma(q,n-1)+q^{2n-1}\gamma(q,n)-q^{4n-1}\gamma(q,n)$$

Therefore

$$\gamma(q,n)=\frac{q^{2\ell n-\ell}}{(1-q^{2n})(1-q^{2n-1})}\gamma(q,n-1) \qquad (2.5)$$

and observing that $\gamma(q,0)=1$ we may iterate (2.5) to get

$$\gamma(q,n)=\frac{q^{\ell n^{2}}}{(q;q)_{2n}}.$$

From this

$$F_{\ell}(z;q)=\sum_{n=0}^{\infty}\frac{q^{\ell n^{2}}}{(q;q)_{2n}}z^{n}.$$

Now we can finish the proof since

$$\sum_{n=0}^{\infty}A_{\ell}(n)q^{n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{\ell}(m,n)q^{n}$$
$$= F_{\ell}(1;q)=\sum_{n=0}^{\infty}\frac{q^{\ell n^{2}}}{(q;q)_{2n}}.$$

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This theorem in the case $\ell = 5$ can be seen as:

$$\sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q)_{2n}} = 1 + q^5 + q^6 + 2q^7 + 2q^8 + 3q^9 + \ldots + 17q^{25} + \ldots =$$
$$= 1 + q^{5_5} + q^{6_6} + (q^{7_5} + q^{7_7}) + (q^{8_6} + q^{8_8}) +$$
$$+(q^{9_5} + q^{9_7} + q^{9_9}) + \ldots + (q^{25_5} + q^{25_7} + q^{25_9} + \ldots + q^{25_{23}} + q^{25_{25}} +$$
$$q^{20_6+5_5} + q^{20_8+5_5} + q^{20_{10}+5_5} + q^{19_5+6_6} + q^{19_7+6_6} + q^{18_6+7_5}) + \ldots$$

3. Particular Cases

For $\ell = 1$, by considering identity 79 of Slater [7, p.160]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \cdot \prod_{n=1}^{\infty} (1 - q^{20n-8})(1 - q^{20n-12})(1 - q^{20n})$$

we have the following theorem:

Theorem 3.1. The number of partitions of n with parts in M_1 such that the difference between any two consecutive parts a_i and b_j satisfy $|a_i - b_j| \geq i + j$ is equals the number of partitions in even parts $\not\equiv 0, \pm 8 \pmod{20}$ or distinct odd parts.

Now we illustrate this theorem for $n = 8$ in the table below.

8_2	7+1
8_4	6+2
8_6	5+3
8_8	5+2+1
$7_1 + 1_1$	4+4
$7_3 + 1_1$	4+3+1
$7_5 + 1_1$	4+2+2
$6_2 + 2_2$	3+2+2+1
$5_1 + 3_1$	2+2+2+2

For $\ell = 2$, as a consequence of Theorem 1, due to Andrews and Santos [5, p.92], we can enunciate the following result.

Theorem 3.2. Let $A_2(n)$ denote the number of partitions of n with parts in M_2 such that the difference between any two consecutive parts a_i and b_j satisfy $|a_i - b_j| \geq i+j$; let $\mathcal{A}_2(n)$ denote the number of partitions of n into parts that are $\equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$; let $\mathcal{B}_2(n)$ denote the number of partitions of n that are either even but $\not\equiv 0 \pmod 8$ or distinct odd and $\equiv \pm 3 \pmod 8$; $\mathcal{C}_2(n)$ denote the number of partitions of n such that the even parts are distinct and no even consecutive parts occur, and $2j+1$ is a part only if $2j$ or $2j+2$ occur. Then

$$A_2(n) = \mathcal{A}_2(n) = \mathcal{B}_2(n) = \mathcal{C}_2(n).$$

In the following table we have an illustration of the partitions described by this theorem for $n = 10$.

$A_2(10) = 7$	$\mathcal{A}_2(10) = 7$	$\mathcal{B}_2(10) = 7$	$\mathcal{C}_2(10) = 7$
10_{10}	5+5	10	10
10_8	5+3+2	6+4	8+2
10_6	4+4+2	6+2+2	6+4
10_4	4+2+2+2	5+3+2	6+2+1+1
10_2	3+3+4	4+4+2	4+3 +2+1
$8_2 + 2_2$	3+3+2+2	4+2+2+2	3+ 2+1+...+1
$8_4 + 2_2$	2+2+2+2+2	2+2+2+2+2	2+1+...+1

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