A Note on The Degeneracy of Planar Vector Fields

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Abstract

In this note we discuss the relation between the degeneracy of a given planar vector field and the normalizing processes. With respect to various equivalence relations (smooth conjugacy, formal conjugacy, smooth orbital equivalence, and formal orbital equivalence), we present the corresponding normal forms and classifications.

1 Introduction

Consider a planar vector field given by an ordinary differential equation

$$X: \quad \dot{x} = ax + by + f(x,y), \quad \dot{y} = cx + dy + g(x,y), \tag{1}$$

where at least one of a, b, c and d is different from 0, and where f and g are smooth (C^∞) functions in a neighborhood of (0,0) with neither free nor linear parts.

We begin with the following definitions.

Definition 1.1 *System (1) is said to be smoothly orbitally equivalent (SOE) to the system*

$$Y: \quad \dot{u} = a_1 u + b_1 v + f_1(u,v), \quad \dot{v} = c_1 u + d_1 v + g_1(u,v) \tag{2}$$

if there are a nonzero smooth function $F(x,y)$ and a smooth change of coordinates

$$\Phi: \quad x = \alpha u + \beta v + \xi(u,v), \quad y = \gamma u + \theta v + \eta(u,v), \tag{3}$$

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where ξ and η have neither free nor linear parts, such that Φ reduces the vector field $F \cdot X$ to Y.

If the relations above hold only in the formal category, then system (1) is said to be formally orbitally equivalent (FOE) to system (2).

Definition 1.2 *System (1) is said to be smoothly conjugated (SC) to (2) if there exists a smooth change of coordinates Φ reducing (1) to (2).*

Formal conjugacy (FC) between (1) and (2) can be defined in a similar way.

It is clear that SOE implies FOE and SC results in FC.

Since the linear part of system (1) is always SOE to one of the following systems:

$$
\begin{aligned}
L_1: &\quad \dot{x} = x + y, \quad \dot{y} = y; \\
L_2: &\quad \dot{x} = \alpha x + y, \quad \dot{y} = -x + \alpha y; \\
L_3: &\quad \dot{x} = \lambda x, \quad \dot{y} = y; \\
L_4: &\quad \dot{x} = y, \quad \dot{y} = 0,
\end{aligned}
\tag{4}
$$

therefore throughout the paper we assume that the linear part of system (1) has been normalized to one of the above linear parts (4).

To classify all the planar systems with a fixed linear part, one needs to consider the corresponding nonlinear parts. The latter characterizes the degeneracy of the systems. The main aim of this note is to expose, with respect to the above mentioned categories, the relation between the degeneracy of a given system and its normalization. In turns out that, in certain cases, the categories of the normalizing processes have to become *lower* if the system is too *degenerated*. By examining the corresponding resonant normal form, we can explicitly describe the degeneracy of the system. Consequently, we can obtain the normal form. Also when possible, we shall explain, from geometric point of view, the algebraic conditions describing the genericity conditions.

The motivation of this note is due to [6, 1, 10]. In [6] the normal form of a planar vector field is given with respect to orbital equivalence. It is known that, except the case where both eigenvalues are 0, any planar vector field X around an isolated singular point and with a nonvanishing linear part is orbitally reducible to a polynomial normal form. We note, however, that the normal forms given there are not the simplest ones in the sense that the reduced polynomials are not of the lowest possible degrees. Also in [6] the categories of normalizing processes can be ameliorated, say, from FOE to SOE, in certain cases. we feel that the investigation on the degeneracy of a system helps one to clarify these categories.

The results and the methods involved in this note can be easily generalized to higher dimensions. The primary techniques employed here are mainly developed by [2, 3, 8] and are exhibited in [10] in more details.

We will use the following definitions:

Definition 1.3 *A vector field X* $\dot{x} = Ax + \cdots$ *is called 0-resonant if there is no nontrivial relations of the form* $k_1\lambda_1 + \cdots + k_n\lambda_n = 0$ *between the eigenvalues* $(\lambda_1, \ldots, \lambda_n)$ *of the matrix A, where* $k_1, \ldots k_n$ *are nonnegative integers.*

X is called 1-resonant if all the relations $l_1\lambda_1 + \cdots + l_n\lambda_n = 0$ *are corollaries of a fixed relation* $k_1\lambda_1 + \cdots + k_n\lambda_n = 0$.

Definition 1.4 *A term* $x_1^{k_1} \ldots x_n^{k_n} \frac{\partial}{\partial x_j}$, *where* k_i *are nonnegative integers with the sum* $k_1 + \cdots + k_n \geq 2$, *is called a resonant monomial if*

$$\lambda_j = k_1\lambda_1 + \cdots + k_n\lambda_n. \tag{5}$$

Relation (5) is called a *resonant relation.* A vector field X is called *non-resonant* vector field if the eigenvalues admit no resonant relations.

It is clear that a non-resonant vector field is necessarily 0-resonant, but not vice versa. For 2-dimensional vector fields we have the following cases.

- Vector fields with linear parts L_1, L_2 ($\alpha \neq 0$), L_3 (λ is irrational) are non-resonant.
- Vector fields with linear parts L_3 (if $\lambda = 2, 3, \ldots$) are 0-resonant.
- Vector fields with linear parts L_2 ($\alpha = 0$), L_3 ($\lambda = 0, -p/q$, p, q are positive integers and $(p, q) = 1$) are 1-resonant.
- Vector fields with linear part L_4 are neither 0-resonant nor 1-resonant, since both eigenvalues are 0.

The following statement is known.

Theorem 1 *Any smooth vector field is FC to its resonant normal form, a normal form containing resonant monomials only.*

Any smooth non-resonant vector field X is SC to its linear part.

Any smooth 0-resonant is SC to a polynomial resonant normal form.

Any smooth 1-resonant is SC to a polynomial resonant normal form, provided it does not belong to a set E of infinite codimension in the set of all 1-resonant vector fields.

Any smooth vector field not belonging to the above cases is not FC to a polynomial normal form.

This theorem contains a series of results obtained in the past decades. For a comprehensive exposition and the relations between them, refer to [10].

One sees that if a vector field X is SC to its linear part (a polynomial, resp.) then it is SOE to its linear part (a polynomial. resp.), since one can always choose the multiplying function F to be identical.

From Theorem 1 we see that to give a classification of planar vector fields one needs only to consider those systems which have resonant monomials, this means the following cases: L_2 with $\alpha = 0$, L_3 with $\lambda = 0, -p/q, m$ $(m = 2, 3, \ldots)$, and L_4.

2 Statement of the results

2.1 O-resonant vector fields

We know from Theorem 1 that any 0-resonant vector field is SC to its resonant normal form. The latter is a polynomial. In 2-dimensional case this can happen only when X has a linear part L_3 with $\lambda = m$ $(m = 2, 3, \ldots)$.

Theorem 2 *Any vector field with eigenvalues* $(m, 1)$ $(m = 2, 3, ...)$ *is SC and SOE to the following normal form*

$$\dot{x} = mx + \delta y^m, \quad \dot{y} = y, \tag{6}$$

where $\delta = \{0, 1\}$.

Note that in (6) vector fields with $\delta = 0$ and $\delta = 1$ are not equivalent. A geometric explanation of this observation is that if $\delta = 1$ then the system has exactly one invariant manifold given by $y = 0$, while if $\delta = 0$ then except this one invariant manifold there are infinity many others having the form $x = cy^m$, where c is a parameter. Also note that a vector field with vanishing δ only means that the nonlinear is degenerated not that it is smoothly linearizable.

For vector fields with generic nonlinear part we have $\delta = 1$. Therefore we have the following

Corollary 2.1 *All planar vector fields with eigenvalues* $(m\lambda, \lambda)$ $(m = 2, 3, ...)$ *and generic nonlinear part are SC (SOE) to each other.*

2.2 1-resonant vector fields

The classification of vector fields in this case involves those systems whose linear part is either L_2 with vanishing α or L_3 where $\lambda = 0, -p/q$. We shall treat these cases separately.

Let X be a smooth vector field having a linear part $(y, -x)$. Then the following statements hold.

Theorem 3 *If X does not belong to an exceptional set E of infinite codimension in the set of all 1-resonant vector fields then there exists an integer $k > 0$ such that X is SC to*

$$\dot{x} = y(1+P(Z)) + \epsilon x Z^k + axZ^{2k}, \quad \dot{y} = -x(1+P(Z)) + \epsilon y Z^k + ayZ^{2k}, \tag{7}$$

and is SOE to

$$\dot{x} = y + ayZ^k + \epsilon x Z^k, \quad \dot{y} = -x - axZ^k + \epsilon y Z^k, \tag{8}$$

where $Z = x^2 + y^2$, P is a polynomial of degree k without constant term, $\epsilon = \pm 1$, and a is a parameter.

If X belongs to the exceptional set E then X is FC to

$$\dot{x} = y(1 + F(Z)), \quad \dot{y} = -x(1 + F(Z)), \tag{9}$$

where F is a formal series of Z without constant term, and X is FOE to its linear part $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.

The invariance of the integer k and the description of the exceptional set E will be explained in the following section.

As to vector fields having L_3 linear part, we have two subcases: $L_3(1) = (0, y)$ and $L_3(2) = (-\frac{p}{q}x, y)$.

Theorem 4 *If X has a linear part $(0, y)$ and if 0 is an isolated singular point then there exists an integer k such that X is SC to*

$$\dot{x} = \epsilon x^{k+1} + ax^{2k+1}, \quad \dot{y} = y(1 + P(x)), \tag{10}$$

and is SOE to

$$\dot{x} = \epsilon x^{k+1}, \quad \dot{y} = y + ax^k y, \tag{11}$$

where P is a polynomial of degree k, $\epsilon = 1$ if k is odd, otherwise $\epsilon = \pm 1$.

Theorem 5 *If X has a linear part $(-\frac{p}{q}x, y)$, where p, q are positive integers with $(p,q) = 1$, then it is SC either to a polynomial system*

$$\dot{x} = -\frac{p}{q}x(1+P(Z))+axZ^k+cxZ^{2k}, \quad \dot{y} = y(1+P(Z))+byZ^k+cyZ^{2k}, \tag{12}$$

where P is a polynomial of degree $k-1$ without constant term, a, b, c are constant such that $qa + pb \neq 0$, $Z = x^q y^p$, or to

$$\dot{x} = -\frac{p}{q}x(1+F(Z)), \quad \dot{y} = y(1+F(Z)), \tag{13}$$

where F is a formal series without constant term. Respectively, X is SOE to

$$\dot{x} = -\frac{p}{q}x + axZ^k, \quad \dot{y} = y + byZ^k \tag{14}$$

or FOE to its linear part.

2.3 Nilpotent singularity

From Theorem 1 one knows that vector field having linear part L_4 is *not* FC to a polynomial. This result is proved in [4, 5]. A brief account of normal forms of such vector fields is as follows. In [8] it is proved that an analytic system with L_4 linear part is FC to $\dot{x} = y + ax^n + \cdots$, $\dot{y} = bx^k + \cdots$. In [7] it was proved that the normal form above can be chosen to be analytic. Departing from this normal form, the authors in [9] discussed the analyticity of orbital normal form and the center-focus problem. Certain study concerning FOE was conducted in [6]. In [7], the FOE of the original system is investigated also.

3 The degeneracy and the exceptional set E

Using elementary reasoning, we can give a proof of the theorems described in the previous section. This involves the well known techniques of normalization of vector fields. Due to the Poincare-Dulac theorem (the first statement in Theorem 1), when normalizing a given vector field, one can depart from the so-called resonant normal form. The latter consists of the resonant monomials only.

In two dimensional case, the proper category (smooth or formal) of a normalization process is known (see Theorem 1). Therefore in this section, instead of proving all the lists of normal forms, we shall give a discussion on the genericity and degeneracy of the nonlinear part of a given system. Nevertheless, to illustrate the methods of normalization, we shall give a brief derivation in one case ($L_3(2)$ case).

The degree of degeneracy of vector field is an invariant of the system, which is reflected by the number k. In particular, if k tends to infinity, then an exceptional set E appears and the categories (SC, FC, SOE, FOE) of normalization change. The exceptional set was noticed in [5, 2].

In what follows, we shall give an explicit description of this exceptional set in terms of the resonant normal form. Since this exceptional set happens only to 1-resonant vector fields, below we shall examine with some details the following cases: L_2 $(\alpha = 0)$, L_3 $(\lambda = 0, -\frac{p}{q})$.

It is worthy pointing out that outside the set E the FC and the SC are equivalent and within this codimensional infinity set there exist germs of vector fields that are FC to each other but are not even topologically equivalent, see [2] for more exploration.

3.1 Vector fields with a center-type linear part

The resonant normal form in this case is given by the following form.

$$\dot{x} = y + yf(Z) + xg(Z), \quad \dot{y} = -x - xf(Z) + yg(Z), \tag{15}$$

where $Z = x^2 + y^2$, f and g are formal series.

If g is not identical zero, then there is an integer k such that $g(Z) = aZ^k + \cdots$ $(a \neq 0)$. It is clear that k is uniquely determined by g and it is invariant of the system. This can be explained as follows. The ring of first integrals of the linear approximation of X is generated by $Z = x^2 + y^2$. The function $Zg(Z)$ is the result of application of the vector field X to the function Z.

In generic case, $k = 1$. In this case the original system is SC to a polynomial vector field of degree 5, which is a focus. In fact, for any finite k the origin is of focus-type. The bigger k is the more degenerated the system is. If $k = \infty$, then g is identical 0. In this case, one can not decide the type of the original system. Due to the possible existence of flat functions in the original system, there exist such systems which are formally conjugated but are not topologically equivalent.

In terms of (15) the exceptional set E contains those vector field such that the Taylor expansion of g vanishes. The result can be reformulated as follows: If X is in E then it is SC to a polynomial and the origin is of focus type. If X is outside E then it is FC to $(y(1 + F(Z)), -x(1 + f(Z))$, where F is a formal series, and no information on the type of the origin can be derived from the reduced normal form.

3.2 Vector fields with linear part $(0, y)$

Very similar to the case discussed above, in this case there exists an exceptional set E, too. Because the center-manifold in this case is 1-dimensional, the restriction of X to the center manifold has the form $\dot{x} = f(x)$. In this case, it is known that the exceptional set E contains those vector fields where the Taylor expansion of f above vanishes. Outside E, the formal conjugacy and the smooth one are one thing.

If f is not identical zero, then there exists a uniquely determined k such that $f(x) = ax^k + \cdots$. Therefore the resonant normal form is as follows:

$$\dot{x} = \pm x^k + \cdots, \quad \dot{y} = y + yg(x). \tag{16}$$

The simplified normal form of this system, up to SC, is given by:

$$\dot{x} = \pm x^k + ax^{2k-1}, \quad \dot{y} = y(1 + P(x)), \tag{17}$$

where P is a polynomial of degree $k - 1$.

If X has a generic nonlinear part, then $k = 2$. Consequently, the system is 3-jet determined.

To get the SOE normal form, one can multiply a function to eliminate the terms $x^{2k-1}\frac{\partial}{\partial x}$ and $P(x)y\frac{\partial}{\partial y}$. This multiplication, however, will lead to the appearance of the term $x^k y\frac{\partial}{\partial y}$. The final orbital normal form is (11).

Since we are interested in the systems having an isolated singular point, therefore the Taylor expansion of f is supposed not to vanish. This means that the system is outside the exceptional set. The normal forms we obtained are always in SC and SOE categories.

3.3 Vector fields with linear part $(-\frac{p}{q}x, y)$

Due to the relation $q\lambda_1 + p\lambda_2 = 0$, it is easy to see that any system with such a linear part has infinite many resonant relations

$$\lambda_1 = \lambda_1 + l(q\lambda_1 + p\lambda_2), \ \ \lambda_2 = \lambda_2 + l(q\lambda_1 + p\lambda_2), \tag{18}$$

for any $l = 1, 2, \ldots$. Consequently, the resonant normal form takes the form

$$\dot{x} = x(-\frac{p}{q} + f(Z)), \quad \dot{y} = y(1 + g(Z)), \tag{19}$$

where $Z = x^q y^p$.

To consider the normal form of such systems, denote $F(Z) := qf(Z) + pg(Z)$ (remind that $q\lambda_1 + p\lambda_2 = 0$). Then it is known (see [10]) that if F does not vanish, i.e., there exists an integer k such that $F(Z) = aZ^k + \cdots$, $a \neq 0$, then X is SC to a polynomial. Certain calculation shows that it is SC to system (12).

The degeneracy of X can be described by k, whose invariance can be explained similarly: the ring of first integrals of the linear approximation of X is generated by Z, and the function $ZF(Z)$ is the result of application of the vector field to the function Z.

If $k = \infty$, i.e, F is identically 0, then in this case, the normal form of X is not reducible to a polynomial but to (13). The normalization process, however, can still be chosen to be SC. This is due to the Chen's theorem (see [1]) which states that if two germs of vector fields at a hyperbolic singular point are FC then they are SC.

Below we derive the orbital normal form of X, illustrating some basic methods of normalization. Without loss of generality, we assume that the nonlinear part of X is generic, this means that we assume $k = 1$.

Take the resonant normal form

$$\dot{x} = -\frac{p}{q}x + a_1 xZ + a_2 xZ^2 + \cdots \quad \dot{y} = y + b_1 xZ + b_2 xZ^2 + \cdots \tag{20}$$

where $qa_1 + pb_1 \neq 0$, $Z = x^q y^p$. To obtain the orbital normal form, we need only to normalize the $2(p+q)+1$ jet, since it is already known that the system is SC to

$$\dot{x} = -\frac{p}{q}x + a_1 xZ + axZ^2, \quad \dot{y} = y + b_1 xZ + axZ^2, \tag{21}$$

which means that all the higher order terms can be removed.

To orbitally normalize the $2(p+q)+1$ jet, we first multiply X with the function $F = 1 + c_1 Z$, where $c_1 = \frac{a_2 q + b_2 p}{a_1 q + b_1 p}$, then make a change of coordinates $(x, y) \to (x + \alpha x Z, y + \beta y Z)$, where α, β are parameters to be decided. This change of coordinates will keep the $p+q+1$ jet unchanged while the homogeneous terms of degree $2(p+q)+1$ are transformed to the same terms with different coefficients. The elimination of these terms depends on the solvability of α and β from a homological equation

$$(a_1 - \frac{p}{q}c_1)\beta - (b_1 + c_1)\alpha = \frac{a_2 + a_1 c_1}{p}. \tag{22}$$

It is clear that this equation is solvable because at least one of the numbers $(a_1 - \frac{p}{q}c_1)$ and $(b_1 + c_1)$ is not zero. Therefore we obtain the normal form (14).

The FOE normal form when X is in the set E can be studied in a similar way.

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