Maximal semigroups in semi-simple Lie groups

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Abstract

The maximal semigroups with non-empty interior in a semi-simple Lie group with finite center are characterized as compression semigroups of subsets in the flag manifolds of the group. For this purpose a convexity theory, called here $\mathcal{B}$-convexity, based on the open Bruhat cells is developed. It turns out that a semigroup with nonempty interior is maximal if and only if it is the compression semigroup of the interior of a $\mathcal{B}$-convex set.

Key words: Semigroups, semi-simple Lie groups, flag manifolds, convexity.
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1 Introduction

The purpose of this paper is to characterize the maximal semigroups with nonempty interior in semi-simple Lie groups with finite center. The principal result is Theorem 5.4 which gives a precise description of the maximal semigroups through their actions on the flag manifolds of the group.

When studying semigroups embedded into groups many different questions have a natural formulation and solution by means of the knowledge of the maximal semigroups on a specific group. This makes the problem of determine the maximal semigroups one of the major problems in the theory of semigroups. For semigroups

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in Lie groups J. Lawson [7], appealing to the Levi decomposition of a Lie algebra, divided the task of classifying – or at least understanding – the maximal semigroups, by considering two main classes namely the semigroups of solvable type and those of semi-simple type, according to the kind of Lie group containing them. In order to understand the maximal semigroups in a general Lie group G it is required to have a classification of these two types, and then mix them up in G. In [7] Lawson himself provided a classification of the maximal semigroups with nonempty interior in solvable groups: There is a one-to-one correspondence between the maximal subsemigroups and the half-spaces in the Lie algebra bounded by a hyperplane subalgebra. Thus for solvable groups the maximal semigroups have an algebraic nature. This classification is extended to compact extensions of solvable groups in [7] (see also Hilgert and Neeb [5]), and to semigroups in lattices of solvable groups (see do Rocio and San Martin [10]).

In a semi-simple Lie group G with finite center it was proved in San Martin and Tonelli [13] that any maximal semigroup $S \subset G$ with nonempty interior is a compression semigroup of a subset C of one of the minimal flag manifolds of G:

$$S = S_C = \{g \in G : gC \subset C\}.$$

However in order to have a complete picture of the maximal semigroups in G it is required to find the appropriate family of sets C such that S_C is indeed maximal. In [13] this was made only for the real rank one simple Lie groups. This paper provides the appropriate sets for general semi-simple groups, generalizing the rank one case. The approach is through a convexity theory for subsets of the flag manifolds. Precisely, we say that a subset of a flag manifold is $\mathcal{B}$-convex if it is the intersection of the open Bruhat cells containing it. This notion of convexity is formally defined by a convex hull operator on subsets. This operator in turn comes from a duality operator mapping subsets of a flag manifold into subsets of the dual flag manifold. Once this convexity theory is settled we prove that a semigroup with nonvoid interior in G is maximal if and only if it is the compression semigroup of the interior of a $\mathcal{B}$-convex set in a minimal flag manifold. This same characterization also holds for partial maximal semigroups in the following sense: From [13] we know that there are different classes of semigroups with nonempty interior in a semi-simple Lie group, namely, one class for each flag manifold of the group (see Section 4 below). A partial maximal semigroup (Θ-maximal in the text) is a semigroup which is maximal within the class given by a flag manifold. These partial maximal semigroups are also described by compression and $\mathcal{B}$-convexity, but now on the flag manifolds different from the minimal ones.

Some simple examples show that the $\mathcal{B}$-convex sets may be rather arbitrary subsets. For instance for a real rank one group any subset of the flag manifold is $\mathcal{B}$-convex. Although in general $\mathcal{B}$-convexity may be a stronger property, this shows the existence of a great profusion of nonconjugate maximal semigroups in semi-simple groups, making it hard – if feasible – to have a classification of them. There is anything one can do about this. It is in the realm of the structure of semi-simple Lie groups. However a further development of the theory of $\mathcal{B}$-convex sets may provide decisive tools in the investigation and application of the theory of semigroups.

2 Preliminaries

In this section we set the notations and basic facts about semi-simple Lie algebras and the associated flag manifolds which are used throughout the paper.

Let $\mathfrak{g}$ be a noncompact semi-simple Lie algebra. We first make some standard choices in $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ the associated Cartan decomposition with $\mathfrak{k}$ standing for the subalgebra of θ-fixed points. Select a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ and let Π stand for the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$. For a root $\alpha \in \Pi$ its root space is denoted by $\mathfrak{g}_\alpha$. Choose a simple system of roots $\Sigma \subset \Pi$ and denote by Π^+ the set of positive roots spanned by Σ. We let $\mathfrak{a}^+$ stand for the Weyl chamber associated to Π^+ and

$$\mathfrak{n}^\pm = \sum_{\alpha \in \Pi^\pm} \mathfrak{g}_\alpha$$

for the nilpotent subalgebras associated with Π^+ and $\Pi^- = -\Pi^+$ respectively. Denote by $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

The subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ is the standard minimal parabolic subalgebra of $\mathfrak{g}$. More generally, if $\Theta \neq \Sigma$ is a subset of Σ we denote by $\mathfrak{p}_\Theta$ the parabolic subalgebra

$$\mathfrak{p}_\Theta = \mathfrak{n}^- (\Theta) \oplus \mathfrak{p}.$$

Here $\mathfrak{n}^- (\Theta)$ is the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle \Theta \rangle$, where $\langle \Theta \rangle$ is the set of positive roots generated by Θ. Of course, $\mathfrak{p} = \mathfrak{p}_\emptyset$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We assume always that G has finite center. In this case the subgroup $K = \exp \mathfrak{k}$ is compact. For $g \in G$ and $X \in \mathfrak{g}$ we put $g \cdot X$ for the adjoint action of g in X. The parabolic subgroup P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G:

$$P_\Theta = \{g \in G : g \cdot \mathfrak{p}_\Theta = \mathfrak{p}_\Theta\}.$$

Its Lie algebra is $\mathfrak{p}_\Theta$. The flag manifold $\mathbb{B}_\Theta = G/P_\Theta$ is realized as the set $\{g \cdot \mathfrak{p}_\Theta : g \in G\}$ of parabolic subalgebras conjugate to $\mathfrak{p}_\Theta$. Alternatively, let $\mathfrak{n}_\Theta^+$ stand for the nilpotent radical (nilradical) of $\mathfrak{p}_\Theta$. Explicitly, $\mathfrak{n}_\Theta^+ = \sum_\alpha \mathfrak{g}_\alpha$ with the sum extended through the positive roots outside $\langle\Theta\rangle$. It is well known that the normalizer of $\mathfrak{n}_\Theta^+$ in $\mathfrak{g}$ and G are $\mathfrak{p}_\Theta$ and P_Θ respectively. Hence $\mathbb{B}_\Theta$ is realized also as the subset $\{g \cdot \mathfrak{n}_\Theta : g \in G\}$ of subalgebras conjugate to $\mathfrak{n}_\Theta^+$.

From these standard constructions the set of flag manifolds become parameterized by the proper subsets of the fixed simple system of roots Σ. If $\Theta_1 \subset \Theta_2$ are subsets of Σ then $P_{\Theta_1} \subset P_{\Theta_2}$ so there is a natural fibration $\mathbb{B}_{\Theta_1} \to \mathbb{B}_{\Theta_2}$ given by $gP_{\Theta_1} \to gP_{\Theta_2}$. The maximal flag manifold $\mathbb{B}_\emptyset$ fibers over all $\mathbb{B}_\Theta$. It will be denoted simply by $\mathbb{B}$. We denote these fibrations by π, indistinctly of the specific flag manifolds. If they are to be emphasized the projection is written $\pi_{\Theta_2}^{\Theta_1} : \mathbb{B}_{\Theta_1} \to \mathbb{B}_{\Theta_2}$.

In the sequel it will be required the notion of the flag manifold dual to $\mathbb{B}_\Theta$: Let $\mathcal{W}$ be the Weyl group of G and denote by $w_0 \in \mathcal{W}$ its *principal involution*, that is, the element of maximal length as a product of reflections with respect to the simple roots in Σ. Alternatively w_0 is the only element of $\mathcal{W}$ such that $w_0(\Sigma) = -\Sigma$. It is well known that $w_0 = -\iota$ where ι is an involutive automorphism of the Dynkin diagram associated with Σ. For the sake of simplicity we put $\Theta^* = \iota(\Theta)$, if $\Theta \subset \Sigma$. The flag manifold $\mathbb{B}_{\Theta^*}$ is said to be *dual* to $\mathbb{B}_\Theta$. This notion is independent of the choice of Σ.

Put $N^\pm = \exp \mathfrak{n}^\pm$. The decomposition of $\mathbb{B}_\Theta$ into the N^--orbits is the Bruhat decomposition of $\mathbb{B}_\Theta$. These orbits are given by $N^- w \cdot \mathfrak{p}_\Theta$, with $w \in \mathcal{W}$, so that its number is $|\mathcal{W}/\mathcal{W}_\Theta|$ where $\mathcal{W}_\Theta$ stands for the subgroup of $\mathcal{W}$ generated by the reflections with respect to the simple roots in Θ. Just one of these orbits is open and dense in $\mathbb{B}_\Theta$, namely $N^- \cdot \mathfrak{p}_\Theta$. We refer to this orbit as an open (Bruhat) cell in $\mathbb{B}_\Theta$. This open cell has an alternative description through incidence with a nilpotent subalgebra, which will be largely used in the sequel. Let $\mathfrak{c}_\Theta$ be the nilpotent subalgebra spanned by the root spaces complementary to $\mathfrak{p}_\Theta$ in $\mathfrak{g}$:

$$\mathfrak{c}_\Theta = \sum_\alpha \mathfrak{g}_\alpha$$

with the sum extended through the negative roots outside $-\langle\Theta\rangle$. Since the Cartan involution θ takes a root α into $-\alpha$, it follows that $\mathfrak{c}_\Theta = \theta\left(\mathfrak{n}_\Theta^+\right)$. However, $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$ and $\mathfrak{n}^+$ normalizes $\mathfrak{n}_\Theta^+$ hence $\mathfrak{c}_\Theta$ is normalized by $\mathfrak{n}^-$ and thus by N^-.

Lemma 2.1 *For a parabolic subalgebra $\mathfrak{q} \in \mathbb{B}_\Theta$ the following statements are equivalent:*

1. $\mathfrak{q}$ *belongs to the open cell $N^- \cdot \mathfrak{p}_\Theta$,*
2. $\mathfrak{q} \cap \mathfrak{c}_\Theta = 0$ *and*
3. $\mathfrak{n} \cap \mathfrak{c}_\Theta = 0$ *where $\mathfrak{n}$ is the nil radical of $\mathfrak{q}$.*

Proof: Take $w \in \mathcal{W}$ with $w \cdot \mathfrak{p}_\Theta \neq \mathfrak{p}_\Theta$. Since w interchanges root spaces we have that $\dim (w \cdot \mathfrak{p}_\Theta \cap \mathfrak{c}_\Theta) \geq 1$. Now N^- normalizes $\mathfrak{c}_\Theta$. Hence

$$N^- \cdot (w \cdot \mathfrak{p}_\Theta \cap \mathfrak{c}_\Theta) \subset \mathfrak{c}_\Theta$$

therefore any $\mathfrak{q} \in N^- w \cdot \mathfrak{p}_\Theta$ has nontrivial intersection with $\mathfrak{c}_\Theta$. On the other hand if $\mathfrak{q} = n \cdot \mathfrak{p}_\Theta$ with $n \in N^-$ then $\mathfrak{q} \cap \mathfrak{c}_\Theta = 0$ for otherwise $n^{-1} \cdot (\mathfrak{q} \cap \mathfrak{c}_\Theta) = \mathfrak{p} \cap \mathfrak{c}_\Theta$ would have positive dimension. This shows the equivalence between the first two statements.

The last equivalence follows the same way from the fact that $w \cdot \mathfrak{n}_\Theta^+ \cap \mathfrak{c}_\Theta \neq 0$ if $w \cdot \mathfrak{p}_\Theta \neq \mathfrak{p}_\Theta$ (see [14, Prop. 1.1.2.13]). $\square$

In the sequel we say that a subset $\sigma \in \mathbb{B}_\Theta$ is an open cell if $\sigma = g(N^- \cdot \mathfrak{p}_\Theta)$ for some $g \in G$. Of course any such open cell is the open orbit of a group conjugate to N^-. By the above lemma an open cell is realized as the set of parabolic subalgebras $\mathfrak{q} \in \mathbb{B}_\Theta$ which have null intersection with a conjugate of $\mathfrak{c}_\Theta$. Now we recognize the set of conjugates of $\mathfrak{c}_\Theta$ as the flag $\mathbb{B}_{\Theta^*}$ dual to $\mathbb{B}_\Theta$. In fact, since $\mathfrak{c}_\Theta = \theta\left(\mathfrak{n}_\Theta^+\right)$ it is the nilradical of the parabolic subalgebra $\theta(\mathfrak{p}_\Theta)$. Hence the conjugates of $\mathfrak{c}_\Theta$ are in one-to-one correspondence with a flag manifold $\mathbb{B}_{\Theta'}$. To see that $\Theta' = \Theta^*$ observe that the restriction of $w_0\theta$ to $\mathfrak{a}$ is the involution ι. Hence $w_0\theta(\mathfrak{p}_\Theta) = \mathfrak{p}_{\Theta^*}$. This shows that the set of conjugates of $\theta(\mathfrak{p}_\Theta)$ is $\mathbb{B}_{\Theta^*}$ and thus this is the flag manifold of the conjugates of $\mathfrak{c}_\Theta$.

Notation: The set of open Bruhat cells in $\mathbb{B}_\Theta$ is denoted by $\mathcal{B}_\Theta$ and its bijection with $\mathbb{B}_{\Theta^*}$ by $x \in \mathbb{B}_{\Theta^*} \mapsto \sigma_x \in \mathcal{B}_\Theta$. The complement of σ_x is denoted with $\kappa_x = \mathbb{B}_\Theta \setminus \sigma_x$.

It follows from the definitions that if $g \in G$ and $x \in \mathbb{B}_{\Theta^*}$ then $g\sigma_x = \sigma_{gx}$ and $g\kappa_x = \kappa_{gx}$. Also, any projection $\pi : \mathbb{B}_\Theta \to \mathbb{B}_{\Theta'}$ is equivariant so that $\pi(\sigma) \in \mathcal{B}_{\Theta'}$ if σ is an open cell in $\mathbb{B}_\Theta$.

From Lemma 2.1 it follows that if $\mathfrak{p} \in \mathbb{B}_\Theta$ and $\mathfrak{q} \in \mathbb{B}_{\Theta^*}$ then $\mathfrak{p} \in \sigma_{\mathfrak{q}}$ if and only if $\mathfrak{n}(\mathfrak{p}) \cap \mathfrak{n}(\mathfrak{q}) = 0$ where $\mathfrak{n}(\mathfrak{p})$ stands for the nilradical of $\mathfrak{p}$. This implies at once the following statement.

Proposition 2.2 *Let $x \in \mathbb{B}_\Theta$ and $y \in \mathbb{B}_{\Theta *}$. Then $x \in \sigma_y$ if and only if $y \in \sigma_x$.*

In the sequel we say that an element in $\mathfrak{g}$ (respectively in G) is split-regular if it is conjugate to some $H \in \mathfrak{a}^+$ (respectively $h \in A^+ = \exp \mathfrak{a}^+$). More generally, $X \in \mathfrak{g}$ will be said to be Θ-regular if it is conjugate to $H \in \mathrm{cl}\,(\mathfrak{a}^+)$ such that

$$\Theta = \{\alpha \in \Sigma : \alpha\,(H) = 0\}.$$

Analogously, $g \in G$ is Θ-regular if $g = \exp X$ with X a Θ-regular element of the Lie algebra. Of course split-regularity and $\emptyset$-regularity are the same thing. If $h \in G$ is Θ'-regular then for any flag manifold $\mathbb{B}_\Theta$, with $\Theta' \subset \Theta$ there exists $x \in \mathbb{B}_\Theta$ and $\sigma \in \mathcal{B}_\Theta$ with $x \in \sigma$ and such that $h^i y \to x$ for all $y \in \sigma$. When this holds we say that x is the attractor of h and σ its stable manifold. In particular split-regular elements have attractors and stable manifolds in any flag manifold. We denote by $\sigma\,(h)$ the stable manifold of the regular element h.

More generally for a split-regular h its set of fixed-points in the maximal flag manifold $\mathbb{B}$ is in bijection with $\mathcal{W}$. These fixed-points are the same for every element in the Weyl chamber A^+ containing h. Hence each Weyl chamber settles a bijection of $\mathcal{W}$ with a subset of $\mathbb{B}$. Sometimes it is convenient to emphasize which subset of $\mathbb{B}$ is being considered. We do this by putting a subscript A^+. Thus $\mathcal{W}_{A^+}$ stands for the Weyl group viewed as the subset of A^+-fixed points in $\mathbb{B}$. For $h \in A^+$ a fixed-point b is related to some $w \in \mathcal{W}$ under the bijection. When this is the case we say that b is the fixed-point of type w of h.

The following lemma shows that for any pair (x, σ) with $x \in \sigma$ one can find a regular element having x as attractor and σ as stable manifold. It will be used frequently in the study of maximal semigroups.

Lemma 2.3 *Take $\sigma \in \mathcal{B}_\Theta$ and $x \in \sigma$. Then there is a Θ-regular element $h \in G$ such that x is its attractor and $\sigma = \sigma\,(h)$.*

Proof: Let b_0 be the base point of $\mathbb{B}_\Theta = G/P_\Theta$ and $\sigma = N^- b_0$. If $\Theta' \subset \Theta$ then b_0 is the attractor for any Θ'-regular element in the closure of the Weyl chamber A^+ and σ is the stable manifold. Given $x \in \sigma$ there exists $n \in N^-$ such that $x = nb_0$. So that if $h \in \mathrm{cl}A^+$ is Θ'-regular then $h_1 = nhn^{-1}$ has x as attractor and σ as stable manifold. This shows the lemma for this specific σ. Since G is transitive on $\mathcal{B}_\Theta$ the lemma follows by conjugation with arbitrary $g \in G$. □

3 $\mathcal{B}$-convexity

Roughly speaking a subset C of a flag manifold $\mathbb{B}_\Theta$ is said to be $\mathcal{B}$-convex provided C is the intersection of the open Bruhat cells containing it. This concept of convexity is easier to develop with the aid of a convex hull operator on subsets of the flag manifolds and a duality operator * that assigns to a subset C of a flag $\mathbb{B}_\Theta$ a subset C^* of the dual flag manifold $\mathbb{B}_{\Theta^*}$. Precisely,

$$C^* = \{x \in \mathbb{B}_{\Theta^*} : C \subset \sigma_x\}. \tag{1}$$

Of course this duality operator can be defined also for a subset $D \subset \mathbb{B}_{\Theta^*}$ giving rise to its dual $D^* \subset \mathbb{B}_\Theta$. Hence it makes sense to write C^{**}, which is contained in $\mathbb{B}_\Theta$. We put $\mathrm{co}_{\mathcal{B}}(C) = C^{**}$ and call this subset the *$\mathcal{B}$-convex hull* of C.

Accordingly C is said to be *$\mathcal{B}$-convex* if $C = \mathrm{co}_{\mathcal{B}}(C)$.

Following Goodman and Pollack [3] a convex hull operator $\mathrm{co}(\cdot)$ deserving this name must satisfy:

1. $C \subset \mathrm{co}(C)$ for any subset C,
2. $\mathrm{co}(\cdot)$ is the identity on singletons,
3. $\mathrm{co}(\cdot)$ is increasing with respect to inclusion of sets, and
4. $\mathrm{co}(\cdot)$ is idempotent.

Let us discuss briefly these properties for the $\mathcal{B}$-convex hull operator. For the first one we distinguish the cases where C^* is empty or not. Clearly the dual $\emptyset^*$ of the empty set is the whole dual flag manifold so that $\mathrm{co}_{\mathcal{B}}(C) = C^{**} = \mathbb{B}_\Theta$ if $C \subset \mathbb{B}_\Theta$ and $C^* = \emptyset$. Hence $C \subset \mathrm{co}_{\mathcal{B}}(C)$ in this case. On the other hand a nonempty subset C is said to be *admissible* if $C^* \neq \emptyset$, i.e., if $C \subset \sigma_y$ for some $y \in \mathbb{B}_{\Theta^*}$. For an admissible C its $\mathcal{B}$-convex hull is seen to be the intersection of the open cells containing it. In fact, $C^{**} = \{y \in \mathbb{B}_\Theta : C^* \subset \sigma_y\}$. By Proposition 2.2, $x \in \sigma_y$ if and only if $y \in \sigma_x$. Since $C^* \neq \emptyset$, it follows that $y \in C^{**}$ if and only if $y \in \sigma_x$ for all $x \in C^*$. But any Bruhat cell containing C is σ_x for some $x \in C^*$, so that for an admissible subset there is the alternative definition

$$\mathrm{co}_{\mathcal{B}}(C) = \bigcap \{\sigma \in \mathcal{B}_\Theta : C \subset \sigma\}. \tag{2}$$

Of course this implies that $C \subset \mathrm{co}_{\mathcal{B}}(C)$. Furthermore we note that if C is $\mathcal{B}$-convex then either $C = \emptyset, \mathbb{B}_\Theta$ or C is admissible, for otherwise $\mathrm{co}_{\mathcal{B}}(C) = \mathbb{B}_\Theta$.

Since it is irrelevant to our purposes here we do not dwell on the $\mathcal{B}$-convexity of the singletons. We just note that if $x, y \in \mathbb{B}_\Theta$ then there exists $\sigma \in \mathcal{B}_\Theta$ with $x \notin \sigma$ and $y \in \sigma$ so that $\{y\}$ is indeed $\mathcal{B}$-convex. Finally the last two of the above listed properties follow from the following statements about the duality operator:

Proposition 3.1 *For a flag manifold $\mathbb{B}_\Theta$ it holds:*

1. *If $C_1 \subset C_2 \subset \mathbb{B}_\Theta$ then $C_1^* \supset C_2^*$.*

2. *Let $C \subset \mathbb{B}_\Theta$. Then C^* is $\mathcal{B}$-convex in $\mathbb{B}_{\Theta^*}$.*

Proof:

1. Assuming that $C_1 \subset C_2$, take $x \in C_2^*$. Then $C_2 \subset \sigma_x$ so that $C_1 \subset \sigma_x$. This implies that $x \in C_1^*$.

2. If C is not admissible then $C^* = \emptyset, \mathbb{B}_{\Theta^*}$ so that its $\mathcal{B}$-convexity is trivial. Assuming that $C^* \neq \emptyset$ we must check that $C^* = (C^*)^{**}$. The inclusion $C^* \subset (C^*)^{**}$ is equivalent $C^* \subset \mathrm{co}_\mathcal{B}(C^*)$, showed above. On the other hand take $y \in (C^*)^{**}$. Then $x \in \sigma_y$ for every $x \in C^{**}$. In particular $x \in \sigma_y$ for all $x \in C$ because C is contained in C^{**}. But this means that $y \in C^*$, showing that $(C^*)^{**} \subset C^*$.

□

From this proposition we get easily the following properties of the operator $\mathrm{co}_\mathcal{B}(\cdot)$:

Proposition 3.2 *For a flag manifold $\mathbb{B}_\Theta$ it holds:*

1. *If $C_1 \subset C_2$ then $\mathrm{co}_\mathcal{B}(C_1) \subset \mathrm{co}_\mathcal{B}(C_2)$.*

2. *If $C \subset \mathbb{B}_\Theta$ then $\mathrm{co}_\mathcal{B}(C) \subset \mathrm{co}_\mathcal{B}(\mathrm{co}_\mathcal{B}(C))$.*

Proof: The second statement is a consequence of the first one and the inclusion $C \subset \mathrm{co}_\mathcal{B}(C)$. The first property follows immediately from 1. in the previous proposition. □

3.1 Examples

The examples below illustrates that the $\mathcal{B}$-convex sets may be either rather arbitrary sets or sets which resemble the standard convex sets in affine spaces or in Riemannian manifolds.

1. In case $\mathfrak{g}$ is a Lie algebra with real rank one there is just one flag manifold $\mathbb{B}$ which is diffeomorphic to a sphere in some dimension. The Bruhat decomposition of $\mathbb{B}$ has two components the open one and its complement which is a singleton. Thus $\mathcal{B}$ consists of the subsets $\mathbb{B}\setminus\{x\}$, $x \in \mathbb{B}$. Therefore any subset of $\mathbb{B}$ is $\mathcal{B}$-convex.

2. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. The flag manifolds are the standard manifolds of flags of subspaces in $\mathbb{R}^n$. In particular the Grassmannians, including the projective space, are flag manifolds of Lie groups associated with $\mathfrak{sl}(n, \mathbb{R})$. Let us focus attention to the Grassmannians $\mathrm{Gr}_k(n)$ of k-dimensional subspaces of $\mathbb{R}^n$. A direct check at the isotropy subalgebras of the $\mathrm{Sl}(n, \mathbb{R})$-action on the Grasssmannians shows that the dual of $\mathrm{Gr}_k(n)$ is the Grassmannian $\mathrm{Gr}_{n-k}(n)$ of subspaces having complementary dimension. In more concrete terms this duality is given by incidence between k-dimensional and $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. Indeed an open cell is the stable manifold of the attractor for the action of a split regular element h in the group. In the present case h is a diagonalizable matrix in $\mathrm{Sl}(n, \mathbb{R})$ having positive and distinct eigenvalues. If $\{e_1, \ldots, e_n\}$ is a basis of eigenvectors of h then the subspace spanned by $\{e_1, \ldots, e_k\}$ is the attractor of h in $\mathrm{Gr}_k(n)$. Its stable manifold is easily seen to be the open and dense subset of k-dimensional subspaces transversal to $\mathrm{span}\{e_{k+1}, \ldots, e_n\}$. This implies that for each $U \in \mathrm{Gr}_{n-k}(n)$ its associated open cell is
$$\sigma_U = \{V \in \mathrm{Gr}_k(n) : V \cap U = 0\},$$
while κ_U is the set of k-dimensional subspaces meeting U nontrivially. It follows that $\emptyset \neq C \subset \mathrm{Gr}_k(n)$ is admissible if and only if there is a $(n-k)$-dimensional subspace U such that $V \cap U = 0$ for all $V \in C$. Note that as in the case of rank one groups there are rather arbitrary $\mathcal{B}$-convex subsets in the Grassmannians. In fact, for any admissible $D \subset \mathrm{Gr}_{n-k}(n)$, its dual D^* is $\mathcal{B}$-convex in $\mathrm{G}_k(n)$.

   For $k = 1$ we can single out a nice class of $\mathcal{B}$-convex sets, namely the classical convex subsets in the projective space $\mathbb{P}^{n-1}$: Let $W \subset \mathbb{R}^n$ be a pointed convex cone and denote by $\bar{W}$ the set of lines in $\mathbb{P}^{n-1}$ contained in W. Since W is pointed $\bar{W}$ is admissible. Also, W is the intersection of the half-spaces in $\mathbb{R}^n$ containing it. Hence $\bar{W}$ is $\mathcal{B}$-convex in $\mathbb{P}^{n-1}$. Of course not every $\mathcal{B}$-convex set is constructed this way from a convex cone.

3. We continue with $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. Let $\mathbf{r} = (r_1 < \cdots < r_m)$ be a sequence of integers with $1 \leq r_1$ and $r_m \leq n-1$ and denote by $\mathbb{F}(\mathbf{r})$ the manifold of flags

$$(V_1 \subset \cdots \subset V_m)$$

of subspaces of $\mathbb{R}^n$ with $\dim V_i = r_i$. Put $\bar{\mathbf{r}} = (n - r_1 < \cdots < n - r_m)$. Then $\mathbb{F}(\bar{\mathbf{r}})$ is the flag manifold dual to $\mathbb{F}(\mathbf{r})$. As in the Grassmannian case this can be seen either by looking at the isotropy subalgebras or by verifying directly that the open cells are given by incidence between the subspaces in a flag. Indeed, if $U = (U_1 \subset \cdots \subset U_m) \in \mathbb{F}(\bar{\mathbf{r}})$ then

$$\sigma_U = \{(V_1 \subset \cdots \subset V_m) : V_i \cap U_i = 0, i = 1, \ldots, m\}$$

is an open cell in $\mathbb{F}(\mathbf{r})$.

3.2 Topology

Up to this point we have considered $\mathcal{B}$-convexity for arbitrary subsets of the flag manifolds, looking at the incidence of parabolic subalgebras only. Now we consider some topological properties of the duality and $\mathcal{B}$-convex hull operators.

Since a flag manifold $\mathbb{B}_\Theta$ is a homogenous space of G, it is endowed with the quotient topology, rendering it a compact metrizable space. This topology is given also by the embedding $\mathbb{B}_\Theta$ in a Grassmannian, either by identifying it with the subalgebras conjugate to $\mathfrak{p}_\Theta$ or to $\mathfrak{n}_\Theta^+$. Here the topology in a Grassmannian is the standard one. A basic property of this topology is: Let L be a vector space with $\dim L = n$. Denote by $\mathrm{Gr}_k(L)$ the Grassmannian of k-dimensional subspaces of L. Suppose that $\xi_0 \in \mathrm{Gr}_k(L)$ and $\eta_0 \in \mathrm{Gr}_{n-k}(L)$ are transversal, i.e., $\xi_0 \cap \eta_0 = 0$. Then there are neighborhoods $A \ni \xi$ and $B \ni \eta$ in $\mathrm{Gr}_k(L)$ and $\mathrm{Gr}_{n-k}(L)$, respectively, such that $\xi \cap \eta = 0$ for all $\xi \in A$ and $\eta \in B$.

Now recall that an open cell $\sigma_\mathfrak{q}$, $\mathfrak{q} \in \mathbb{B}_{\Theta^*}$, is the set of parabolic subalgebras in $\mathbb{B}_\Theta$ which are transversal to the nilradical $\mathfrak{n}(\mathfrak{q})$ of $\mathfrak{q}$. Since the dimension of $\mathfrak{n}(\mathfrak{q})$ complements the dimension of any $\mathfrak{p} \in \mathbb{B}_\Theta$, the above transversality property implies the

Lemma 3.3 *Let $x_0 \in \mathbb{B}_\Theta$ and $y_0 \in \mathbb{B}_{\Theta^*}$ be such that $y_0 \in \sigma_{x_0}$ and $x_0 \in \sigma_{y_0}$. Then there are neighborhoods $U \ni x$ and $V \ni y$ in $\mathbb{B}_\Theta$ and $\mathbb{B}_{\Theta^*}$ respectively such that $x \in \sigma_y$ and $y \in \sigma_x$, for all $x \in U$ and $y \in V$.*

Another basic property of the topology in the flag manifolds is related to sequences in the complements κ_y of the open cells σ_y:

Lemma 3.4 *Let $y_j \in \mathbb{B}_{\Theta^*}$ be a sequence with $\lim y_j = y$. If $x \in \kappa_y$ then there is a sequence $x_j \in \kappa_{y_j}$ such that $\lim x_j = x$.*

Proof: By transitivity of G in $\mathbb{B}_{\Theta^*}$ there exists a sequence $g_j \in G$ with $g_j \to 1$ and such that $y_j = g_j y$. The required sequence is $x_j = g_j x$. In fact, $x_j \in \kappa_{y_j} = g_j \kappa_y$ and $x_j \to x$. □

From these lemmas we get the following topological properties of the duality operator which are basic for the study of maximal semigroups.

Proposition 3.5 *Suppose that $C \subset \mathbb{B}_\Theta$ is compact and admissible. Then C^* is open.*

Proof: Suppose that $C \neq \emptyset \neq C^*$ and take $x \in C$ and $y \in C^*$. From Lemma 3.3 above there are neighborhoods $U_x \ni x$ and $V_x \ni y$ such that $z \in \sigma_w$ for all $z \in U_x$ and $w \in V_x$. By compactness there is a finite covering

$$C \subset U_{x_1} \cup \cdots \cup U_{x_l}.$$

Then $V = V_{x_1} \cap \cdots \cap V_{x_l}$ is a neighborhood of y contained in C^*. □

Proposition 3.6 *Suppose that $C \subset \mathbb{B}_\Theta$ is admissible and $\mathrm{int}C \neq \emptyset$. Then $\mathrm{cl}\,(C^*) \subset \sigma_x$ for all $x \in \mathrm{int}C$. Hence $\mathrm{cl}\,(C^*) \subset (\mathrm{int}C)^*$ and $\mathrm{cl}\,(C^*)$ is admissible.*

Proof: Take $x \in \mathrm{int}C$. Let $y \in \mathrm{cl}\,(C^*)$ and $y_j \in C^*$ be such that $\lim y_j = y$. We must check that $x \in \sigma_y$. Suppose to the contrary that $x \in \kappa_y$. Then by Lemma 3.4 there is a sequence $x_j \in \kappa_{y_j}$ with $\lim x_j = x$. This implies that $x_j \in \mathrm{int}C$ for large j. But this contradicts the fact that $y_j \in C^* \subset (\mathrm{int}C)^*$. □

Proposition 3.7 *Let $C \subset \mathbb{B}_\Theta$ be open and such that $\mathrm{cl}C$ is admissible. Then C^* is closed and $\mathrm{int}\,(C^*) = (\mathrm{cl}C)^*$.*

Proof: Since C is open, Proposition 3.6 implies that $\mathrm{cl}\,(C^*) \subset C^*$ so that C^* is closed. Furthermore Proposition 3.1 implies that $(\mathrm{cl}C)^* \subset C^*$. But $(\mathrm{cl}C)^*$ is open hence $(\mathrm{cl}C)^* \subset \mathrm{int}\,(C^*)$. For the reverse inclusion suppose that there exists $x \in \mathrm{int}\,(C^*) \setminus (\mathrm{cl}C)^*$. Then $x \in \kappa_y$ for some $y \in \mathrm{cl}C$. Take a sequence $y_j \in C$ such that $\lim y_j = y$. By Lemma 3.4 there exists a sequence $x_j \in \kappa_{y_j}$ with $\lim x_j = x$. Hence, for large j, $x_j \in \mathrm{int}\,(C^*) \subset C^*$ and $x_j \in \kappa_{y_j}$ with $y_j \in C$, which is a contradiction. □

Applying this proposition twice we get the following information about the $\mathcal{B}$-convex hull of a closed subset.

Proposition 3.8 *Let $C \subset \mathbb{B}_\Theta$ be closed admissible subset with $\mathrm{int}C \neq \emptyset$. Then $\mathrm{co}_\mathcal{B}\,(C)$ is closed and has nonempty interior*

$$\mathrm{int}\,(\mathrm{co}_\mathcal{B}\,(C)) = (\mathrm{cl}\,(C^*))^*\,.$$

Proof: Proposition 3.5 implies that C^* is open hence $\mathrm{co}_\mathcal{B}\,(C) = C^{**}$ is closed. The above proposition applied to C^* implies that

$$\mathrm{int}\,(C^{**}) = (\mathrm{cl}\,(C^*))^*\,.$$

This open set is not empty because $\mathrm{cl}\,(C^*)$ is admissible as follows from Proposition 3.6. □

3.3 Invariance

The relevance of $\mathcal{B}$-convexity for semigroups in G stays in the following invariance properties of the dual and the $\mathcal{B}$-convex hull operators.

Proposition 3.9 *Let $g \in G$ and $C \subset \mathbb{B}_\Theta$. Then $(gC)^* = g\,(C^*)$.*

Proof: Take a parabolic subalgebra $\mathfrak{p} \in C^*$ and denote by $\mathfrak{n}$ its nilradical. By definition $\mathfrak{q} \cap \mathfrak{n} = 0$ for every parabolic subalgebra $\mathfrak{q} \in C$. Now $g \cdot \mathfrak{n}$ is the nilradical of $g \cdot \mathfrak{p}$, and

$$g \cdot \mathfrak{q} \cap g \cdot \mathfrak{n} = g \cdot (\mathfrak{q} \cap \mathfrak{n}) = 0$$

if $\mathfrak{q} \in C$. This implies that $g \cdot \mathfrak{p} \in (gC)^*$ and hence that $g\,(C^*) \subset (gC)^*$. Applying this inclusion to gC and g^{-1} we have $g^{-1}\,((gC)^*) \subset C^*$ so that $(gC)^* \subset g\,(C^*)$, concluding the proof. □

Corollary 3.10 *Let $g \in G$ and $C \subset \mathbb{B}_\Theta$ be such that $gC \subset C$. Then $g^{-1}(C^*) \subset C^*$.*

Proof: By Proposition 3.2 $(gC)^* \supset C^*$. Hence by the above proposition $g(C^*) \supset C^*$ which is equivalent to $g^{-1}(C^*) \subset C^*$. □

Corollary 3.11 *Let $g \in G$ and $C \subset \mathbb{B}_\Theta$. Then $g(\mathrm{co}_{\mathcal{B}}(C)) = \mathrm{co}_{\mathcal{B}}(gC)$. Therefore gC is $\mathcal{B}$-convex if C is $\mathcal{B}$-convex.*

Proof: Follows from the proposition and the equality $\mathrm{co}_{\mathcal{B}}(C) = C^{**}$. □

We can state now that the $\mathcal{B}$-convex hull operator maps invariant subsets into invariant subsets. This will be essential in the description of maximal semigroups.

Proposition 3.12 *Let $g \in G$ and $C \subset \mathbb{B}_\Theta$ be such that $gC \subset C$. Then $g(\mathrm{co}_{\mathcal{B}}(C)) \subset \mathrm{co}_{\mathcal{B}}(C)$.*

Proof: Follows immediately from the previous corollary and Proposition 3.2. □

Finally we have the following localization type property of the $\mathcal{B}$-convex sets:

Proposition 3.13 *The family of open $\mathcal{B}$-convex sets is a basis for the topology of $\mathbb{B}_\Theta$.*

Proof: Let $C \subset \mathbb{B}_{\Theta^*}$ be a compact admissible subset with $\mathrm{int} C \neq \emptyset$. From the previous section we know that C^* is open and its closure is admissible. Clearly C^* is an open $\mathcal{B}$-convex set. From it we generate a basis for the open sets in $\mathbb{B}_\Theta$. First take $x \in C^*$ and an open cell $\sigma \supset \mathrm{cl} C^*$. By Lemma 2.3 there exists a split-regular $h \in G$ such that x is its attractor and $\sigma = \sigma(h)$. The sequence h^k contracts σ into x as $k \to +\infty$. Since $\mathrm{cl}(C^*)$ is a compact subset of σ, the contraction is uniform in $\mathrm{cl}(C^*)$. This means that for any neighborhood U of x there exists a $k_0 > 0$ such that $h^k C^* \subset U$ for $k \geq k_0$. This shows that the open $\mathcal{B}$-convex sets form a basis for the neighborhoods of x. The corollary follows then by transitivity of G and the fact that $g \in G$ maps $\mathcal{B}$-convex sets into $\mathcal{B}$-convex sets. □

4 Semigroups

In this section we consider the action on the flag manifolds of semigroups in semi-simple Lie groups. We complement the results of San Martin [11] and San Martin and Tonelli [13], paving the way for the characterization of the maximal semigroups.

4.1 Topological introduction

Before looking at the semigroup actions on the flag manifolds we recall some terminology of topological nature which hold in a more general context. In this subsection we let G be a topological group acting continuously in a compact topological space M. Let $S \subset G$ be a semigroup with $\mathrm{int} S \neq \emptyset$.

Its action on M induces the pre-order relation $x \preceq y$ if $y \in \mathrm{cl}\,(Sx)$, $x, y \in M$. Let $\sim$ be the equivalence relation associated with $\preceq$, namely $x \sim y$ if $x \preceq y$ and $y \preceq x$. The pre-order in M induces a partial order in the quotient $M/\sim$ which is also denoted by $\preceq$.

A *control set* for S in M is an equivalence class D of $\sim$ having the property that there exists $x \in D$ and $g \in \mathrm{int} S$ with $gx = x$. Given a control set D the fixed-point set

$$D_0 = \{x \in D : \exists g \in \mathrm{int} S, gx = x\}$$

is known to be open and dense in D. It is named the *core* or *set of transitivity* of D (see [13]). This second name comes from the fact that for every $x, y \in D_0$ there exists $g \in S$ such that $gx = y$. We denote by $\mathcal{D}\,(S)$ the set of control sets of S. It is partially ordered by $\preceq$ in $M/\sim$. In case M is compact there are *invariant control sets*. These are the control sets which are maximal with respect to $\preceq$. They are closed subsets of M. The same way there are minimal control sets. They are open and coincide with the cores of the invariant control sets of the inverse semigroup $S^{-1} = \{g^{-1} : g \in S\}$.

The domain of attraction $\mathcal{A}\,(D)$ of a control set D is defined by

$$\mathcal{A}\,(D) = \{x \in M : \exists g \in S, gx \in D\}.$$

For a subset C contained in M we denote by S_C its *compression semigroup* in G:

$$S_C = \{g \in G : gC \subset C\}.$$

A quick glance at this expression is enough to show that if $C = \mathrm{cl}\,(\mathrm{int} C)$ then $S_C = S_{\mathrm{int}(C)}$. We refer to Colonius and Kliemann [2] for a detailed development of these concepts in the context of control systems.

4.2 Flag manifolds

We return here to the flag manifold setting with S a semigroup with nonvoid interior in the semi-simple group G.

Consider for a moment the maximal flag manifold $\mathbb{B} = \mathbb{B}_{\emptyset}$. From [13] we know that for each $w \in \mathcal{W}$ there exists a control set $D(w)$ such that $x \in D(w)_0$ if and only x is the w-fixed point for some split-regular $h \in \mathrm{int}S$. Moreover, any control set D is $D(w)$ for some $w \in \mathcal{W}$. The assignment $w \mapsto D(w)$ permits to single out, from S, a flag manifold $\mathbb{B}(S)$ as follows: Take a split-regular $h \in \mathrm{int}S$ and denote by $A^+ = \exp \mathfrak{a}^+$ the Weyl chamber containing h. Recall that we write $\mathcal{W}_{A^+}$ to emphasize the bijection of $\mathcal{W}$ with subsets of $\mathbb{B}$. Let $1 \in \mathcal{W}_{A^+}$ be the identity. Then the control set $D(1)$ is the only invariant control set in $\mathbb{B}$. The same way the control set $D(w_0)$ is the only minimal control set in $\mathbb{B}$ where w_0 is the principal involution of $\mathcal{W}$.

The subset $\mathcal{W}_{A^+}(S) = \{w \in W : D(w) = D(1)\}$ is a parabolic subgroup of $\mathcal{W}_{A^+}$, that is, it is generated by the reflections with respect to the simple roots in a proper subset $\Theta(S) \subset \Sigma$. Here Σ is the simple system of roots associated with $\mathfrak{a}^+$. We put $\mathbb{B}(S) = \mathbb{B}_{\Theta(S)}$. A decisive property of this special flag manifold is that the invariant control set of S on it is an admissible subset, i.e., is contained in open Bruhat cells. Precisely,

Proposition 4.1 *With the above notations let $C \subset \mathbb{B}(S)$ be the invariant control set. Then C is contained in the stable manifold $\sigma(h)$ for any split-regular $h \in \mathrm{int}S$. Moreover if $\Theta \subset \Theta(S)$ and $\pi : \mathbb{B}_\Theta \to \mathbb{B}(S)$ is the canonical fibration then $\pi^{-1}(C)$ is the invariant control set for S in $\mathbb{B}_\Theta$.*

Proof: See Proposition 4.8 and Theorem 4.3 in [13]. □

In the sequel we say that the semigroup is of *type* Θ if $\Theta(S) = \Theta$, i.e., $\mathbb{B}(S) = \mathbb{B}_\Theta$. We emphasize that any proper semigroup with nonempty interior is of type Θ for some Θ. Furthermore if $S \subset T$ are semigroups with nonempty interior then any control set of S is contained in just one control set of T, and T is of type $\Theta' \supset \Theta$ if S is of type Θ.

Another information provided by the subgroup $\mathcal{W}_{A^+}$ concerns the number of control sets in the flag manifold $\mathbb{B}_\Theta$. It is given by the order of the double coset space $\mathcal{W}_{A^+} \backslash \mathcal{W} / \mathcal{W}_\Theta$, where $\mathcal{W}_\Theta$ is the parabolic subgroup generated by the reflections in Θ.

For a semigroup of type Θ its invariant control set in $\mathbb{B}_\Theta$ is an admissible subset which is the closure of its interior. The next proposition complements this statement by showing that every subset of $\mathbb{B}_\Theta$ having these properties is the invariant control set of some semigroup of type Θ.

Proposition 4.2 *Suppose that the admissible subset $C \subset \mathbb{B}_\Theta$ satisfies $C = \mathrm{cl}\,(\mathrm{int}C)$. Then the compression semigroup*

$$S_C = \{g \in G : gC \subset C\}$$

has nonempty interior. Moreover C is the invariant control set of S_C in $\mathbb{B}_\Theta$, $C_0 = \mathrm{int}C$ and S_C is of type Θ.

Proof: Take $x \in \mathrm{int}C$ and let σ be an open cell containing C. By Lemma 2.3 there exists a split-regular $h \in G$ such that x is its attractor and $\sigma = \sigma\,(h)$. The sequence h^k contracts σ into x as $k \to +\infty$. Since C is a compact subset of σ, the contraction is uniform in C. This means that for any neighborhood U of x there exists $k_0 > 0$ such that $h^k C \subset U$ for $k \geq k_0$. In particular if we take $U \subset C$ we find that $g = h^{k_0}$ belongs to S_C. Furthermore the subset $\{f : f\,(C) \subset U\}$ is open in the compact-open topology on the continuous maps of $\mathbb{B}_\Theta$. By the continuity of the G-action we have then that $g \in \mathrm{int}S_C$ showing the first part of the proposition.

For the second statement note that C is invariant under S_C. Moreover, we found a split-regular $g \in \mathrm{int}S$ having x as attractor for arbitrary $x \in \mathrm{int}C$. This implies that C is the invariant control set of S_C because the core of the invariant control set contains the attractors for the split-regular elements in $\mathrm{int}S$. $\square$

5 Maximal semigroups

A subsemigroup S of a group L is said to maximal if it is not a group and there is no semigroup $T \neq L$ containing S properly. A well known fact in the theory of subsemigroups of topological groups is that any semigroup with interior points is contained in a maximal semigroup, which by force is closed. See Hilgert, Hofmann and Lawson [4] for a proof using the Lemma of Zorn.

For semigroups with nonempty interior in semi-simple Lie groups we can enlarge the notion of maximality by taking into account the type of the semigroup. As before let G be a semi-simple Lie group.

Definition 5.1 *We say that a semigroup $S \subset G$ with* $\text{int} S \neq \emptyset$ *is* Θ*-maximal or maximal with respect to* $\mathbb{B}_\Theta$ *if it is of type* Θ *and is not properly contained in any semigroup of type* Θ.

It will be proved below that the Θ-maximal semigroups are essentially the compression semigroups of the $\mathcal{B}$-convex sets in $\mathbb{B}_\Theta$. Before providing the proof for this fact we make the following remarks:

Let S be a Θ-maximal semigroup and denote by C its invariant control set in $\mathbb{B}_\Theta$. Since C is S-invariant if follows that $S \subset S_C$ where S_C is the compression semigroup

$$S_C = \{g \in G : gC \subset C\}.$$

By Proposition 4.2, S_C is of type Θ. This shows that if S is Θ-maximal then S is the compression semigroup of its invariant control set in $\mathbb{B}_\Theta$. Suppose there is $\Theta' \neq \Sigma$ containing Θ properly and let $\pi : \mathbb{B}_\Theta \to \mathbb{B}_{\Theta'}$. Then $\pi(C)$ is admissible in $\mathbb{B}_{\Theta'}$. Moreover $\text{int}(\pi(C))$ is dense in $\pi(C)$ because π is an open map. Hence $S_{\pi(C)}$ is of type Θ' by Proposition 4.2. Since π is equivariant under the actions of G in $\mathbb{B}_\Theta$ and $\mathbb{B}_{\Theta'}$ it follows that $S \subset S_{\pi(C)}$. This inclusion is proper. In fact, the invariant control set of $S_{\pi(C)}$ in $\mathbb{B}_\Theta$ is $\pi^{-1}(\pi(C))$ because $S_{\pi(C)}$ is of type Θ'. But C is admissible hence $C \neq \pi^{-1}(\pi(C))$ so that $\pi^{-1}(\pi(C))$ cannot be the invariant control set of S in $\mathbb{B}_\Theta$. This shows that any semigroup of type Θ is contained properly in a semigroup of type $\Theta' \supset \Theta$ if $\Theta \neq \Theta'$. In particular a Θ-maximal semigroup is not maximal unless Θ is maximal in Σ, that is, the complement of a singleton. In this case $\mathbb{B}_\Theta$ is a minimal flag manifold.

Conversely, if Θ is maximal and S is a Θ-maximal semigroup then S is maximal. In fact, if $T \subset S$ and $T \neq S$ then T is of type $\Theta' \supset \Theta$. Since S is Θ-maximal this implies that $\Theta' \neq \Theta$, but then T can not be a proper semigroup.

Now, thanks to the invariance of the $\mathcal{B}$-convex hull of a subset it follows easily that a Θ-maximal semigroup is the compression semigroup of a $\mathcal{B}$-convex set in $\mathbb{B}_\Theta$:

Proposition 5.2 *Suppose that S is a Θ-maximal semigroup and denote by C its invariant control set in $\mathbb{B}_\Theta$. Put $K = \text{cl}(\text{int}(\text{co}_\mathcal{B}(C)))$. Then $C = K$ and*

$$S = S_C = \{g \in G : gC \subset C\}.$$

Proof: If $g \in S$ then $gC \subset C$ so that Proposition 3.12 ensures that $g(\text{co}_\mathcal{B}(C)) \subset \text{co}_\mathcal{B}(C)$. By continuity $gK \subset K$. Hence S is contained in the compression semigroup S_K of K. By definition of a semigroup of type Θ, C is admissible in $\mathbb{B}_\Theta$. This implies that K is contained in an open cell σ of $\mathbb{B}_\Theta$. It follows that K is a nonempty admissible subset satisfying $K = \mathrm{cl}\,(\mathrm{int}K)$. Therefore Proposition 4.2 implies that S_K is of type Θ. Now by assumption S is Θ-maximal. Hence $S = S_K$. Invoking Proposition 4.2 again we have that the invariant control set of S_K is K so that $C = K$ concluding the proof. □

This proposition has the following converse which ensures that the compression semigroup of the interior of a $\mathcal{B}$-convex set is maximal.

Proposition 5.3 *Let $C \subset \mathbb{B}_\Theta$ be a proper closed $\mathcal{B}$-convex set with* $\mathrm{int}C \neq \emptyset$. *Put* $K = \mathrm{cl}\,(\mathrm{int}C)$. *Then the compression semigroup S_K is Θ-maximal.*

Proof: By definition of $\mathcal{B}$-convexity C is admissible. Proposition 3.8 then implies that K is admissible. Since $K = \mathrm{cl}\,(\mathrm{int}K)$ it follows from Proposition 4.2 that $\mathrm{int}S_K \neq \emptyset$, S_K is of type Θ and K is the invariant control set of S_K. To see the Θ-maximality take a semigroup T of type Θ with $S_K \subset T$. Denote by D the invariant control set of T in $\mathbb{B}_\Theta$. From $S_K \subset T$ it follows that $K \subset D$. Now, S_K is a compression semigroup and D is T-invariant. Hence it is enough to show that $K = D$ to get $T \subset S_K$ and thus $S_K = T$.

We prove first that $D \subset \mathrm{co}_{\mathcal{B}}\,(K)$. Suppose to the contrary that there exists $y \in D \setminus \mathrm{co}_{\mathcal{B}}\,(K)$. By definition of $\mathcal{B}$-convexity there exists an open cell $\sigma \in \mathcal{B}_\Theta$ such that $K \subset \sigma$ and $y \notin \sigma$. Take $x \in \mathrm{int}K$. From Lemma 2.3 there is a split-regular $h \in G$ such that x is its attractor and $\sigma = \sigma\,(h)$. Arguing as in the proof of Proposition 4.2 we can assume, after substituting h by some of its powers h^p, $p \geq 1$, that $h \in \mathrm{int}S_K$.

The limit $y_0 = \lim_{j\to+\infty} h^j y$ is a fixed point of h different from the attractor x because $y \notin \sigma\,(h)$. Since $h \in \mathrm{int}S_K$ there exists a control set, say E, of S_K such that $y_0 \in E_0$. The fact that y_0 is not the attractor of h implies that $E \neq K$. On the other hand $h \in T$, $y \in D$ and D is closed and T-invariant. Hence $y_0 \in D$. But E is entirely contained in a control set of T. Therefore $E \subset D$.

Now, both S_K and T are of type Θ so that they have the same number of control sets in $\mathbb{B}_\Theta$. Since any control set of S_K is contained in a control set of T, the existence of $E \neq K$ with $K, E \subset D$ is a contradiction. This shows that $D \subset \mathrm{co}_{\mathcal{B}}\,(K)$.

Therefore $\mathrm{int}D \subset \mathrm{int}\,(\mathrm{co}_{\mathcal{B}}\,(K))$. But $\mathrm{int}\,(\mathrm{co}_{\mathcal{B}}\,(K)) = \mathrm{int}C$ by Proposition 3.8. On the other hand $D = \mathrm{cl}\,(\mathrm{int}D)$ because it is the invariant control set of a semigroup with nonvoid interior. Hence $D \subset \mathrm{cl}\,(\mathrm{int}C) = K$. This implies that $T = S_K$, showing that S_K is Θ-maximal. □

We summarize the previous remarks and the above two propositions in the following final characterization of maximal semigroups in semi-simple Lie groups.

Theorem 5.4 *A semigroup S is Θ-maximal if and only if there is a $\mathcal{B}$-convex set C with* $\mathrm{int}C \neq \emptyset$ *such that $S = S_K$, the compression semigroup of $K = \mathrm{cl}\,(\mathrm{int}C)$. In this case K is the invariant control set of S in $\mathbb{B}_\Theta$ and $\mathrm{co}_{\mathcal{B}}\,(K) \subset C$.*

A semigroup S is maximal if and only if $\mathbb{B}_\Theta$ is a minimal flag manifold and S is Θ-maximal.

6 Miscellanea

In this section we prove further results about maximal semigroups and provide some examples.

6.1 Duality and minimal control set

Since a Θ-maximal semigroup S is the compression semigroup of its invariant control set C every object related to S is in principle obtained from C. We indicate here how the control sets of S on the flag manifolds are obtained from C, determining in detail the minimal control set from the duality operator. The minimal control set is the core of the invariant control set of $S^{-1} = \{g^{-1} : g \in S\}$ so we start by discussing this semigroup. Clearly S^{-1} has nonvoid interior if and only if $\mathrm{int}S \neq \emptyset$. A consequence of Corollary 4.6 in [13] is that $\mathbb{B}\,(S^{-1})$ is the flag manifold dual to $\mathbb{B}\,(S)$. Since there are imprecisions in the statement and in the proof of that corollary we offer here a version of it.

Proposition 6.1 *Take a split-regular $h \in \mathrm{int}S$ and let $A^+ = \exp \mathfrak{a}^+$ be the Weyl chamber containing h. Then*

$$\mathcal{W}_{A^-}\left(S^{-1}\right) = \mathcal{W}_{A^+}\,(S) \tag{3}$$

where $A^- = (A^+)^{-1}$.

Proof: Let b_0 be the attractor of h and w_0 the principal involution with respect to $\mathfrak{a}^+$. We have $w_0 A^+ = A^-$ and that $w_0 b_0$ is the repeller of h that is the attractor of h^{-1}. Let C and C^- be the invariant control set for S and S^{-1} in $\mathbb{B}$, respectively. By definition $w \in \mathcal{W}_{A^+}\,(S)$ if and only if $D\,(w) = C$. This means that $wb_0 \in C$

because wb_0 is the w-fixed-point of h and hence $wb_0 \in D(w)$. By Theorem 4.5 in [13] $D(ww_0) = D(w_0)$. In fact, this theorem ensures that $\mathcal{W}_{A^+}(S)\, ww_0 = \mathcal{W}_{A^+}(S)\, w_0$ is a consequence of $w \in \mathcal{W}_{A^+}(S)$. Since w_0 is the principal involution $D(w_0)$ is the minimal control set, which is given by C_0^-. Then we get from $D(ww_0) = D(w_0)$ that $ww_0b_0 \in C^-$. On the other hand $ww_0b_0 = w(w_0b_0)$ is the w-fixed point for h^{-1} because w_0b_0 is its attractor. Hence $ww_0b_0 \in C^-$ implies that $w \in \mathcal{W}_{A^-}(S^{-1})$.

Therefore we have $\mathcal{W}_{A^+}(S) \subset \mathcal{W}_{A^-}(S^{-1})$. The reverse inclusion follows from this after remarking that $S = (S^{-1})^{-1}$ and $A^+ = (A^-)^{-1}$. □

From this proposition we can define $\mathbb{B}(S)$ and $\mathbb{B}(S^{-1})$ by taking the same Weyl chamber A^+ as reference. In doing this it emerges that $\mathbb{B}(S^{-1})$ is the dual of $\mathbb{B}(S)$.

Take a split-regular $h \in \mathrm{int}S$ and assume without loss of generality that $h \in A^+$. If Σ is the associated simple system of roots then $\mathcal{W}_{A^+}(S)$ is generated by reflections with respect to the subset $\Theta(S) \subset \Sigma$. By formula (3) $\mathcal{W}_{A^-}(S^{-1})$ is generated by the same set of reflections. However by definition of $\mathcal{W}_{A^-}(S^{-1})$ we must look the generators of this subgroup in the subsets of $-\Sigma$. This is of course $-\Theta$. Hence the parabolic subalgebra associated to $\mathcal{W}_{\mathfrak{a}^-}(S^{-1})$ is

$$\mathfrak{p}_\Theta^- = \mathfrak{p} + \mathfrak{n}^+(\Theta) \tag{4}$$

where $\mathfrak{n}^+(\Theta)$ is the subalgebra spanned by $\mathfrak{g}_\alpha$ with $\alpha \in -\langle -\Theta \rangle = \langle \Theta \rangle$ and $\mathfrak{p}$ is the standard minimal parabolic subalgebra. Then $B(S^{-1}) = G/P_\Theta^-$. Now $w_0(-\Theta) = \iota\Theta$ and $w_0\mathfrak{p}_\Theta^- = \mathfrak{p}_{\pi\Theta}$ where

$$\mathfrak{p}_{\pi\Theta} = \mathfrak{p} + \mathfrak{n}^-(\iota\Theta)$$

and $\mathfrak{n}^-(\pi\Theta)$ is spanned by $\mathfrak{g}_{-\alpha}$ with $\alpha \in \langle \iota\Theta \rangle$. Hence $\mathbb{B}(S^{-1}) = \mathbb{B}_{\iota(\Theta)}$ the dual of $\mathbb{B}(S) = \mathbb{B}_\Theta$. Summarizing

Proposition 6.2 *The flag $\mathbb{B}(S^{-1})$ is the dual to $\mathbb{B}(S)$.*

Returning to the maximal semigroups suppose that S is Θ-maximal. Then S^{-1} is Θ^*-maximal. In fact, by Proposition 6.2, S^{-1} is of type Θ^*. If $T \supset S^{-1}$ is a semigroup of type Θ^* then $S \subset T^{-1}$ and T^{-1} is of type Θ. Hence $S = T^{-1}$ showing that S^{-1} is Θ^*-maximal. With this in mind we can describe S^{-1} as a compression semigroup.

Proposition 6.3 *Let S be a Θ-maximal semigroup and denote by C its invariant control set in $\mathbb{B}_\Theta$. Then the invariant control set of S^{-1} in $\mathbb{B}_{\Theta^*}$ is $\mathrm{cl}(C^*)$. Moreover S^{-1} is the compression semigroup $S_{\mathrm{cl}(C^*)}$.*

Proof: From the S-invariance of C it follows that C^* is invariant under S^{-1} (see Corollary 3.10). Hence $\mathrm{cl}\,(C^*)$ is S^{-1}-invariant so that $S \subset S_{\mathrm{cl}(C^*)}$. But $\mathrm{cl}\,(C^*) = (\mathrm{int}C)^*$ hence by Theorem 5.4, $S_{\mathrm{cl}(C^*)}$ is Θ^*-maximal. The equality $S^{-1} = S_{\mathrm{cl}(C^*)}$ follows then by the Θ^*-maximality of S^{-1}. □

This proposition allows the determination of the minimal control sets of the maximal semigroup $S = S_C$. In fact, in any flag manifold the minimal control set of S is the set of transitivity of the invariant control set of S^{-1}. Keeping the above notations, the invariant control set of S^{-1} in $\mathbb{B}_{\Theta^*}$ is $D = \mathrm{cl}\,(C^*)$ and its set of transitivity is $D_0 = \mathrm{int}\,(\mathrm{cl}\,(C^*))$, which contains C^* densely. Moreover, let $\pi : \mathbb{B} \to \mathbb{B}_{\Theta^*}$ be the fibration from the maximal flag manifold. Then $\pi^{-1}(D)$ is the invariant control set for S^{-1} in $\mathbb{B}$ and its core $\pi^{-1}(D_0)$ (see Proposition 4.1). Also, if $\mathbb{B}_{\Theta'}$ is any flag manifold, the projection $\pi_{\Theta'} : \mathbb{B} \to \mathbb{B}_{\Theta'}$ maps control sets and their cores into control sets and cores respectively. Hence the minimal control set for S in $\mathbb{B}_{\Theta'}$ is $\pi_{\Theta'}(\pi^{-1}(D_0))$. Since the projections between the flag manifolds and their inverse images preserve closures and interiors of subsets we get the minimal control set as the interior of the closure of $\pi_{\Theta'}(\pi^{-1}(C^*))$.

The subset $\pi_{\Theta'}(\pi^{-1}(C^*))$ is easily described in terms of incidence of parabolic subalgebras and their nilradicals: Think of a point $x \in \mathbb{B}_{\Theta^*}$ as being the nilradical of the corresponding parabolic subalgebra. Viewing the elements of $\mathbb{B}$ as minimal parabolic subalgebras the fiber $\pi^{-1}\{x\}$ is the set of minimal parabolic subalgebras containing x. On the other hand, if $y \in \mathbb{B}$ then $\pi_{\Theta'}(y)$ is the only parabolic subalgebra in $\mathbb{B}_{\Theta'}$ containing y. Hence the parabolic subalgebras in $\pi_{\Theta'}(\pi^{-1}\{x\})$ contain x. Reciprocally if $z \in \mathbb{B}_{\Theta'}$ is a parabolic subalgebra containing x then there is a minimal parabolic subalgebra $y \in \pi_{\Theta'}^{-1}\{z\}$ containing x so that $z \in \pi_{\Theta'}(\pi^{-1}\{x\})$. Therefore $\pi_{\Theta'}(\pi^{-1}\{x\})$ is the set of parabolic subalgebras in $\mathbb{B}_{\Theta'}$ containing the nilradical x. Thus from the previous paragraph we can state:

Proposition 6.4 *Let $S = S_C$ be a Θ-maximal semigroup. Given a flag $\mathbb{B}_{\Theta'}$ denote by $\overline{C^*}$ the set of parabolic subalgebras in $\mathbb{B}_{\Theta'}$ containing the nilradical of the parabolic subalgebras in $C^* \subset \mathbb{B}_{\Theta^*}$. Then the minimal control set of S in $\mathbb{B}_{\Theta'}$ is* $\mathrm{int}\left(\mathrm{cl}\left(\overline{C^*}\right)\right)$.

We mention by pass that the other control sets, or more precisely their cores, are determined from the invariant and the minimal control sets. This is true not only for maximal semigroups but for an arbitrary semigroup S with nonvoid interior. The idea is that for any control D of S there is a control set D^- of S^{-1} such that $(D^-)_0 = D_0$. The intersection of their domains of attraction (under the actions of S and S^{-1} respectively) is D_0. Now in [12] it was proved that the domain of attraction of a control set $D(w)$ of S is built from the minimal control set and an algebraic property of w, namely its minimal decomposition as product of simple reflections. In a symmetric way the domain of attraction of $D(w)^-$ depends only on w and the minimal control set of S^{-1}, that is, the invariant control set of S. With this construction it is possible to describe the cores of the control sets by incidence of parabolic subalgebras. Since this is not specific for maximal semigroups we leave outside the details.

6.2 Maximal semigroups containing a given semigroup

As mentioned above any semigroup with nonvoid interior in a topological group is contained in a maximal one. This very general fact can be improved in our context by means of Theorem 5.4. Starting with a semigroup S of type Θ let C be its invariant control set in $\mathbb{B}_\Theta$. Then $\mathrm{co}_{\mathcal{B}}(C)$ is S-invariant and the arguments in the proof of Proposition 5.2 ensure that S is contained in the Θ-maximal semigroup S_K where $K = \mathrm{cl}(\mathrm{int}(\mathrm{co}_{\mathcal{B}}(C)))$. Also, if $\Theta \subset \Theta'$ then the projection $\pi : \mathbb{B}_\Theta \to \mathbb{B}_{\Theta'}$ is defined and the same argument applied to $\pi(C)$ instead of C shows that S is contained in a Θ'-maximal semigroup. In particular we recover the general result that there exists a maximal semigroup containing S.

In general a semigroup S of type Θ can be contained in several Θ'-maximal semigroups if $\Theta \subset \Theta'$, according to the $\mathcal{B}$-convex sets left invariant by S. The following statement shows that sometimes there is uniqueness of the maximal semigroup containing S.

Proposition 6.5 *Let $S = S_C$ be a Θ-maximal semigroup with $C = \mathrm{cl}(\mathrm{int}C)$ a $\mathcal{B}$-convex set. Suppose that for $\Theta \subset \Theta'$, $\pi(C)$ is $\mathcal{B}$-convex in $\mathbb{B}_{\Theta'}$. Then $S_{\pi(C)}$ is the only Θ'-maximal semigroup containing S.*

Proof: From Theorem 5.4 it follows that C is the invariant control set of S in $\mathbb{B}_\Theta$ hence the S-invariant control set in $\mathbb{B}_{\Theta'}$ is $\pi(C)$. In particular $\pi(C)$ is S-invariant so that $S \subset S_{\pi(C)}$. By assumption $\pi(C)$ is $\mathcal{B}$-convex. Moreover, $\mathrm{int}(\pi(C))$ is dense in $\pi(C)$ because π is an open map. Applying Theorem 5.4 again it follows that $S_{\pi(C)}$ is indeed Θ'-maximal. Now let T be a Θ'-maximal semigroup containing S. The the invariant control set of T in $\mathbb{B}_{\Theta'}$, say D, contains $\pi(C)$. Of course $T = S_{\pi(C)}$ if $D = \pi(C)$. On the other hand the arguments in the proof of Proposition 5.3 show that $D \neq \pi(C)$ contradicts the assumption that T is of type Θ'. □

6.3 Examples

6.3.1 Total positivity

A square matrix with real entries is said to be totally positive provided its minors of all orders are nonnegative numbers. It is well known that the set of totally positive matrices in $\mathrm{Sl}\,(n,\mathbb{R})$ is a semigroup with nonvoid interior. We consider here the maximality properties of a semigroup slightly larger than T: An $n \times n$ matrix is said to be sign-regular if for every $k = 1, \ldots, n-1$, its minors of order k have the same sign. The semigroup $\bar{T}$ of sign-regular matrices clearly contains T. It is a compression semigroup as shows the following constructions.

Let $\Lambda_k = \bigwedge^k \mathbb{R}^n$ be the k-fold exterior product of $\mathbb{R}^n$. The Grassmannian $\mathrm{Gr}_k\,(n)$ embeds into the projective space of Λ_k as the set of lines spanned by the decomposable elements. Analogously the Grassmannian $\mathrm{Gr}_k^+\,(n)$ of oriented k-dimensional subspaces, which is a two-fold covering of $\mathrm{Gr}_k\,(n)$ embeds in a sphere of Λ_k. For $g \in \mathrm{Sl}\,(n,\mathbb{R})$ denote also by g the induced linear map of Λ_k. Both Grassmannians $\mathrm{Gr}_k\,(n)$ and $\mathrm{Gr}_k^+\,(n)$ are invariant under $g \in \mathrm{Sl}\,(n,\mathbb{R})$.

Let $\beta_1 = \{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and $\beta_k = \{e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ where $I = (i_1 < \cdots < i_k)$ the basis induced in Λ_k. This basis is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$ in Λ_k coming from the standard inner product in $\mathbb{R}^n$. The positive orthant in Λ_k is determined by the inequalities $\langle e_I, \cdot \rangle \geq 0$. We denote by $\mathcal{O}_k$ its intersection with the oriented Grassmannian $\mathrm{Gr}_k^+\,(n)$:

$$\mathcal{O}_k = \{v \in \mathrm{Gr}_k^+\,(n) : \langle v, e_I \rangle \geq 0 \text{ for all } I\}.$$

Consider the compression semigroup

$$T_k = \{g \in \mathrm{Sl}\,(n,\mathbb{R}) : g\mathcal{O}_k \subset \mathcal{O}_k\}.$$

Since the k-minors of g are the entries $\langle ge_I, e_J \rangle$ of the matrix of g_k with respect to β_k, it follows that $g \in T_k$ if and only if all its minors of order k are nonnegative. Hence

$$T = T_1 \cap \cdots \cap T_{n-1}.$$

Put $C_k = \pi\,(\mathcal{O}_k)$ where $\pi : \mathrm{Gr}_k^+\,(n) \to \mathrm{Gr}_k\,(n)$ is the canonical projection and set

$$\bar{T}_k = \{g \in \mathrm{Sl}\,(n,\mathbb{R}) : gC_k \subset C_k\}.$$

It is easily checked that $g \in \bar{T}_k$ if and only if either $g \in T_k$ or all the k-minors of g are negative. Hence

$$\bar{T} = \bar{T}_1 \cap \cdots \cap \bar{T}_{n-1}.$$

Now we verify that C_k is $\mathcal{B}$-convex. This will be a consequence of

Lemma 6.6 *For $V \notin \operatorname{int}(C_k)$ let $V^\perp$ be its orthocomplement in $\mathbb{R}^n$. Then $V^\perp \notin C_k^*$, i.e., there exists $W \in C_k$ with $\dim\left(W \cap V^\perp\right) \geq 1$.*

Proof: Take a basis $\{v_1, \ldots, v_k\}$ of V and let $v = v_1 \wedge \cdots \wedge v_k$ be the associated decomposable vector in Λ_k.

If V is in the boundary of C_k then $\langle v, e_I \rangle = 0$ for some basic element $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. Put $E_I = \operatorname{span}\{e_{i_1}, \ldots, e_{i_k}\}$. Then $\langle v, e_I \rangle = 0$ is equivalent to $\dim\left(E_I \cap V^\perp\right) \geq 1$. Since $E_I \in C_k$ this shows the lemma in case $V \in C_k$.

Assume that $V \notin C_k$ and consider the continuous map

$$f_v : w \in \operatorname{Gr}_k^+(n) \longmapsto \langle v, w \rangle \in \mathbb{R}.$$

By definition of C_k it follows that $v \notin \pm\mathcal{O}_k$ so that there are indices I, J such that $\langle v, e_I \rangle > 0$ and $\langle v, e_J \rangle < 0$. Let A be the subgroup of diagonal matrices with positive eigenvalues. This subgroup is connected and leaves invariant the orthant $\mathcal{O}_k$. Moreover it is easy to find $g, h \in A$ and $z \in \mathcal{O}_k$ such that $g^i z \to e_I$ and $h^i z \to e_J$ as $i \to +\infty$. Hence f_v assumes positive and negative values in Az, implying that there is $w = w_1 \wedge \cdots \wedge w_k$ in $\mathcal{O}_k$ such that $\langle v, w \rangle = 0$. Put $W = \operatorname{span}\{w_1, \ldots, w_k\}$. Then $\langle v, w \rangle = 0$ means that $\dim\left(W \cap V^\perp\right) \geq 1$. Since $W \in C_k$ this shows the lemma. □

This lemma shows immediately that if an $(n-k)$-dimensional subspace U belongs to C_k^* then its orthocomplement $U^\perp$ belongs to $\operatorname{int}(C_k)$. Reciprocally, take $V, W \in \operatorname{int}(C_k)$ and choose bases $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ of V and W respectively such that $v = v_1 \wedge \cdots \wedge v_k$ and $w = w_1 \wedge \cdots \wedge w_k$ are in $\operatorname{int}(\mathcal{O}_k)$. Then $\langle v, w \rangle > 0$ because $\mathcal{O}_k$ is in an orthant defined by an orthonormal basis. Hence $V \cap W^\perp = 0 = W \cap V^\perp$ so that $V^\perp, W^\perp \in C_k^*$. Therefore

$$C_k^* = \{V^\perp : V \in \operatorname{int}(C_k)\}.$$

The above lemma also shows that $V \notin C_k^{**}$ if $V \notin C_k$ so that $C_k = C_k^{**}$ is $\mathcal{B}$-convex. Therefore,

Proposition 6.7 *$\bar{T}_k$ is maximal for all $k = 1, \ldots, n-1$.*

We leave aside further discussions about the semigroup $\bar{T}$, but mention that a similar approach shows for any sequence $\mathbf{r} = (r_1 < \cdots < r_m)$, the semigroup

$$\bar{T}_{\mathbf{r}} = \bar{T}_{r_1} \cap \cdots \cap \bar{T}_{r_m}$$

is maximal with respect to $\mathbb{F}(\mathbf{r})$. In particular $\bar{T}$ is maximal with respect to the maximal flag manifold.

We refer to Ando [1] for a survey about totally positivity matrices. See also Lusztig [8] and references therein for a generalization to semi-simple groups.

6.3.2 A class of compression semigroups

The following example is a particular instance of the compression semigroups considered by Hilgert and Neeb [6]. Let Q be a quadratic form in $\mathbb{R}^n$ with matrix

$$\begin{pmatrix} 1_{k\times k} & 0 \\ 0 & -1_{(n-k)\times(n-k)} \end{pmatrix}$$

Denote by β the corresponding nondegenerate bilinear form. Let $C \subset \mathrm{Gr}_k(n)$ be the set of subspaces where Q is positive semi-definite and consider the compression semigroup S_C as a subsemigroup of $\mathrm{Sl}(n,\mathbb{R})$. The continuity of Q ensures that $C = \mathrm{cl}(\mathrm{int} C)$. Moreover, let $U \in \mathrm{Gr}_{n-k}(n)$ be such Q is negative definite on U. Then Q is negative definite in any subspace of U. This implies that $V \cap U = 0$ for all $V \in C$. Hence $C \subset \sigma_U$ so that C is admissible and $U \in C^*$. Therefore S_C has nonempty interior and is of type $\mathrm{Gr}_k(n)$.

Denote by $D \subset \mathrm{Gr}_{n-k}(n)$ the set of subspaces where Q is negative definite. We have just seen that $D \subset C^*$ or equivalently $C \subset D^*$. We claim that $C = D^*$. To check this use the well known fact that if $W \subset \mathbb{R}^n$ is a subspace with $\dim W \leq n-k$ and such that Q is negative definite in W then it extends to a subspace $U \supset W$ with $\dim U = n-k$ and Q negative definite in U.

Now suppose that there exists $V \in D^*$ such that Q is not positive semi-definite in V. Then there is a subspace $W \subset V$ where Q is negative definite. Since W extends to an element of D this contradicts the fact that V is transversal to every element of D. Hence $D^* \subset C$ and $C = D^*$.

Therefore C is $\mathcal{B}$-convex which implies that S_C is maximal of type $\mathrm{Gr}_k(n)$, and hence maximal in $\mathrm{Sl}(n,\mathbb{R})$.

6.4 Remarks and questions

Although Theorem 5.4 gives an exact characterization of the maximal semigroups in terms of $\mathcal{B}$-convexity it is far from of being conclusive for the full understanding of the maximal semigroups. Specially in what concerns specific classes of semigroups, like e.g. connected semigroups, infinitesimally generated semigroups, etc. For deeper insights into the maximal semigroups our results must be followed by a further development of the geometry of the $\mathcal{B}$-convex sets and their compression semigroups. Below we list some natural questions and remarks pointing to this direction.

1. From the work of Lawson [7] one knows that a maximal semigroup S in a solvable group G is total in the sense that $G = S \cup S^{-1}$. This property does not

hold for semigroups in semi-simple groups because of the existence of an open set of compact elements. However one can ask whether a maximal semigroup is total with respect to a flag manifold $\mathbb{B}_\Theta$, in the sense that $\mathbb{B}_\Theta$ is the union of the S-control sets. With this kind of totality the action of S on the flag manifold is completely clear since one knows the action inside the control sets. At this regard we mention that under totality the proof Proposition 5.3 would be simplified. In fact, the main point there is to show that a point outside the invariant control set is in the domain of attraction of another control set.

2. If S is connected then its invariant control set (in any homogeneous space of G) is connected. This suggests to investigate the compression semigroups S_C with C convex and connected. In general S_C is not connected. This is shown for instance by the compression semigroup in $\mathrm{Sl}(2,\mathbb{R})$ of an interval in the projective line $\mathbb{P}^1$. It has two connected components $\pm\mathrm{Sl}^+(2,\mathbb{R})$, where $\mathrm{Sl}^+(2,\mathbb{R})$ is the semigroup of 2×2 matrices with positive entries. However $\mathrm{Sl}^+(2,\mathbb{R})$ is connected and maximal with this property. Similar facts may occur in general: There might be a class of connected $\mathcal{B}$-convex sets which are invariant control sets of semigroups which are maximal with the property of being connected. This development certainly goes through the study of the connected $\mathcal{B}$-convex sets and the $\mathcal{B}$-convex hull of connected sets, which in general may not be connected. Of course the same kind of questions make sense for Θ-maximal semigroups.

3. Similar remarks apply to the infinitesimally generated semigroups. Here one of the basic questions seems to be characterization of the maximal semigroups (and corresponding $\mathcal{B}$-convex sets) whose tangent wedge generate a semigroup with same invariant control set (see D. Mittenhuber [9]).

4. It looks like that Proposition 6.5 can be improved by showing that the projection of a $\mathcal{B}$-convex set is $\mathcal{B}$-convex, at least for large classes of $\mathcal{B}$-convex sets.

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