Hopf-Zero Bifurcations of Reversible Vector Fields

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Abstract: In this paper we consider symmetric equilibriums of reversible vector fields having eigenvalues $(0, \alpha i, -\alpha i)$. The objective of the paper is to analyze the dynamics of such systems around these critical points. The main results of the paper include a complete list of all normal forms, versal unfoldings, and bifurcation diagrams of codimensional one case. Important conclusions on existence of homoclinic and heteroclinic orbits, invariant tori and symmetric periodic orbits are obtained, too.

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1 Introduction

Let X be a (germ of) C^∞ vector field defined on $\mathbb{R}^n, 0$ given by an ordinary differential equation

$$X: \quad \dot{x} = F(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where $F(x)$ is a smooth function. Assume that X has an equilibrium point at the origin, i.e., $F(0) = 0$. Vector field (1) is called time-reversible if there is a germ of a smooth involution $\phi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ ($\phi \circ \phi = id.$) such that the relation

$$F(\phi(x)) = -\phi'(x) \cdot F(x), \quad x \in \mathbb{R}^n, 0 \tag{2}$$

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holds. Moreover, X is said to be of type (n,k) if the dimension of the fixed point set of φ, $S = Fix\{\phi\}$, is equal to k. It is clear that $0 \le k \le n$.

The objective of this article is to study symmetric critical points of generic one-parameter families of reversible vector fields of type $(3,1)$, with emphasis on topological classification of such systems. Since in $(3,1)$-type case any involution ϕ is C^∞ conjugated to $\phi_0(x,y,z) = (-x,-y,z)$ (see [10]), therefore throughout the paper we assume that the systems considered are ϕ_0 reversible. Remind that in terms of this local coordinate system the fixed point set is given by $S = \{x = 0, y = 0\}$.

Given a ϕ_0 reversible system X, it is easy to see that the 1-jet of X always takes the form $(\mu_1 + az)\frac{\partial}{\partial x} + (\mu_2 + bz)\frac{\partial}{\partial y} + (cx + dy)\frac{\partial}{\partial z}$, where μ_1, μ_2, a, b, c and d are parameters. Since we are interested in the local behavior of a reversible vector field around a symmetric critical point, that is, if $X(0) = 0$ then $0 \in Fix\ \varphi_0$, we assume that μ_1 and μ_2 vanish (for a study of families of vector fields depending on certain parameters, please see [6, 7]). Consequently, we know that in generic case the eigenvalues of such systems are either $(0, \alpha, -\alpha)$ or $(0, \alpha i, -\alpha i)$, where α is a nonzero real parameter. The present article is devoted to the systems of the latter case, i.e., having eigenvalues $(0, \alpha i, -\alpha i)$.

We notice that some study on bifurcation and classification for general (not necessarily reversible) systems having eigenvalues $(0, \pm\alpha i)$ has been carried out (see, for example, [6, 12]). Systems having such eigenvalues but in other contexts, say, divergent free systems, also attracted attentions (see [2]). In particular, one can see shortly that divergent free systems in $\mathbb{R}^3$ in generic case is topologically orbitally equivalent to one specified type of reversible system (see X_4 in equation (11)). In [3] there is an exposition between various contexts.

In the context of reversible vector fields, different types of systems have been examined. In [14] all reversible systems having $(2,1)$ type are classified, in [4] $(2,0)$ type, and in [9] $(3,2)$ type. In each case, topological classification and the respective normal forms of the symmetric singularities of lower codimensions are presented. For applications of $(3,1)$ type reversible vector fields, please refer to [11], where some models from physics and hydrodynamics are discussed.

The techniques employed in [14, 9] are to perform special changes of coordinates around the singularity such that the analysis of the full system can be transfered to a study of the contact between a general system and a codimension 1 submanifold in $\mathbb{R}^n$. These techniques can be generalized to the $(n, n-1)$-type (see [14] for more details). In $(3,1)$-type case, however, these techniques are not applicable any longer and, different methods should be pursued.

The main ideas of the present paper are from [6, 5, 1, 8]. Namely, we perform formal changes of coordinates to reduce X to the resonant normal form, and the latter, as usual, is furthermore rewritten in the form of a cylindrical polar coordinates (together with possible time-rescale). Since in the cylindrical polar coordinates the azimuthal coordinate takes the form $\dot{\theta} = \pm 1$, therefore to analyze the qualitative behavior of such systems we can drop the azimuthal coordinate and only consider the restricted planar systems. Moreover, due to the independence of the polar coordinates on the azimuthal one, this allows us to perform "blowing-ups". On the other hand, since the restricted planar system satisfies an inequality of Lojasiewicz type, therefore by [5] we know that the results concerning the classification of the 2-dimensional singularities of the restricted system hold in C^0 conjugacy category instead of merely C^0 equivalence. To carry out these procedures and to extract the dynamical properties of the original system from the reduced one, we found out, however, that some essential difference and difficulties arise because of the reversibility. Firstly, the reversibility assumption generally imposes certain constraints on the resonant normal form, which in turn results in some degeneracy of the restricted planar vector field. For example, in the reduced system, even when the nonlinear part is generic, all the terms having degree 3 do not appear. This happens typically to the (3,1)-type of reversible vector fields, but it does bring some cumbersome calculation and difficulties in finding the reduced normal forms, and one needs to perform more "blowing-ups". The second difference is more mathematical. In studying the reduced planar vector field, we should take the reversibility of the original system into consideration to determine the dynamics patterns. More precisely, we shall face the center-focus problem in the unfolded planar system (see for instance case $X^{+}_{3,-,\frac{1}{4}}$ in the diagram) and we process as follows. Notice that a pair of singularities in the unfolded system in fact exactly give rise to a symmetric periodic solution of the original system therefore the orbit cannot be an attractor or a repellor. It follows that the singularities must be of center type. This implies that the original system has a family of invariant tori. Similarly, when the unfolded 2D systems admit symmetric singularities of saddle type then we can draw conclusion that the original systems have homoclinic orbits (see for instance case $X^{-}_{3,+,\frac{1}{4}}$ in the diagram) or heteroclinic orbits (see for instance case $X^{-}_{5,+}$ in the diagram). We emphasize that the phenomena described above are due to the reversal symmetry of the original system. In other words, a general vector field (not necessarily reversible) with the same linear part may not enjoy such properties. Therefore the approach employed in the paper is not a simple application of [6, 2].

The main results of this paper include a classification and qualitative behavior of symmetric singularities occurring generically for 1-parameter families of (3,1)-type vector fields. Also from the classification we can draw some interesting conclusions about the existence of invariant tori, periodic orbits, homoclinic and heteroclinic orbits for vector fields considered. In this paper we also give a complete list of bifurcation diagrams.

The rest of the paper is arranged as follows: In section 2 we first present a preliminary and review some important properties of reversible vector fields. Then we give the normal form on which our discussion relies. We exhibit all results in section 3. In section 4 we shall give a brief discussion on the genericity conditions as well as the proof of the results. Section 5 contains the classification of all cases. In this section, we also draw some conclusions about the existence of invariant tori and homoclinic orbits. The last part of the paper illustrates the bifurcation diagrams. We remark that in these diagrams the arrows indicating the direction of the flow of the vector fields do not necessarily reflect the direction of that of the original system. This is because in deriving the orbital C^0 normal form we may multiply a nonvanishing negative function.

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2 Preliminaries

In this section we first clarify some terms and recall the related properties concerning reversible systems. Then we process our objects to the desired forms for further study.

2.1 Basic facts

Let X be a ϕ-reversible system on $\mathbb{R}^n$. A critical point p of X is called symmetric critical point if it lies on S_ϕ, the fixed point set of the involution ϕ.

The following statements are known.

- The phase portrait of X is symmetric with respect to S_ϕ.
- Any periodic orbit γ is symmetric if and only if $S_\phi \bigcap \gamma \neq 0$. If $\gamma(t)$ is a solution of X then so is $\phi\gamma(-t)$.
- Any symmetric critical point or symmetric periodic orbit cannot be an attractor or a repellor.

The criteria of the proposed classification is based on the following definition.

Definition 2.1 *Two vector fields X and Y at their singularities are said to be C^0 conjugated in neighborhoods of the singular points if there is a homeomorphism carrying one singular point into the other and conjugating the local phase flows of the systems at these singular points. They are called orbitally C^0 equivalent if there is a homeomorphism mapping the local phase curves of X into that of Y.*

Definition 2.2 *We say that vector field X defined on $\mathbb{R}^n, 0$ satisfies an inequality of Lojasiewicz type if there exists an integer $k > 0$ and $c > 0$ such that $\|X(x)\| \geq c\|x\|^k$ for all x in a neigbourhood of the singular point.*

The following statements can be deduced from [5].

Theorem 1 *Assume that a smooth vector field X defined on $\mathbb{R}^2, 0$ satisfies an inequality of Lojasiewicz type and has a characteristic orbit. Then the singularity is finitely determined for C^0 conjugacy. Moreover, if two finitely determined singularities on the plane are C^0-equivalent, then they are also C^0 conjugated.*

2.2 Reduced normal form

Let $\mathscr{X}$ be the space of germs (at 0) of C^r ϕ_0 reversible vector fields having eigenvalues $(0, \pm\alpha i)$. Assume that $\mathscr{X}$ is endowed with the C^r-topology, $r > 3$.

Then for any vector field $X \in \mathscr{X}$, due to the relation (2), one can show with some calculation that X takes the form

$$X: \begin{cases} \dot{x} = a_0 z + a_1 x^2 + a_2 y^2 + a_3 z^2 + a_4 xy + \cdots \\ \dot{y} = b_0 z + b_1 x^2 + b_2 y^2 + b_3 z^2 + b_4 xy + \cdots \\ \dot{z} = c_0 x + d_0 y + c_1 xz + c_2 yz + \cdots \end{cases} \tag{3}$$

where a_0, b_0, c_0 and d_0 are constant. Note that in terms of this coordinates, to have eigenvalues of the form $(0, \pm\alpha i)$, the inequality $a_0 c_0 + b_0 d_0 < 0$ must be satisfied. Indeed, it is easy to see that the equality $a_0 c_0 + b_0 d_0 = 0$ leads to three identical eigenvalues, and the corresponds systems are more degenerated, which shall not be the objects of the present paper.

Since under a linear change of coordinates (together with a multiplication of a constant, if necessary), one can always reduce the linear part of the vector field (3) to the form

$$j^1 X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

therefore throughout the paper, we assume that the linear part of (3) has been normalized to the above form.

System (3) can be put into resonant normal form (see [15]). To realize so, we rewrite the involution ϕ_0 in the form $\phi_0(x, \xi) = (-x, -\bar{\xi})$, where $\xi = y + iz$. The resonant normal form means that X takes the form

$$\dot{x} = f(x, r^2), \quad \dot{\xi} = \xi g(x, r^2), \tag{4}$$

where $r^2 = \xi\bar{\xi}$, f is a real function and g a function having complex coefficients. It is easy to see that the reversibility of X leads to the following equalities:

$$f(x, r^2) = f(-x, r^2), \quad \overline{\xi g(x, r^2)} = -\bar{\xi} g(-x, r^2). \tag{5}$$

It follows from these relations that f is even in x and g can be decomposed into the form $g(x, r^2) = x g_1(x^2, r^2) + i(\alpha + g_2(x^2, r^2))$, where $\alpha \neq 0$. In other words,

X has the following form.

$$\begin{cases} \dot{x} = f(x^2, r^2) \\ \dot{y} = z(\alpha + g_2(x^2, r^2)) + yxg_1(x^2, r^2) \\ \dot{z} = -y(\alpha + g_2(x^2, r^2)) + zxg_1(x^2, r^2). \end{cases} \tag{6}$$

Multiplying the above system by a function $h = \frac{1}{\alpha + g_2(x^2, r^2)}$ first and then putting the multiplied system into cylindrical polar system, we have $\dot{x} = \tilde{f}(x^2, r^2)$, $\dot{r} = xrg(x^2, r^2)$, $\dot{\theta} = \pm 1$. Equivalently, the following expansion holds.

$$X: \quad \begin{cases} \dot{x} = \delta_1 x^2 + \delta_2 r^2 + \delta_3 x^4 + \delta_4 r^4 + \alpha x^2 r^2 + \cdots \\ \dot{r} = cxr + \beta x^3 r + \gamma x r^3 + \cdots \\ \dot{\theta} = \pm 1 \end{cases} \tag{7}$$

where $\delta_1 = \{0, 1\}, \delta_2 = \{0, \pm 1\}$, $\delta_3, \delta_4, \alpha, \beta, \gamma$ and c are parameters. Note that this normal form contains no θ-dependent terms. Thus by treating the azimuthal coordinate as time, we arrive at the reduced form

$$\tilde{X}: \quad \begin{cases} \dot{x} = \delta_1 x^2 + \delta_2 r^2 + \delta_3 x^4 + \delta_4 r^4 + \alpha x^2 r^2 + \cdots \\ \dot{r} = cxr + \beta x^3 r + \gamma x r^3 + \cdots \end{cases} \tag{8}$$

Convention: Throughout the paper when we say $X \in \mathscr{X}$ is orbitally topologically equivalent to a 2-dimensional system $\tilde{X}$ of the form (8), we always mean that X has been processed to the cylindrical polar system (7) and there the azimuthal coordinate has been dropped out. In other words, for a given system $X \in \mathscr{X}$ we shall denote by $\tilde{X}$ the corresponding reduced 2D normal form of X. We also introduce the notation $\tilde{\mathscr{X}}$ which stands for the set of $\tilde{X}$ with $X \in \mathscr{X}$. Therefore there is a 1-1 correspondence, i.e., (7) and (8), between $\mathscr{X}$ and $\tilde{\mathscr{X}}$.

3 Statement of the Results

In terms of (8), we introduce the following notation.

$$\begin{aligned} \Sigma_0^1 = & \ \{X \in \mathscr{X} : \delta_1 \neq 0, \ \delta_2 = 1, c \neq 1\}, \\ \Sigma_0^2 = & \ \{X \in \mathscr{X} : \delta_1 \neq 0, \ \delta_2 = -1, c \neq 0, 1\}, \\ \Sigma_0 = & \ \Sigma_0^1 \bigcup \Sigma_0^2. \end{aligned} \tag{9}$$

It is not hard to see that the algebraic conditions described in Σ_0 characterize the genericity conditions. The derivation of these conditions will be explained in the next section. In other words, the following statement is true.

Theorem 2 *Vector fields orbitally equivalent to those of Σ_0 form an open and dense set in $\mathscr{X}$.*

In generic case, the classification and normal forms of such systems are known. For completeness we recall these results. There are slightly different expositions in [13, 6]. The following rearrangement is of more convenience in our further discussion.

Theorem 3 *The set Σ_0 can be divided into the following five subsets $\Sigma_0(i)$, $(i = 1, 2, 3, 4, 5)$, such that any two vector fields belonging to the same subset are orbitally topologically equivalent:*

$$\begin{array}{l} \Sigma_0(1) = \{X \in \mathscr{X} :\ \delta_1 = 1,\ \delta_2 = -1,\ c > 1\} \\ \Sigma_0(2) = \{X \in \mathscr{X} :\ \delta_1 = 1,\ \delta_2 = 1,\ c < 1\} \\ \Sigma_0(3) = \{X \in \mathscr{X} :\ \delta_1 = 1,\ \delta_2 = -1,\ 0 < c < 1\} \\ \Sigma_0(4) = \{X \in \mathscr{X} :\ \delta_1 = 1,\ \delta_2 = -1,\ c < 0,\} \\ \Sigma_0(5) = \{X \in \mathscr{X} :\ \delta_1 = 1,\ \delta_2 = 1,\ c > 1\} \end{array} \tag{10}$$

As to the normal form, we have the following results. Remind the convention described at the end of the previous section.

Theorem 4 *In generic case, any vector field $Y \in \mathscr{X}$ is C^0 orbitally equivalent to one of the normal forms X where*

$$\begin{array}{ll} \tilde{X}_1: & \tilde{X} = (x^2 - r^2, 2xr) \\ \tilde{X}_2: & \tilde{X} = (x^2 + r^2, \frac{1}{2}xr) \\ \tilde{X}_3: & \tilde{X} = (x^2 - r^2, \frac{1}{2}xr) \\ \tilde{X}_4: & \tilde{X} = (x^2 - r^2, -xr) \\ \tilde{X}_5: & \tilde{X} = (x^2 + r^2, 2xr). \end{array} \tag{11}$$

The corresponding normal forms of 3D can be obtained by pulling the above 2D normal forms back to the form $X_i = (\tilde{X}_i, \pm 1)$. Also due to theorem 1 we know that the C^0 orbital equivalence in the above theorem can be improved to C^0 conjugacy if only planar singularities are involved.

When the genericity conditions are violated then we get degenerated vector fields. The codimensional 1 singularity of X means that one (and only one) of the following conditions

$$\delta_1 = 0, \quad \delta_2 = 0, \quad c = 0, \quad \delta_1 = c = 1, \tag{12}$$

is satisfied and at the same time the higher order terms are generic. To give a classification of all codimensional 1 singularities, we need to investigate the higher order terms. We shall show that the topological classification in codimensional 1 case depends on the parameters $\delta_3, \delta_4, \alpha, \beta, \gamma$ and c. Like in the generic case, we introduce the following sets which characterize the topological types of the singularities of the 3D systems. The topological invariance of these algebraic conditions describing these sets will be explained in the following section.

$$\begin{array}{l}
\Sigma_1(1.1) = \{X \in \mathcal{X} : \ \delta_1 = 0, \ \delta_2 = 1, \ \delta_3 = 1, \ c < 0\} \\
\Sigma_1(1.2) = \{X \in \mathcal{X} : \ \delta_1 = 0, \ \delta_2 = 1, \ \delta_3 = -1, \ c < 0\} \\
\Sigma_1(2.1) = \{X \in \mathcal{X} : \ \delta_1 = 0, \ \delta_2 = 1, \ \delta_3 = 1, \ c > 0\} \\
\Sigma_1(2.2) = \{X \in \mathcal{X} : \ \delta_1 = 0, \ \delta_2 = 1, \ \delta_3 = -1, \ c > 0\} \\
\Sigma_1(3.1) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = 1, \ c < 0\} \\
\Sigma_1(3.2) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = -1, \ c < 0\} \\
\Sigma_1(3.3) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = 1, \ 0 < c < \frac{1}{2}\} \\
\Sigma_1(3.4) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = -1, \ 0 < c < \frac{1}{2}\} \\
\Sigma_1(3.5) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = 1, \ \frac{1}{2} < c < 1\} \\
\Sigma_1(3.6) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = -1, \ \frac{1}{2} < c < 1\} \\
\Sigma_1(3.7) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = 1, \ c > 1\} \\
\Sigma_1(3.8) = \{X \in \mathcal{X} : \ \delta_2 = 0, \ \delta_1 = 1, \ \delta_4 = -1, \ c > 1\} \\
\Sigma_1(4.1) = \{X \in \mathcal{X} : \ \delta_1 = c = 1, \ \delta_2 = 1\} \\
\Sigma_1(4.2) = \{X \in \mathcal{X} : \ \delta_1 = c = 1, \ \delta_2 = -1\} \\
\Sigma_1(5.1) = \{X \in \mathcal{X} : \ c = 0, \ \delta_1 = 1, \ \delta_2 = -1, \ \beta + \gamma > 0\} \\
\Sigma_1(5.2) = \{X \in \mathcal{X} : \ c = 0, \ \delta_1 = 1, \ \delta_2 = -1, \ \beta + \gamma < 0\}
\end{array} \tag{13}$$

Denote by Σ_1 the union of the above 16 sets, i.e.,

$$\Sigma_1 := \bigcup \Sigma_1(i,j). \tag{14}$$

Let $\mathscr{X}_1 = \mathscr{X} - \Sigma_0$. Then similar to the generic case, we have the following

Theorem 5 *Vector fields which are orbitally equivalent to those of Σ_1 form an open and dense set in $\mathscr{X}_1$. Σ_1 is a codimensional 1 embedded submanifold of $\mathscr{X}$.*

As to the singularities of generic 1-parameter families of reversible vector fields of $\mathscr{X}$ we prove the following results.

Theorem 6 *1) Two vector fields X and Y in $\mathscr{X}_1$ are C^0 orbitally equivalent if and only if they belong to the same subset $\Sigma_1(i,j)$ of (13).*

2) Any one-parameter family $\tilde{X}^\lambda$, with $X^0 \in \Sigma_1$, in generic case (transversal to Σ_1) is C^0 orbitally equivalent to one of the following 16 normal forms.

$$\begin{array}{rl} \tilde{X}^\lambda_{1,\pm}: & (\lambda x^2 + r^2 \pm x^4, -xr); \\ \tilde{X}^\lambda_{2,\pm}: & (\lambda x^2 + r^2 \pm x^4, xr); \\ \tilde{X}^\lambda_{3,\pm,c}: & (x^2 + \lambda r^2 \pm r^4, cxr); \\ \tilde{X}^\lambda_{4,\pm}: & ((1+\lambda)x^2 \pm r^2, xr); \\ \tilde{X}^\lambda_{5,\pm}: & (x^2 - r^2, \lambda xr \pm xr^3), \end{array} \tag{15}$$

where c takes one of the following values $\{-1, \frac{1}{4}, \frac{3}{4}, 2\}$, and λ is the unfolding parameter.

By recalling theorem 1 we have the following

Corollary 3.1 *Any planar singularity $\tilde{X} \in \tilde{\mathscr{X}}$, where $X \in \Sigma_1$, in generic case is C^0 conjugated to one of the normal form (15) where the unfolding parameter λ is 0.*

Notice that in the theorem we do not require that the normal form $\tilde{X}^\lambda$ is continuous with respect to the parameter. Also remind that the second statement of the theorem does not imply that the corresponding 3D unfoldings $X^\lambda = (\tilde{X}^\lambda, 1)$ are C^0 stable in the space of all one-parameter families of vector fields of $\mathscr{X}$.

In other words, the normal forms $X^0 = (\tilde{X}^0, \pm 1)$, where $\tilde{X}^0$ is from (15) with $\lambda = 0$, only give a topological classification of singularities of 3D systems, not the classification of the unfolding systems.

A list of 2D bifurcation figures are attached at the end of the article.

4 Proof of The Main Theorem

4.1 Comments on the generic case

It is known from [13] that the system (8) in generic case is 2-determined with respect to C^0 conjugacy. Below we verify that the algebraic conditions described in (9) coincide with the genericity conditions of this system in our terminology. In other words, if one of these conditions in (12) is violated then (8) is not generic. To show this, consider the 2-jet of $\tilde{X}$: $\dot{x} = \delta_1 x^2 + \delta_2 r^2$, $\dot{r} = cxr$, where δ_1 and δ_2 have been rescaled so that $\delta_1 = 0, 1$, and $\delta_2 = 0, \pm 1$. This system, after once blowing-up under $x = \rho\cos\theta, r = \rho\sin\theta$, can be put into the following form

$$\begin{cases} \dot{\rho} = \rho(\delta_1 \cos^3\theta + (\delta_2 + c)\cos\theta \sin^2\theta) \\ \dot{\theta} = -\delta_2 \sin^3\theta + (c - \delta_1)\sin\theta\cos^2\theta. \end{cases} \tag{16}$$

Note that for generic δ_2 and $(c - \delta_1)$ one has $\delta_2(c - \delta_1) \neq 0$. If $\delta_2(c - \delta_1) < 0$ then system (16) has only one singular point $(0, 0)$. In this case it is easy to see that the blown-up system is hyperbolic if and only if $\delta_1 \neq 0$. If $\delta_2(c - \delta_1) > 0$ then system (16) has three singular points $(0, 0)$, $(0, \pm arctan\sqrt{\frac{c-\delta_1}{\delta_2}})$. A little calculation shows that in this case system (16) is hyperbolic at all these singular points if and only if $c \neq 0$. Rearranging these relations, we verified the genericity conditions specified in (9).

4.2 Codimension 1 case

¿From the previous discussion we know that if one of the conditions (12) is broken then the vector field is degenerated. In this part we first precise the conditions under which the system is of codimension 1. This means that certain genericity conditions should be imposed on the coefficients of other terms. Obviously, we need only to consider the following four cases.

1. $\delta_1 = 0,\ \delta_2 \neq 0,\ c \neq 0;$
2. $\delta_2 = 0,\ \delta_1 \neq 0,\ c \neq 0, 1;$
3. $\delta_1 = c = 1,\ \delta_2 \neq 0;$
4. $c = 0,\ \delta_1 \neq 0,\ \delta_2 \neq 0;$

Because of the similarity of these cases, in what follows we only treat one case in more details. In all the other cases we shall primarily focus on the difference from this one.

Taking case (1), we consider the following items: the genericity conditions imposed on the higher order terms, the unfolding of the system, and the bifurcation of the unfolded system. Remember in all the 2D cases, the discussion should be in the category of C^0 conjugacy, not the orbital equivalence, this is because we have treated the azimuthal coordinate as time. In fact one can check that all the calculation below abides by this rule.

Let $\delta_1 = 0$, $\delta_2 = \pm 1, c \neq 0$. Note that in this case we can always scale x, thus preserve time, such that $\delta_2 = 1$ and δ_3 takes one of the following values: $0, 1, -1$. Therefore, the equation (8) takes the following form

$$\begin{cases} \dot{x} = r^2 + \delta_3 x^4 + \delta_4 r^4 + \alpha x^2 r^2 + \cdots \\ \dot{r} = cxr + \beta x^3 r + \gamma x r^3 + \cdots \end{cases} \tag{17}$$

where $c \neq 0$ and the dots denote the terms of degrees higher than 4.

It is a straightforward exercise to check that the above system is of codimension 1 in $\tilde{\mathscr{X}}$ if and only if $\delta_3 \neq 0$. If $\delta_3 \neq 0$ then according to signs of δ_3 and c we can divide (17) into four different cases: $c > 0, \delta_3 = \pm 1$; $c < 0, \delta_3 = \pm 1$. Performing blowing-ups, one can show that these four cases are topologically different from each other.

Below we prove that we can choose $\tilde{X}^{\lambda}_{1,\pm}$ and $\tilde{X}^{\lambda}_{2,\pm}$ (see 15) as the corresponding unfoldings.

First we clarify the equivalence of two 1-parameter families of vector fields. We say $X^{\lambda} \sim Y^{\mu}$, if there exists a function h: $(-\epsilon, \epsilon) \to (-\epsilon, \epsilon)$, where ϵ is small and $h(\lambda) = \mu$, such that X^{λ} is conjugated to Y^{μ}.

Let $\tilde{X}_{\mu}$ be an unfolding such that $\tilde{X}_0$ is in one of the first four sets of (13). Assume that, say, $\tilde{X}_{\mu_0} \in \Sigma_1(1,1)$. Then $\delta_1(\mu_0) = 0$, and for any μ_1 and μ_2 such that $(\mu_1 - \mu_0)(mu_2 - \mu_0) < 0$, one has $\tilde{X}_{\mu_1}, \tilde{X}_{\mu_2} \in \Sigma_0$, and $\tilde{X}_{\mu_1}$ is topologically different from $\tilde{X}_{\mu_2}$. This fact implies that for $\mu_1 > \mu_0$ (resp. $\mu_1 < \mu_0$) there are only two possibilities: $\delta_1 > 0$ (resp. $\delta_1 < 0$), or $\delta_1 < 0$ (resp. $\delta_1 > 0$). Take the first case (the other possibility can be treated exactly in the same way). Then for any $\mu > \mu_0$ one has $\tilde{X}_{\mu} \in \Sigma_0(i)$ and $\mu < \mu_0$ one has $\tilde{X}_{\mu} \in \Sigma_0(j)$, where $i \neq j$. The reparameterizing $\mu(\lambda) = \lambda - \mu_0$ gives us the unfolding $\tilde{X}^{\lambda}_{1,+}$.

As to the validity of Theorem 5 it is sufficient to notice that Σ_1 given by (14) is a submanifold of codimensional 1 of $\mathscr{X}$. The proof of this fact is omitted here.

Next we shall briefly discuss the remaining cases, specifying the primary difference from the previous case.

In case (2), i.e., $\delta_2 = 0,\ \delta_1 = 1,\ c \neq 0, 1$, we can rescale x and r such that δ_4 takes one of the values $0, \pm 1$. In this case, when blowing-up the corresponding system, one can see that if $c = \frac{1}{2}$ then the vector field considered shall have

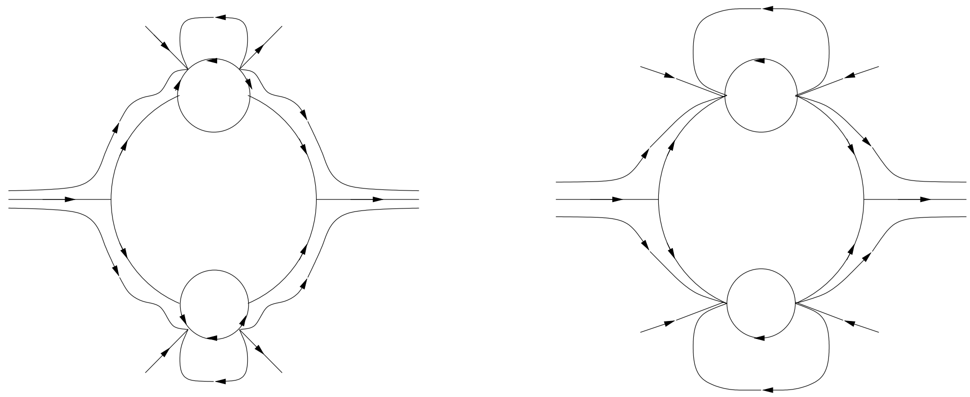


Figure 1: Blow-ups of cases $\tilde{X}^0_{3,-,\frac{1}{4}}$ and $\tilde{X}^0_{3,-,\frac{3}{4}}$

higher codimension. Correspondingly, we have eight subcases due to all the possible combinations between $\delta_4 = \pm 1$ and c lies in $(-\infty, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$ and $(1, \infty)$. Since c takes values from these four sets, consequently, there is no modality in the classification. The unfolding is given by $\tilde{X}^\lambda_{3,\pm,c}$ (see 15).

Remark 4.1 *Remind that in the attached bifurcation figures, the phase portraits of the cases* $\tilde{X}^\lambda_{3,-,\frac{1}{4}}$ *and* $\tilde{X}^\lambda_{3,-,\frac{3}{4}}$ *seem to be identical. Their blowing-ups at* $\lambda = 0$*, however, show that these two cases are topologically different. The two blow-ups are shown in figure 1.*

Case (3), i.e., $\delta_1 = c = 1, \delta_2 \neq 0$, can be treated in a similar way. The unfoldings are given by $\tilde{X}^\lambda_{4,\pm}$ (see 15).

To study case (4), i.e., $c = 0,\ \delta_1 \neq 0,\ \delta_2 \neq 0$, one only to consider the higher order terms of $\dot{r}$. The essential difference between this case and the previous ones is that in this case one needs to show the invariance of $(\beta + \gamma)$, which can be done by blowing-up the system considered. According to the signs of $(\beta + \gamma)$ we have two subcases. The unfoldings are given by $\tilde{X}^\lambda_{5,\pm}$ (see 15).

5 Symmetric periodic orbits, invariant tori, homoclinic and heteroclinic orbits

In this section based on the classification of the 2D systems obtained in the previous sections, we shall give an analysis of the original 3D systems, specifying the existence of symmetric periodic orbits, homoclinic orbits, invariant tori and other important properties.

5.1 Symmetric periodic orbits and invariant tori

Consider the classification and the unfoldings of the reduced 2D system. From the bifurcation diagrams we see that in the unfolded systems $\tilde{X}^-_{3,+,-1}$, $\tilde{X}^+_{3,-,\frac{1}{4}}$, $\tilde{X}^+_{3,-,\frac{3}{4}}$, and $\tilde{X}^+_{3,-,\frac{3}{2}}$ the corresponding blown-up systems are of center-focus type. In each case, we have a pair of singularities in the unfolded system and we have a center-focus problem. Remind that these two symmetric singularities of the 2D system in fact correspond to a symmetric periodic orbit of the original 3D system. Consequently, the orbit cannot be an attractor or a repellor (see Section 2.1). This implies that the type of the 2D unfolded system is a center. In other words, we have two families of circles centered around the singularities. Therefore the corresponding 3D unfolded system admits a family of invariant tori which link these two families of tori. It is clear that the 3D unfolded system has no singularities on these invariant tori, and the 3D *pull-back* $X = (\tilde{X}_{\cdot}, \pm 1)$, where $\tilde{X}_{\cdot}$ takes the 2D normal form with $\lambda = 0$, only gives a classification of 3D singularities of the systems, not the 3D unfoldings.

Besides the cases mentioned above there are other cases where symmetric periodic orbits exist: $\tilde{X}^-_{3,+,\frac{1}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{2}}$ $\tilde{X}^+_{3,-,-1}$, $\tilde{X}^-_{5,+}$ and $\tilde{X}^+_{5,-}$.

5.2 Homoclinic and Heteroclinic orbits

In the 2D bifurcation diagram we see that in the cases $\tilde{X}^-_{3,+,\frac{1}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{2}}$ and $\tilde{X}^+_{3,-,-1}$ the unfolded systems also have a pair of symmetric singularities. These singularities, however, are of saddle type. Moreover, one can see that except the case $\tilde{X}^+_{3,-,-1}$ in all the other cases in a neighborhood of γ there exist a family of homoclinic orbits tending to 0.

The other cases where homoclinic orbits exist are $\tilde{X}^+_{1,+}$, $\tilde{X}^0_{1,-}$, $\tilde{X}^+_{1,-}$, $\tilde{X}^-_{3,-,\frac{1}{4}}$, $\tilde{X}^0_{3,-,\frac{1}{4}}$, $\tilde{X}^-_{3,-,\frac{3}{4}}$, $\tilde{X}^0_{3,-,\frac{3}{4}}$, $\tilde{X}^\lambda_{3,-,\frac{3}{2}}$, $\tilde{X}^\lambda_{4,-}$, $\tilde{X}^-_{5,+}$ and $\tilde{X}^0_{5,+}$

In a similar way, we know that in the cases $\tilde{X}^-_{5,+}$ and $\tilde{X}^+_{5,-}$ the unfolded 2D problem has two pairs of symmetric singularities. For example, from case $\tilde{X}^+_{5,-}$ we deduce that for positive λ, the unfolding parameter, $\tilde{X}^\lambda_{5,-} \in \tilde{\mathfrak{X}}$ has a loop connecting the two periodic orbits and the symmetric equilibrium point P_λ. Moreover, there is at P_λ a one parameter family of heteroclinic orbits.

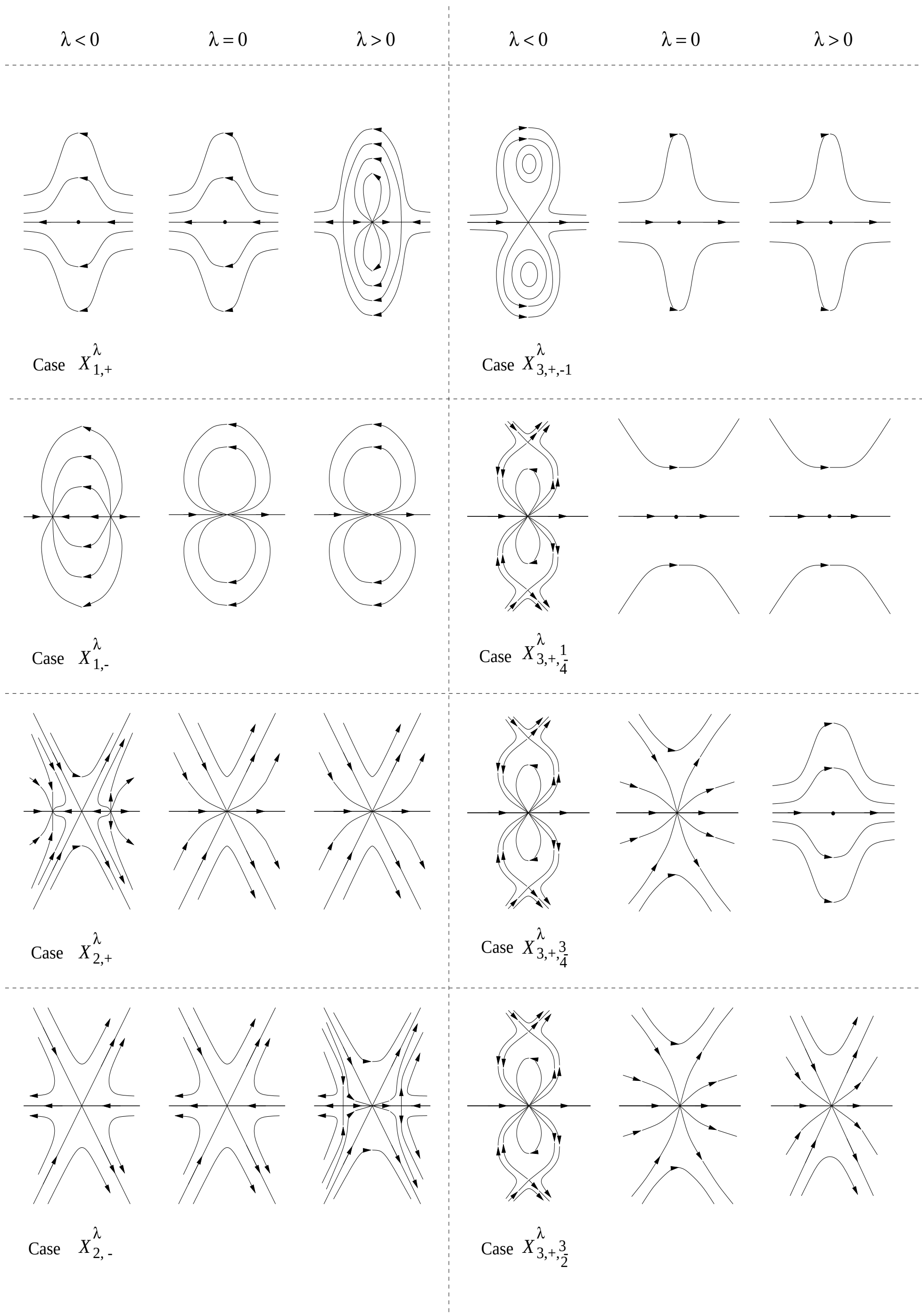
In the case $\tilde{X}^-_{1,-}$ there are also a family of heteroclinic orbits.

Note that the 3D systems corresponding to $\tilde{X}^+_{5,-}$ possess homoclinic as well as heteroclinic orbits.

References

[1] G. Belitskii, Equivalence and normal forms of germs of smooth mappings, Russian Math. Surveys **33**, 107-177, 1978.

[2] H. W. Broer, Quasi periodic flow near a codimension one singularity of a divergence free vector field in dimension three, Lecture Notes in Math. **898**, Springer, Berlin, 1981.



$\lambda < 0$
$\lambda = 0$
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$\lambda = 0$
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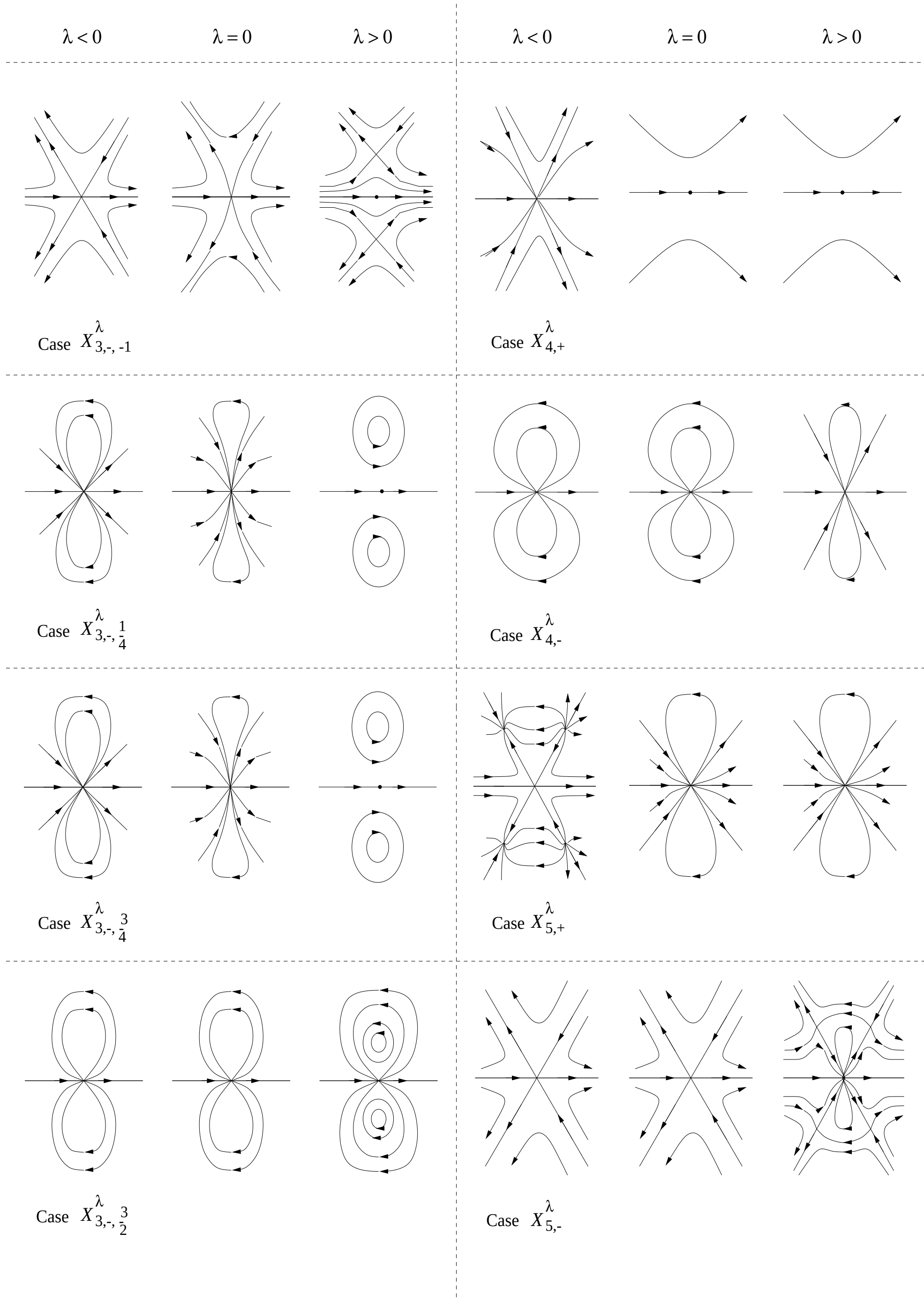
Case $X^{\lambda}_{1,+}$
Case $X^{\lambda}_{3,+,-1}$
Case $X^{\lambda}_{1,-}$
Case $X^{\lambda}_{3,+,\frac{1}{4}}$
Case $X^{\lambda}_{2,+}$
Case $X^{\lambda}_{3,+,\frac{3}{4}}$
Case $X^{\lambda}_{2,-}$
Case $X^{\lambda}_{3,+,\frac{3}{2}}$


$\lambda < 0$
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Case $X^{\lambda}_{3,-,-1}$
Case $X^{\lambda}_{4,+}$
Case $X^{\lambda}_{3,-,\frac{1}{4}}$
Case $X^{\lambda}_{4,-}$
Case $X^{\lambda}_{3,-,\frac{3}{4}}$
Case $X^{\lambda}_{5,+}$
Case $X^{\lambda}_{3,-,\frac{3}{2}}$
Case $X^{\lambda}_{5,-}$

[3] H. W. Broer, G. B. Huitema and M. B. Sevryuk, *Quasi-periodic motions in families of dynamical systems: order amidst chaos*, Lecture Notes in Math. **1645**, Springer, Berlin, 1996.

[4] C. Buzzi, Generic one-parameter families of reversible vector fields, in Real and complex singularities, Editors: J.W. Bruce and F. Tari, Research Notes in Mathematics, **V. 412**, Chapman and Hall/CRC, 202-214, 2000.

[5] F. Dumortier, Singularities of vector fields, *Monografias de Matematica,* **No. 32**, IMPA (Brazil), 1978.

[6] J. Guckenheimer and P. Holmes, Nonlinear Oscillations. Dynamical Systems, and Bifurcations of vector fields, Springer-Verlag, 1983.

[7] F. Khechichine, Familles génériques à quatre paramètres de champs de vecteurs quadratiques dans le plan: Singularité à partie linéaire nulle, Ph.D thesis, U. of Bourgogne, 1991.

[8] J. F. Mattei and M. A. Teixeira, Vector fields in the vicinity of a circle of critical points, *Trans. Am. Math. Soc.*, **V. 297**, No. 1, 1986.

[9] J. C. R. Medrado and M. A. Teixeira, Symmetric singularities of reversible vector fields in dimension three, *Phys. D*, **V. 112** , 122-131, 1998.

[10] D. Montgomery and L. Zippin, Topological transformations groups, *Interscience, New York*, 1995.

[11] J. A. G. Roberts and G. R. W. Quispel, Chaos and time-reversal symmetry. Order and chaos in reversible dynamical systems. *Phys. Rep.*, **216**, 63-177, 1992.

[12] M. B. Sevryuk, Reversible systems, *Lecture Notes in Math.* **1211**, 1986.

[13] F. Takens, Normal Forms for Certain Singularities of Vector Fields, *Ann. Inst. Fourier*, **23**, 163-195, 1973.

[14] M. A. Teixeira, Singularities of reversible vector fields, *Physica D*, **100**, 101-118, 1997.

[15] J. Yang, Polynomial normal forms of vector fields, *Dr. Thesis, Technion-Israel*, 1997.

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