On the stability of flag manifolds

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Abstract

In this paper we derive a second variation of the energy formula for harmonic closed Riemann surfaces into flag manifolds. Then we discuss the stability for a important class of maps with respect to a large class of metrics on $F(n)$ obtained via a perturbation of the Kähler ones.
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§1 Introduction

We study in this note the stability equations for maps (called Eells-Wood) from closed Riemann surfaces into $(F(n), ds^2_{\Lambda=(\lambda_{ij})})$ where $ds^2_{\Lambda=(\lambda_{ij})}$ is a Borel-type metric. The stability properties of the Eells-Wood maps with respect to a very large class of metrics obtained by perturbing the Kähler ones through a precisely defined procedure (6.2 Lemma) is obtained. The content of this paper in particular, extends and corrects some results in my paper [18].

The modern study of harmonic surfaces in Riemannian homogenenous manifolds started with Calabi [8], Chern [9] and Eells [12] and now, after Uhlenbeck [20] is very well understood in the particular case of harmonic 2-spheres into a homogeneous symmetric space. A crucial step was to complexify the problem, and this was done by Eells-Wood [13], Din-Zakarewski [10] and Glaser-Stora [14].

The case where the target manifold is a homogeneous non-symmetric space, like flag manifolds, received considerable less attention. Some of the first works in this direction, were [15] and [18].

Black's book [3] discusses this case, relating this study with the understanding of f-structures on flag manifolds which is very closely connected with Eells-Wood's theorem [13], so this study provides a natural relationship between theory of twistors and harmonic maps into flags. The main interest in this case relies heavily in its connection with symmetric spaces like Grassmannians, as well their similarities with the variational approach to problems in low dimensional topology ([2], [11]).

In this paper we derive:

5.2 Theorem (Second variation of energy)

Let $\phi = (\pi_1, \ldots, \pi_n) : M^2 \longrightarrow (F(n), ds^2_{\Lambda=(\lambda_{ij})})$ be an arbitrary harmonic map.

$$\text{Then } \frac{d^2}{dt^2}\Big|_{t=0} E(\phi_t) = I^{\phi}_{\Lambda}(q) = 4Re \int_M \left\langle q.A^{\Lambda}_z, \frac{\partial q}{\partial z} \right\rangle v_g + 2Re \sum_{ij} \lambda_{ij} \int_M \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g.$$

Using a very algebraic result (6.2. Lemma) we can deduce several stability results:

6.5. Theorem: Let $\psi = (\pi_1, \ldots, \pi_n) : M^2 \to (F(n), J, ds^2_{\Lambda'=(\lambda'_{ij})})$ be an Eells-Wood map, where $\Lambda' = (\lambda'_{ij})$ is the following perturbation of a Kähler metric ($\Lambda = (\lambda_{ij})$:

$$\lambda'_{ij}(i<j) = \begin{cases} \lambda_{ij} \quad \text{if} \quad j = 2, \dots, n-1 \\ \\ \lambda_{ij} + \varepsilon_k, \varepsilon_k \geq 0 \quad \text{for} \quad 1 \leq k \leq \ell = \dfrac{(n-1)(n-2)}{2} \end{cases}$$

Then ψ is stable.

6.7. Theorem. Let $\psi = (\pi_1, \dots, \pi_n) : M^2 \to (F(n), J, ds^2_{\Lambda=(\lambda'_{ij})})$ be an Eells-Wood map. We consider the following perturbation of a Kähler metric $\Lambda = (\lambda_{ij})$:

$$\lambda'_{i_k j_k}(i_k < j_k) = \begin{cases} \lambda_{i_{k_0} j_{k_0}} - \varepsilon_{k_0}, \varepsilon_{k_0} > 0, k_0 \in \left[1, \dfrac{(n-1)(n-2)}{2}\right] \cap I\!N \\ \\ \lambda_{i_k j_k} \quad \text{otherwise} \end{cases}$$

Then ψ is not stable.

Or more generally:

6.8. Theorem. Let $\psi = (\pi_1, \dots, \pi_n) : M^2 \to (F(n), J, ds^2_{\Lambda'=(\lambda'_{ij})})$ be an Eells-Wood map. Furthermore we consider the perturbation $\Lambda' = (\lambda'_{ij})$ of a Kähler metric $\Lambda = (\lambda_{ij})$ given by:

$$\lambda'_{ij}(i<j) = \begin{cases} \lambda_{ij} \quad \text{if} \quad j = i+1 \\ \\ \lambda_{ij} - \varepsilon_k, \varepsilon_k > 0, 1 \leq k \leq \dfrac{(n-1)(n-2)}{2}. \end{cases}$$

Then ψ is not stable.

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§2 Complex differential geometry of $F(n)$

Let $F(n) = \{(L_1, \ldots, L_n); L_i$ is an uni-dimensional subspace of $\mathbb{C}^n, L_i \boxdot L_j, \bigoplus_{i=1}^{n} L_i = \mathbb{C}^n\}$. ALgebraically, $F(n) = \dfrac{U(n)}{T}$ where $U(n) = \{A \in \mathbb{C}_n \cong M(n \times n; \mathbb{C}); A.\overline{A}^t = A.A^* = I\}$ and T is any maximal torus of $U(n)$, i.e. $T = \underbrace{U(1) \times \cdots \times U(1)}_{n \text{ times}}$. Let $p = T(F(n))_{(T)} = F(n)_{(T)}$ where $u(n) = \{X \in \mathbb{C}_n; X + X^* = 0\} \cong p \oplus \underbrace{u(1) \oplus \cdots \oplus u(1)}_{n \text{ times}}$.

2.1. Definition: $J : p \to p$ such that $J^2 = -I$ is called an almost complex structure.

Using the natural $U(n)$ action on $F(n)$, we see according to Borel-Hirzebruch [4] that there are $2^{\binom{n}{2}}$ such that invariant almost complex structures.

For each invariant J we have naturally associated a tournament τ_J with n players $\{1, \ldots, n\}$. More precisely: we consider

$$J([a_{ij}] = (a'_{ij}) \qquad \text{where}$$

$i \to j (i < j) \;\Leftrightarrow\; a'_{ij} = \sqrt{-1} a_{ij}$ or

$i \leftarrow j (i < j) \;\Leftrightarrow\; a'_{ij} = -\sqrt{-1} a_{ij}$.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{C}^n$ and $E_i =$ the subspace of $\mathbb{C}^n$ generated by e_i. We have:

2.2. Lemma: $u(n)^{\mathbb{C}} \cong gl(n, \mathbb{C}) \cong \mathbb{C}_n$.

Proof: Clearly $u(n)^{\mathbb{C}} \subseteq \mathbb{C}_n$. Reciprocally, given $X \in \mathbb{C}_n$, we can find $A, B \in u(n)$ such that $X = A + \sqrt{-1}B$. For this it is enough to take $A = \dfrac{X - X^*}{2}$ and $B = \dfrac{X + X^*}{2\sqrt{-1}}$.

Now we have:

$$\begin{aligned} u(n)^{\mathbb{C}} &\cong \mathbb{C}_n = Hom(\mathbb{C}^n, \mathbb{C}^n) \cong \overline{\mathbb{C}}^n \otimes \mathbb{C}^n = \\ &\cong (\overline{E}_1 \oplus \cdots \oplus \overline{E}_n) \otimes (E_1 \oplus \cdots \oplus E_n) \cong (\overline{E}_1 \otimes E_1 \oplus \cdots \oplus \overline{E}_n \otimes E_n) \\ &\oplus (\bigoplus_{i<j} \overline{E}_i E_j \oplus \overline{E}_j E_i) \quad \text{where} \quad \overline{E}_i E_j \quad \text{means} \quad \overline{E}_i \otimes E_j \end{aligned}$$

Therefore if $E_j = \{ae_j, a \in \mathbb{C}\}$ so $\overline{E}_j = \{\overline{a}e_j, a \in \mathbb{C}\}$.

Let $D_{ij}^{\mathbb{C}} = \overline{E}_i E_j \oplus \overline{E}_j E_i$. For example in $F(3)$:

$$D_{12}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; a_{12}, a_{21} \in \mathbb{C} \right\}$$

We consider $D_{ij} = D_{ij}^{\mathbb{C}} \cap u(n)$. We have $p^{1,0} \cong \bigoplus_{\substack{i \leftarrow j \\ (i<j)}} \overline{E}_i E_j$ and $p^{0,1} \cong \bigoplus_{\substack{i \rightarrow j \\ (i<j)}} \overline{E}_j E_i$.

So given $q : M^2 \to u(n)$ we have naturally defined $\dfrac{\partial q}{\partial z} \in T(M)^* \otimes p^{1,0}$ and $\dfrac{\partial q}{\partial \overline{z}} \in T(M)^* \otimes p^{0,1}$.

We will now study the $U(n)$-invariant metrics on $F(n)$ (they are called Borel type metrics).

Let $A, B \in F(n)_{(T)}$. We define: $ds^2_{\Lambda=(\lambda_{ij})}(A, B) := \sum_{i,j} \lambda_{ij} tr(E_i A E_j B^*)$, where $\lambda_{ij} = \lambda_{ji} > 0, \lambda_{ii} = 0, \forall i, j$. We notice that $\lambda_{ij} = 1, \forall i \neq j$ is just the metic induced on $F(n)$ by the Killing form of $U(n)$.

Borel in [4] described precisely the set of invariant Kahler metrics which are up to permutation, given by the following symmetric matrix:

$$\Lambda = \begin{pmatrix} 0 & \lambda_1 & \lambda_1+\lambda_2 & \cdots & & \lambda_1+\cdots\lambda_{n-1} \\ & 0 & \lambda_2 & \lambda_2+\lambda_3 & \cdots & \cdot \\ & & 0 & \lambda_3 & & \\ & * & & & & \\ & & * & \cdot & & \cdot \\ & * & & & \cdot & \cdot \\ & & & & & \\ & & * & & \cdot & \\ & & & & 0 & \lambda_{n-1} \\ & & & & & 0 \end{pmatrix}$$

§3 f-holomorphic maps

We suppose that $h : M^2 \to \mathbb{C}P^{n-1}$ is a non-degenerate holomorphic map. See [13] for more details. Let h be given locally by $u(z) = (u_0(z), u_1(z), \dots, u_{n-1}(z))$, then as it is usual we can define the $k-th$ associate curve of h, called o_k by $o_k : M^2 \to G_{k+1}(\mathbb{C}^n)$ given by: $o_k(z) = u(z) \wedge u'(z) \wedge \cdots \wedge u^{(k)}(z)$. We can prove that o_k is well-defined. Then we consider $h_k : M^2 \to \mathbb{C}P^{n-1}$ as: $h_k = o_k^{\perp}(z) \cap o_{k+1}(z)$. Therefore we have naturally defined maps $\psi : M^2 \to F(n)$ called Eells-Wood maps which are given by:

$$\psi(z) = (h_0(z), h_1(z), \dots, h_{n-1}(z))$$

3.1. Definition. *An* f-structure on $F(n)$ is a section $\mathcal{F}$ of End $(T(F(n)))$ such that $\mathcal{F}^3 + \mathcal{F} = 0$.

3.2. Theorem ([17]). There is a 1-1 correspondence between $U(n)$ invariant f-structures $\mathcal{F}$ on $F(n)$ and $n \times n$ skew-symmetric matrices $\varepsilon(\mathcal{F}) = (\mathcal{F}_{ij})$ with values in the set $\{1, 0, -1\}$ such that rank $\mathcal{F} = \sum_{\mathcal{F}_{ij}=1} (1)^{i+j} = \sum_{\mathcal{F}_{ij}=-1} (1)^{i+j} = \frac{1}{2} \sum_{\mathcal{F}_{ij}\neq 0} (1)^{(i+j)}$

and, $\mathcal{F}$ is an almost complex structure if and only if, $\mathcal{F}_{ij} \neq 0 \quad \forall i \neq j$, where rank $\mathcal{F}$ is defined as the dimension of the eigenspace associated to the eigenvalue $\sqrt{-1}$ of $\mathcal{F}$.

Therefore $F(n)$ has $3^{\binom{n}{2}}$ invariant f-structures.

Now let M^2 be as usual a closed Riemann surface with local complex coordinate z and $\phi : M \to F(n)$. Set $A'_{ij} = \pi_j \circ \dfrac{\partial}{\partial z} \circ \pi_i$ where π_i denotes the orthogonal projection onto E_i. When $i \neq j, A_{ij}$ is called a second fundamental form of ϕ.

3.3. Remark. The notation of the second fundamental is the same as in the paper [6] by Burstall-Salomon.

3.4. Definition. A map $\phi : M \to F(n)$ is said to be subordinate to an ε-matrix (ε_{ij}) if $A'_{ij} = 0$ whenever $\varepsilon_{ij} \neq 1, i \neq j$. We recall that $\varepsilon_{ij} \in \{-1, 0, 1\}$ for any $i \neq j$.

Similarly to [19 , pg 90] we can see that $\phi : M \to F(n)$ is f-homorphic relative to an invariant f-structure $\mathcal{F}$ on $F(n)$ if and only if, it is subordinate to $\varepsilon(\mathcal{F})$.

3.5. Definition. An invariant f-structure $\mathcal{F}$ on $F(n)$ with the property $[\mathcal{F}_+, \mathcal{F}_-] \subset \underbrace{u(1) \oplus \cdots \oplus u(1)}_{n \text{ times}}$ will be called a horizontal f-structure, where

$\mathcal{F}_+ = \sqrt{-1}$ eigenspace

$\mathcal{F}_- = -\sqrt{-1}$ eigenspace

3.6. Definition: $\phi : M \to F(n)$ is said to be an equi-harmonic map if $\phi : (M, g) \to (F(n), ds^2_{\Lambda=(\lambda_{ij})})$ is harmonic for any Borel type metric $ds^2_{\Lambda=(\lambda_{ij})}$.

According Black's theorem [3] we have:

3.7. Theorem ([3]). Suppose that $\phi : M^2 \to F(n)$ is subordinate to an horizontal f-structure. Then $\phi = (\pi_1, \ldots, \pi_n)$ is an equi-harmonic map and each $\pi_j : M^2 \to \mathbb{C}P^{n-1}$ is harmonic for $j = 1, 2, \ldots, n$.

3.8. Corollary ([18]). The Eells-Wood maps $\psi : M^2 \to F(n)$ are equi-harmonic.

Proof: Let $\phi : M^2 \to \mathbb{C}P^{n-1}$ be a full isotropic harmonic map. Then according to [6] its diagram is:

$$\underset{\pi_1}{\bullet} \longrightarrow \underset{\pi_2}{\bullet} \longrightarrow \bullet \cdots \longrightarrow \underset{\pi_n}{\bullet} \longrightarrow 0$$

Hence $\psi = (\pi_1, \ldots, \pi_n) : M^2 \to F(n)$ is subordinate to the horizontal ε-matrix

$$\begin{pmatrix} 0 & 1 & 0 & & \cdots & & 0 \\ -1 & 0 & 1 & & \cdots & & 0 \\ 0 & -1 & 0 & 1 & \cdots & & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & & & & & & 1 \\ 0 & & & & & -1 & 0 \end{pmatrix}$$

Then by 3.7. Theorem we have ψ is equi-harmonic.

§4 Harmonic map equations

Let $\Phi : M^2 \to U(n)$ be the lift map of $\phi : M \to F(n)$, i.e., $\phi = \pi \circ \Phi$ where $\pi : U(n) \to \dfrac{U(n)}{T} = F(n)$ is the natural projection. Let $e_1, \ldots, e_n$ the canonical basis of $\mathbb{C}^n$. We denote by π_j the matrix of the orthogonal projection on E_j, where E_j denotes the subspace of $\mathbb{C}^n$ generated by e_j.

Then $\pi_j : M \to gl(n,\mathbb{C})$ satisfies that $A'_{j_i}(e_1,\ldots,e_n) = (e_1,\ldots,e_n)A_z^{ij}$, where $A_z^{ij} = \pi_i \dfrac{\partial \pi_j}{\partial z}$. For $V \in \Gamma(\phi^*(T(F(n))))$, we set $q = \phi^*\beta(V)$, where $\phi^*\beta : \phi^*(TF(n)) \to M \times u(n)$ is the pull-back of the Maurer-Cartan form.

Define a variation of ϕ by:

$$\phi_t(x) := (\exp(-tq)\Phi)$$

Denote associate objects by $\pi_j(t)$, $A_z^{ij}(t)$, etc. Then we have:

4.1. Lemma a) $\delta\pi_j = \dfrac{\partial}{\partial t}\Big|_{t=0}\pi_j(t) = [\pi_j, q]$.
b) $\dfrac{\partial}{\partial z}[\pi_j, q] = \left[\dfrac{\partial \pi_j}{\partial z}, q\right] + \left[\pi_j, \dfrac{\partial q}{\partial z}\right]$.
c) $\delta(A_z^{ij}) = \dfrac{\partial}{\partial t}\Big|_{t=0} A_z^{ij}(t) = [A_z^{ij}, q] - \pi_i \dfrac{\partial q}{\partial z}\pi_j$.

Proof: a) According to the definition we have $\pi_j = \Phi E_j \Phi^*$, where E_j denotes the $n \times n$ matrix having 1 in the (j,j)-position and zero elsewhere. Therefore

$$\pi_j(t) = e^{-tq}\pi_j e^{tq}$$

Hence $\partial\pi_j = \dfrac{d}{dt}\Big|_{t=0}\pi_j(t) = -q\pi_j + \pi_j q = [\pi_j, q]$.
b) It is clear

c) Using a) and b) we get

$$\begin{aligned}
\delta(A_z^{ij}) &= \frac{\partial}{\partial t}\Big|_{t=0} A_z^{ij}(t) = \frac{\partial}{\partial t}\Big|_{t=0}\left(\pi_i(t)\frac{\partial \pi_j(t)}{\partial z}\right) = \\
&= \left(\frac{\partial}{\partial t}\Big|_{t=0}\pi_i(t)\right)\frac{\partial \pi_j}{\partial z} + \pi_i \frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}\Big|_{t=0}\pi_j(t)\right) = [\pi_i, q]\frac{\partial \pi_j}{\partial z} + \\
&+ \pi_i \frac{\partial}{\partial z}[\pi_j, q] = [A_z^{ij}, q] - \pi_i \frac{\partial q}{\partial z}\pi_j,
\end{aligned}$$

where we notice that $\pi_i\pi_j = 0$ whenever $i \neq j$.

The Hermitian inner product on $gl(n,\mathbb{C}) \cong \mathbb{C}_n$ is defined by:

$$\langle A, B\rangle := tr(AB^*), \forall A, B \in \mathbb{C}_b.$$

It is easy to check that $\langle A, B\rangle = \overline{\langle B, A\rangle}, \langle A, [B, C]\rangle = \langle [B^*, A], C\rangle$. In particular $\langle A, B\rangle + \langle B, A\rangle = 2Re\langle A, B\rangle$.

4.2. Lemma. 1) $Re\langle [A^{ji}_{\bar{z}}, A^{ij}_z], q\rangle = 0$.

2) $\langle A^{ij}_z, \pi_i B\pi_j\rangle = \langle A^{ij}_z, B\rangle$

3) $\langle A^{ij}_z, [q, [q, A^{ij}_z]]\rangle = -\langle [q, A^{ij}_z], [q, A^{ij}_z]\rangle$

4) $\left\langle A^{ij}_z, \pi_i \frac{\partial q}{\partial z}\pi_j q\right\rangle = -\left\langle A^{ij}_z q, \pi_i \frac{\partial q}{\partial z}\pi_j\right\rangle$

5) $\left\langle A^{ij}_z, q\pi_i \frac{\partial q}{\partial z}\pi_j q\right\rangle = -\left\langle qA^{ij}_z, \pi_i \frac{\partial q}{\partial z}\pi_j\right\rangle$,

$\forall B \in \mathbb{C}_n, q : M^2 \to u(n)$ where $A^{ij}_{\bar{z}} := \pi_i \frac{\partial \pi_j}{\partial \bar{z}}$.

Proof: 1) We see that $(A^{ji}_{\bar{z}})^* = -A^{ij}_z$, so we have $[A^{ji}_{\bar{z}}, A^{ij}_z]^* = [A^{ji}_{\bar{z}}, A^{ij}_z]$.

Hence $2Re\langle [A^{ji}_{\bar{z}}, A^{ij}_z], q\rangle = \langle [A^{ji}_{\bar{z}}, A^{ij}_z], q\rangle + \langle q, [A^{ji}_{\bar{z}}, A^{ij}_z]\rangle = tr([A^{ji}_{\bar{z}}, A^{ij}_z]q^*) + tr(q[A^{ji}_{\bar{z}}, A^{ij}_z]^*) = -tr([A^{ji}_{\bar{z}}, A^{ij}_z]q) + tr(q[A^{ji}_{\bar{z}}, A^{ij}_z]) = 0$

2) Notice that $\pi_i\pi_j = 0, i \neq j$ and $\pi_i^2 = \pi_i$ so: $\langle A^{ij}_z, \pi_i B\pi_j\rangle = tr(A^{ij}_z \pi_j^* B^* \pi_i^*) = tr\left(\pi_i \frac{\partial \pi_j}{\partial z}\pi_j B^* \pi_i\right) = tr\left(\pi_i \frac{\partial \pi_j}{\partial z}\pi_j B^*\right) = -tr\left(\frac{\partial \pi_i}{\partial z}\pi_j^2 B^*\right) = tr\left(\pi_i \frac{\partial \pi_j}{\partial z} B^*\right) = \langle A^{ij}_z, B\rangle$.

3) $\langle A^{ij}_z, [q, [q, A^{ij}_z]]\rangle = \langle [q^*, A^{ij}_z], [q, A^{ij}_z\rangle = -\langle [q, A^{ij}_z], [q, A^{ij}_z]\rangle$.

4) $\left\langle A^{ij}_z, \pi_i \frac{\partial q}{\partial z}\pi_j q\right\rangle = tr\left(A^{ij}_z q^* \pi_j^* \left(\frac{\partial q}{\partial z}\right)^* \pi_i^*\right) = -tr\left(A^{ij}_z q\pi_j \left(\frac{\partial q}{\partial z}\right)^* \pi_i\right) = -\left\langle A^{ij}_z q, \pi_i \frac{\partial q}{\partial z}\pi_j\right\rangle$.

5) $\left\langle A^{ij}_z q, \pi_i \frac{\partial q}{\partial z}\pi_j\right\rangle = tr\left(A^{ij}_z \pi_j^* \left(\frac{\partial q}{\partial z}\right)^* \pi_i^* q^*\right) = -tr(qA^{ij}_z \pi_j \left(\frac{\partial q}{\partial z}^* \pi_i\right) = -\left\langle qA^{ij}_z, \pi_i \frac{\partial q}{\partial z}\pi_j\right\rangle$.

4.3. Definition. Let $\phi = (\pi_1, \ldots, \pi_n) : (M^2, q) \to (F(n), ds^2_{\Lambda=(\lambda=(\lambda_{ij})})$. We define the energy of ϕ as:

$$E(\phi) := \int_M \sum_{i,j} \lambda_{ij} |A_z^{ij}|^2 v_g$$

We will now compute the Euler-Lagrange equations of our variational problem:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E(\phi_t) &= \int_M \sum_{i,j} \lambda_{ij} \frac{\partial}{\partial t}\Big|_{t=0} |A_z^{ij}(t)|^2 v_g = \\
&= 2Re \int_M \sum \lambda_{ij} \left\langle A_z^{ij}, \frac{\partial}{\partial t}\Big|_{t=0} A_z^{ij}(t) \right\rangle v_g = \\
&= 2Re \int_M \sum \lambda_{ij} \left\langle A_z^{ij}, [A_z^{ij}, q] - \pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle v_g
\end{aligned}$$

so we get $\frac{1}{2} \frac{d}{dt}\Big|_{t=0} E(\phi_t) = I + II$ where

$$\begin{aligned}
I &= Re \int_M \sum \lambda_{ij} \langle A_z^{ij}, [A_z^{ij}, q] \rangle v_g \\
II &= -Re \int_M \sum \lambda_{ij} \left\langle A_z^{ij}, \pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle v_g
\end{aligned}$$

But $I \equiv 0$ due to 4.2. Lemma (part 1). For II, by using Stokes'theorem we get:

$$\begin{aligned}
II &= -Re \int_M \sum \lambda_{ij} \left\langle A_z^{ij}, \frac{\partial q}{\partial z} \right\rangle v_g = \\
&= Re \int_M \sum \lambda_{ij} \left\langle \frac{\partial A_z^{ij}}{\partial z}, q \right\rangle v_g - \\
&- Re \int_M \sum \lambda_{ij} \frac{\partial}{\partial \overline{z}} \langle A_z, q \rangle v_g = \\
&- Re \int_M \left\langle \frac{\partial A_z^{\Lambda}}{\partial \overline{z}}, q \right\rangle v_g,
\end{aligned}$$

where $A_z^{\Lambda} := \sum_{i,j} \lambda'_{ij} A_z^{ij}$. Therefore:

4.4 Proposition. $\phi : (M^2, g) \to (F(n), ds^2_\Lambda)$ is harmonic if and only if, $\dfrac{\partial}{\partial \overline{z}} A^\Lambda_z = 0$ if and only if $\dfrac{\partial A^\Lambda_x}{\partial x} + \dfrac{\partial}{\partial y} A^\Lambda_y = 0$, where

$$A^\Lambda_x := \sum \lambda_{ij} \pi_i \frac{\partial \pi_j}{\partial x}, A^\Lambda_y := \sum \lambda_{ij} \pi_i \frac{\partial \pi_j}{\partial y}$$

Proof: Just apply Nöether's theorem ([1]) to the computations above.

§5 Second variation of energy for maps into $F(n)$

A natural problem is:

"Find for each Borel type metric (or at least for a large number of them) when a harmonic map $\phi = (\pi_1, \ldots, \pi_n) : (M^2, g) \to (F(n), ds^2_{\Lambda = (\lambda_{ij})})$ is stable.

We will see that this question is a very difficult one, and we only will have an interesting answer in the case that $\phi = (\pi_1, \ldots, \pi_n) : (M^2, g) \to (F(n), J, ds^2_{\Lambda = (\lambda_{ij})})$ is an Eells-Wood map. These results extend as well give a correct formula for the second variation of energy as described in [18].

5.1. Lemma. $\left. \dfrac{\partial^2 \pi_i(t)}{\partial t^2} \right|_{t=0} = [[\pi_i, q], q].$

Proof: By definition $\pi_i(t) = e^{-tq} \pi_i e^{tq}$. Then $\dfrac{\partial}{\partial t} \pi_i(t) = -q e^{tq} \pi_i e^{tq} + e^{-tq} \pi_i q e^{tq} \overset{[q, e^{tq}]=0}{=\!=\!=} e^{-tq} [\pi_i, q] e^{tq}$. Hence

$$\left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \pi_i(t) = -q[\pi_i, q] + [\pi_i, q] q = [[\pi_i, q], q].$$

5.2. Theorem (Second variation of energy).

Let $\phi = (\pi_1, \ldots, \pi_n) : (M^2, q) \to (F(n), ds^2_{\Lambda=(\lambda_{ij})})$ an arbitrary harmonic map. Then:

$$\frac{d^2}{dt^2} E(\phi_t)\Big|_{t=0} := I^{\phi}_{\Lambda}(q) = 4Re \int_M \left\langle q A^{\Lambda}_z, \frac{\partial q}{\partial z} \right\rangle v_g +$$
$$+ 2Re \sum_{i,j} \lambda_{ij} \int_M \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g$$

Proof: Using the definition we see that:

$$\frac{d}{dt} E(\phi_t) = 2Re \sum \lambda_{ij} \int_M \left\langle A^{ij}_z(t), \frac{\partial}{\partial t} A^{ij}_z(t) \right\rangle v_g. \qquad \text{Hence :}$$
$$\frac{d^2}{dt^2} E(\phi_t) = 2 \int_M \sum \lambda_{ij} \left\langle \frac{\partial}{\partial t}(A^{ij}_z(t)), \frac{\partial}{\partial t}(A^{ij}_z(t)) \right\rangle +$$
$$+ 2Re \sum \lambda_{ij} \int_M \left\langle A^{ij}_z, \frac{\partial^2}{\partial t^2} A^{ij}_z(t) \right\rangle v_g$$

Therefore:

$$\frac{d^2}{dt^2} E(\phi_t)\Big|_{t=0} = 2 \sum \lambda_{ij} \int_M \left\langle \frac{d}{dt}\Big|_{t=0} A^{ij}_z, \frac{d}{dt}\Big|_{t=0} A^{ij}_z(t) \right\rangle$$
$$+ 2Re \sum \lambda_{ij} \int_M \left\langle A^{ij}_z, \frac{\partial^2}{\partial t^2}\Big|_{t=0} A^{ij}_z(t) \right\rangle v_g = I + II$$

We will analise I and II separately. We begin our study with II. We have:

$$Re \sum \lambda_{ij} \int_M \left\langle A^{ij}_z, \frac{\partial^2}{\partial t^2}\Big|_{t=0} A^{ij}_z(t) \right\rangle v_g =$$
$$= Re \sum \lambda_{ij} \int_M \left\langle A^{ij}_z, \frac{\partial}{\partial t}\Big|_{t=0} \left(\frac{\partial}{\partial t} \left(\pi_i(t) \frac{\partial \pi_j(t)}{\partial z} \right) \right) \right\rangle v_g =$$
$$= Re \sum \lambda_{ij} \int_M \left\langle A^{ij}_z, \frac{\partial}{\partial t}\Big|_{t=0} \left[\left(\frac{\partial \pi_i(t)}{\partial t} \frac{\partial \pi_j(t)}{\partial z} \right) + \right.\right.$$
$$\left.\left. + \; \pi_i(t) \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t} \pi_j(t) \right) \right] \right\rangle v_g =$$

$$= Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}, \frac{\partial^2}{\partial t^2}\Big|_{t=0} \pi_i(t) \frac{\partial \pi_j}{\partial z} + \frac{\partial}{\partial t}\Big|_{t=0} \pi_i(t) \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t}\Big|_{t=0} \pi_j(t) \right) + \right.$$

$$\left. + \pi_i \frac{\partial}{\partial z} \left(\frac{\partial^2}{\partial t^2}\Big|_{t=0} \pi_j(t) \right) \right\rangle v_g =$$

$$= Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}, [[\pi_i, q], q] \frac{\partial \pi_j}{\pi z} + 2[\pi_i, q] \frac{\partial}{\partial z}([\pi_j, q]) + \right.$$

$$\left. + \pi_i \frac{\partial}{\partial z}([[\pi_j, q], q]) \right\rangle v_g = Re \sum \lambda_{ij} \int_M \langle A_z^{ij}, A + B + C \rangle v_g$$

But $A = [[\pi_i, q], q] \frac{\partial \pi_j}{\partial z} = [\pi_i q - q\pi_i, q] \frac{\partial \pi_j}{\partial z} = \pi_i q^2 \frac{\partial \pi_j}{\partial z} - 2q\pi_i q \frac{\partial \pi_j}{\partial z} + q^2 A_z^{ij}$.

On the other hand:

$$B = 2[\pi_i, q] \frac{\partial}{\partial z}([\pi_j, q]) = 2(\pi_i q - q\pi_i) \left(\left[\frac{\partial \pi_j}{\partial z}, q \right] + \right.$$

$$\left. + \left[\pi_j, \frac{\partial q}{\partial z} \right] \right) = 2\pi_i q \frac{\partial \pi_j}{\partial z} q - 2\pi_i q^2 \frac{\partial \pi_j}{\partial z} + 2\pi_i q \pi_j \frac{\partial q}{\partial z} -$$

$$- \pi_i q \frac{\partial q}{\partial z} \pi_j - 2q A_z^{ij} q + 2q\pi_i q \frac{\partial \pi_j}{\partial z} + 2q\pi_i \frac{\partial q}{\partial z} \pi_j$$

and

$$C = \pi_i \frac{\partial}{\partial z}([[\pi_j, q], q]) = A_z^{ij} q^2 - 2\pi_i \frac{\partial q}{\partial z} \pi_j q -$$

$$- 2\pi_i q \frac{\partial \pi_j}{\partial z} q - 2\pi_i q \pi_j \frac{\partial q}{\partial z} + \pi_i \frac{\partial}{\partial z}(q^2) \pi_j + \pi_i q^2 \frac{\partial \pi_j}{\partial z}$$

Hence:

$$A + B + C = q^2 A^{ij} - 2q + A_z^{ij} q^2 + \pi_i \frac{\partial}{\partial z}(q^2) \pi_j -$$

$$- 2\pi \frac{\partial q}{\partial z} \pi_j q - 2\pi_i q \frac{\partial q}{\partial z} \pi_j + 2q\pi_i \frac{\partial q}{\partial z} \pi_j =$$

$$= [q, [q, A_z^{ij}]] + \pi_i \frac{\partial q^2}{\partial z} \pi_j - 2\pi_i \frac{\partial q}{\partial z} \pi_j q -$$

$$- 2\pi_i \frac{\partial q}{\partial z} \pi_j + 2q\pi_i \frac{\partial q}{\partial z} \pi_j.$$

Finally:

$$II = 2Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}, [q,[q,A_z^{ij}]] + \pi_i \frac{\partial q^2}{\partial z} \pi_j - \right.$$
$$\left. -2\pi_i \frac{\partial q}{\partial z} \pi_j q - 2\pi_i q \frac{\partial q}{\partial z} \pi_j + 2q\pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle v_g$$

Then if we apply 4.2. Lemma we obtain:

$$II = -2Re \sum \lambda_{ij} \int_M \langle [q, A_z^{ij}], [q, A_z^{ij}] \rangle v_g$$
$$+4Re \sum \lambda_{ij} \int_M \left\langle [A_z^{ij}, q], \pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle +$$
$$+2Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}, \frac{\partial q^2}{\partial z} \right\rangle -$$
$$-4Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}, \pi_i q \frac{\partial q}{\partial z} \pi_j \right\rangle v_g$$

But since ϕ is harmonic, we can use 4.4. Proposition, hence:

$$Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}, \frac{\partial q^2}{\partial z} \right\rangle v_g = Re \int_M \left\langle A_z^{\Lambda}, \frac{\partial q^2}{\partial z} \right\rangle v_g =$$
$$= -Re \int_M \left\langle \frac{\partial}{\partial \overline{z}} (A_z^{\Lambda}), q^2 \right\rangle v_g = 0.$$

Therefore

$$II = -2Re \sum \lambda_{ij} \int_M \langle [q, A_z^{ij}], [q, A_z^{ij}] \rangle v_g$$
$$-4Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}, \pi_i q \frac{\partial q}{\partial z} \pi_j \right\rangle v_g$$
$$+4Re \sum \lambda_{ij} \int_M \left\langle [q, A_z^{ij}, q], \pi \frac{\partial q}{\partial z} \pi_j \right\rangle v_g.$$

On the other hand, by using again 4.2. Lemma we get:

$$I = 2Re \sum \lambda_{ij} \int_M \left\langle [A_z^{ij}, q] - \pi_i \frac{\partial q}{\partial z} \pi_j, [A_z^{ij}, q] - \pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle v_g$$

$$= 2Re \sum \lambda_{ij} \int_M \langle [A_z^{ij}, q], [A_z^{ij}, q] \rangle v_g$$

$$-4Re \sum \lambda_{ij} \int_M \left\langle [A_z^{ij}, q], \pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle v_g$$

$$+2Re \sum \lambda_{ij} \int_M \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g$$

Finally:

$$\frac{d^2}{dt^2} E(\phi_t) \Big|_{t=0} = I_\Lambda^\phi(q) = I + II =$$

$$-4Re \sum \lambda_{ij} \int_M \left\langle A_z^{ij}], q \frac{\partial q}{\partial z} \right\rangle v_g$$

$$+2Re \sum \lambda_{ij} \int_M \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g$$

$$= 4Re \int_M \left\langle q A_z^\Lambda, \frac{\partial q}{\partial z} \right\rangle v_g + 2Re \sum \lambda_{ij} \int_M \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g.$$

§6 Stability on $F(n)$

6.1. Definition. $\Lambda' = (\lambda'_{ij})$ is said to be a perturbation of $\Lambda = (\lambda_{ij})$ associated to a map $\phi = (\pi_1, \ldots, \pi_n) : M^2 \to F(n)$ if:

(i) $\lambda'_{ij} = \lambda_{ij}$ if $(i,j) \neq (i_1, j_1), (j_1, i_1), \ldots, (i_r, j_r)$ and (j_r, i_r)

(ii) $\lambda'_{i_k j_k} = \lambda_{i_k j_k} + \varepsilon_k > 0$ for $1 \leq k \leq r$

(iii) $A_z^{i_1 j_1} = A_z^{j_1 i_1} = \cdots = A_z^{i_r j_r} = A_z^{j_r i_r} = 0$, where $ds^2_{\Lambda = (\lambda_{ij})}$ and $ds^2_{\Lambda' = (\lambda'_{ij})}$ are Borel type metrics.

We will now deduce a key property of the second variation formula, that will be exploited here.

6.2. Lemma. Set $\phi = (\pi_1, \ldots, \pi_n) : M^2 \to F(n)$ an equi-harmonic map. Then:

$$I^{\phi}_{\Lambda'}(q) = I^{\phi}_{\Lambda}(q) + \int_M 2\varepsilon_1 \left(\left| \pi_{i_1} \frac{\partial q}{\partial z} \pi_{j_1} \right|^2 + \left| \pi_{j_1} \frac{\partial q}{\partial z} \pi_{i_1} \right|^2 \right) + \cdots +$$
$$+ \cdots + 2\varepsilon_r \left(\left| \pi_{i_r} \frac{\partial q}{\partial z} \pi_{j_r} \right|^2 + \left| \pi_{j_r} \frac{\partial q}{\partial z} \pi_{j_r} \right|^2 \right) v_g$$

Proof: $I^{\phi}_{\Lambda'}(q) = 4Re \int_M \left\langle q A^{\Lambda'}_z, \frac{\partial q}{\partial z} \right\rangle v_g + 2Re \sum \lambda'_{ij} \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g.$

But $A^{\Lambda'}_z = A^{\Lambda}_z$ since $A^{i_1 j_1}_z = A^{j_1 i_1}_z = \cdots = A^{i_r j_r}_z = A^{j_r i_r}_z = 0$. Hence:

$$\begin{aligned} I^{\phi}_{\Lambda'}(q) &= 4Re \int_M \left\langle q A^{\Lambda}_z, \frac{\partial q}{\partial z} \right\rangle v_g + \\ &+ 2Re \sum_{i,j} \lambda_{ij} \int_M \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g + \\ &+ 2Re \sum_{k=1}^{r} \varepsilon_k \int_M \left(\left| \pi_{i_k} \frac{\partial q}{\partial z} \pi_{j_k} \right|^2 + \left| \pi_{j_k} \frac{\partial q}{\partial z} \pi_{i_k} \right|^2 \right) v_g = \\ &= I^{\phi}_{\Lambda}(q) + 2Re \sum \varepsilon_k \int_M \left(\left| \pi_{i_k} \frac{\partial q}{\partial z} \pi_{j_k} \right|^2 + \left| \pi_{j_k} \frac{\partial q}{\partial z} \pi_{i_k} \right|^2 \right) v_g \end{aligned}$$

6.3. Definition. A harmonic map $\phi : (M^2, q) \to (F(n), ds^2_{\Lambda})$ is said to be stable if $I^{\phi}_{\Lambda}(q) \geq 0$ for any variation $q : M^2 \to u(n)$.

6.4. Theorem ([16]). Let $\phi : (M^2, J_1, q) \to (F(n), J, ds^2_{\Lambda})$ be a holomorphic map between Kähler manifolds. Then ϕ is harmonic and stable.

We will now study the stability of the Eells-Wood maps with respect to a very large set of Borel type metrics.

6.5. Theorem. Let $\psi = (\pi_1, \ldots, \pi_n) : M^2 \to (F(n), J, ds^2_{\Lambda' = (\lambda'_{ij})})$ be an Eells-Wood map. We consider $\Lambda' = (\lambda'_{ij})$ the following, perturbation of a Kähler metric

($\Lambda = (\lambda_{ij})$:

$$\lambda'_{ij}(i<j) = \begin{cases} \lambda_{ij} \quad \text{if} \quad j = 2, \ldots, n-1 \\ \\ \lambda_{ij} + \varepsilon_k, \varepsilon_k \geq 0, \quad \text{for} \quad 1 \leq k \leq \dfrac{(n-1)(n-2)}{2} \end{cases}$$

Then ψ is stable.

Proof: Let J be an almost complex structure in which $\psi : (M, J_1) \to (F(n), J)$ is holomorphic. Then according to 6.2. Lemma we get:

$$I^{\psi}_{\Lambda'=(\lambda'_{ij})},(q) = I^{\psi}_{\Lambda=(\lambda_{ij})}(q) + \int_M \left[2\varepsilon_1 \left(\left| \pi_1 \frac{\partial q}{\partial z} \pi_3 \right|^2 + \left| \pi_3 \frac{\partial q}{\partial z} \pi_1 \right|^2 \right) + \cdots \right.$$
$$\left. + \cdots + 2\varepsilon_\ell \left(\left| \pi_1 \frac{\partial q}{\partial z} \pi_n \right|^2 + \left| \pi_n \frac{\partial q}{\partial z} \pi_1 \right|^2 \right) \right] v_g \geq 0,$$

since according to Lichnerowicz's theorem (6.4. Theorem) $I^{\psi}_{\Lambda}(q) \geq 0$ and every $\varepsilon_k \geq 0$, for $1 \leq k \leq \ell = \binom{n}{2} - (n-1) = \dfrac{(n-1)(n-2)}{2}$.

6.6. Proposition ([19]). Let $\psi = (\pi_1, \ldots, \pi_n) : (M^2, q) \to (F(n), ds^2_{\Lambda=(\lambda_{ij})})$ an Eells. Wood map where ds^2_{Λ} is a Kähler metric. Then there exists a variation $q : M^2 \to u(n)$ such that $I^{\psi}_{\Lambda}(q) = 0$.

Proof: According to Lichnerowicz's theorem (6.4. Theorem) we have:

$$\frac{d^2}{dt^2} E(\psi_t)\bigg|_{t=0} = I^{\psi}_{\Lambda=(\lambda_{ij})} =$$
$$= 8Re \int_M \left\langle q \left(\sum_{i \to j} \lambda_{ij} A^{ij}_z \right), \frac{\partial q}{\partial z} \right\rangle v_g +$$
$$+4Re \sum_{i \to j} \lambda_{ij} \int_M \left\langle \pi_i \frac{\partial q}{\partial z} \pi_j, \frac{\partial q}{\partial z} \right\rangle v_g$$

In order to make things easier to analise we will consider $n = 3$ and τ_J isomorphic to the canonical tournament, i.e.:

$$i \to j \Leftrightarrow j > i$$

We notice that the general case is a straightforward generalization.

Let $q : M^2 \to u(3)$ given by:

$$q = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \text{therefore}$$

$$\frac{\partial q}{\partial z} = \sum_{i \leftarrow j} \pi_i \frac{\partial q}{\partial z} \pi_j = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

Now:

$$q. \left(\sum_{i \to j} \lambda_{ij} A_z^{ij} \right) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix} . \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix}$$

Therefore $\left\langle q. \left(\sum_{i \to j} \lambda_{ij} A_z^{ij} \right), \frac{\partial q}{\partial z} \right\rangle = 0$. Hence $I_\Lambda^\phi(q) = 0 + 0 = 0$.

We are now in position to prove:

6.5. Theorem. Let $\psi = (\pi_1, \ldots, \pi_n) : (M^2, q) \to (F(n), J, ds^2_{\Lambda' = (\lambda'_{ij})})$ and Eells-Wood map. We consider the following perturbation of a Kähler metric $\Lambda = (\lambda_{ij})$:

$$\lambda'_{ij} = \begin{cases} \lambda_{i_{k_0} j_{k_0}} - \varepsilon_{k_0}, \varepsilon_{k_0} > 0 \quad \text{for} \quad i = i_{k_0} \quad \text{and} \quad j = j_{k_0}, \\ k_0 \in \left[1, \ell = \dfrac{(n-1)(n-2)}{2} \right] \cap I\!N \\ \\ \lambda_{ij} \quad \text{otherwise} \end{cases}$$

Then ψ is not stable.

Proof: According to 6.6. Proposition we have $q : M^2 \to u(n)$ such that $I^{\psi}_{\Lambda=(\lambda_{ij})}(q) = 0$.

Now if we apply 6.2. Lemma we obtain:

$$I^{\psi}_{\Lambda'=(\lambda'_{ij})}) = I^{\psi}_{\Lambda=(\lambda_{ij})}(q) +$$
$$+ \int_M [\, 2\varepsilon_1 \, (\left|\pi_{i_1}\frac{\partial q}{\partial z}\pi_{j_1}\right|^2 + \left|\pi_{j_1}\frac{\partial q}{\partial z}\pi_{i_1}\right|^2 + \cdots + (-2\varepsilon_{k_0}) \left(\left|\pi_{i_{k_0}}\frac{\partial q}{\partial z}\pi_{j_{k_0}}\right|^2 + \left|\pi_{j_{k_0}}\frac{\partial q}{\partial z}\pi_{k_0}\right|^2 \right)$$
$$+ \cdots + 2\varepsilon_r \left(\left|\pi_{i_r}\frac{\partial q}{\partial z}\pi_{j_r}\right|^2 + \left|\pi_{j_r}\frac{\partial q}{\partial z}\pi_{j_r}\right|^2 \right) \,]\, v_g < 0$$

if we choose q such that

$$\pi_{i_{k_0}}\frac{\partial q}{\partial z}\pi_{j_{k_0}} \neq 0 \quad \text{or} \quad \pi_{j_{k_0}}\frac{\partial q}{\partial z}\pi_{i_{k_0}} \neq 0.$$

More generally, we can prove:

6.8. Theorem. Let $\psi = (\pi_1, \ldots, \pi_n) : (M^2, g) \to (F(n), ds^2_{\Lambda'=(\lambda'_{ij})})$ be an Eells-Wood map. Furthermore we consider the perturbation $\Lambda' = (\lambda'_{ij})$ given by:

$$\lambda'_{ij}(i<j) = \begin{cases} \lambda_{ij} \quad \text{if} \quad j = i+1 \\ \\ \lambda_{ij} - \varepsilon_k, \varepsilon_k > 0, j \neq i+1, 1 \leq k \leq \frac{(n-1)(n-2)}{2} \end{cases}$$

where $\Lambda = (\lambda_{ij})$ is a Kähler metric. Then ψ is not stable.

Proof: Let q whose existence is assured by 6.6. Proposition, be such that: $\pi_{i_1}\frac{\partial q}{\partial z}\pi_{j_1} \neq 0$ or $\pi_{j_1}\frac{\partial q}{\partial z} \neq 0, \ldots, \pi_{i_1}\frac{\partial q}{\partial z}\pi_{j_r} \neq 0$ or $\pi_{j_r}\frac{\partial q}{\partial z}\pi_{i_r} \neq 0$.

Now we can again apply 6.2. Lemma so we get:

$$I^{\psi}_{\Lambda'=(\lambda'_{ij})}(q) = I^{\psi}_{\Lambda=(\lambda_{ij})}(q) + \int_M [\, -2\varepsilon_1 \left(\left|\pi_{i_1}\frac{\partial q}{\partial z}\pi_{j_1}\right|^2 + \left|\pi_{j_1}\frac{\partial q}{\partial z}\pi_{i_1}\right|^2 \right) +$$

$$+\cdots+-2\varepsilon_r\left(\left|\pi_{i_r}\frac{\partial q}{\partial z}\pi_{j_r}\right|^2+\left|\pi_{j_r}\frac{\partial q}{\partial z}\pi_{i_r}\right|^2\right)]\,v_g<0.$$

6.9. Corollary ([18]). Let $\psi=(\pi_1,\ldots,\pi_n):(M^2,g)\to(F(n),$ Killing metric$)$ be an Eells-Wood map. Then ψ is not stable.

Proof: Just apply 6.8. Theorem for $\lambda_{12}=\lambda_{23}=\cdots=\lambda_{n(n-1)}=1,\varepsilon_1=1,\ldots,\varepsilon_{\ell=\frac{(n-1)(n-2)}{2}}=n-2.$

6.10. Remark: Our result above implies in particular that an Eells-Wood map $\psi=(\pi_1,\pi_2,\pi_3):M^2\to(F(3),$ Killing metric$)$ is not stable. This fact is interesting because in this case the Killing form metric is (1,2)-sympletic so according to Licherowiz's theorem holomorphic maps are harmonic. We can prove that the Killing form metric is not (1,2)-sympletic on $F(n), n\geq 4$.

In [7] was proved that a stable harmonic map $\phi:S^2\to(G/H,h)$ where $(G/H,h)$ is a symmetric space is $\pm$-holomorphic. We finish this note with the following conjecture:

"There exists $\phi=(\pi_1,\ldots,\pi_n):S^2\to(F(n),ds^2_{\Lambda=(\lambda_{ij})})$ stable but not holomorphic for any invariant almost complex structure".

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