ASYMPTOTIC LIKELIHOOD RATIO TESTING IN ELLIPTICAL LINEAR MEASUREMENT ERROR MODELS

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Summary

The main object of this paper is to consider hypothesis testing in elliptical measurement error models. Bartlett corrected likelihood ratio statistics are considered for several hypotheses of interest. The corrections are obtaines by computing directly expected values of the statistics, without the need of computing cumulants, as is usually the case when deriving such corrections. Orthogonal parametrizations are also derived which are crucial in obtaining the main results.

1. Introduction

We consider the simple regression model with additive measurement errors specified by the equations

$$Y_{ij} = \alpha_j + \beta_j x_{ij} + e_{ij} \tag{1.1}$$

and

$$X_{ij} = x_{ij} + u_{ij}, \tag{1.2}$$

$i = 1, \ldots, n_j$ and $j = 1, \ldots, k$. When the x_{ij} are considered fixed (parameters), then the functional model follows. When the x_{ij} are random variables, then the structural model follows. In this paper, we consider the structural model situation. In this situation, the model (1.1)-(1.2) can be written (Arellano-Valle and Bolfarine, 1995) as

$$\mathbf{Z}_{ij} = \mathbf{a}_j + \mathbf{B}_j \mathbf{r}_{ij}, \tag{1.3}$$

where $\mathbf{Z}_{ij} = (Y_{ij}, X_{ij})'$, $\mathbf{r}_{ij} = (x_{ij}, e_{ij}, u_{ij})'$, $\mathbf{a}_j = (\alpha_j, 0)'$ and $\mathbf{B}_j = [\mathbf{b}_j \ \mathbf{I}_2]$, with $\mathbf{b}_j = (\beta_j, 1)'$ and $\mathbf{I}_m$ being the identity matrix of dimension m. Thus, it follows from (1.3) that the distribution of $\mathbf{Z}_{ij}$ is determined by the distribution of $\mathbf{r}_{ij}$. In the literature, it is typically considered that the random vectors $\mathbf{r}_{ij} = (x_{ij}, e_{ij}, u_{ij})'$, $i = 1, \ldots, n_j, j = 1, \ldots, k$, are independent and $\mathbf{r}_{ij} \sim N_3(\boldsymbol{\eta}_j, \boldsymbol{\Omega}_j)$, where

$$\boldsymbol{\eta}_j = \begin{pmatrix} \mu_{xj} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Omega}_j = \begin{pmatrix} \sigma^2_{xj} & 0 & 0 \\ 0 & \sigma^2_{ej} & 0 \\ 0 & 0 & \sigma^2_{uj} \end{pmatrix}, \tag{1.4}$$

so that $\mathbf{Z}_{ij} \sim N_3(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$, where

(1.5)

$$\boldsymbol{\mu}_j = \mathbf{a}_j + \mathbf{B}_j \boldsymbol{\eta}_j = \begin{pmatrix} \alpha_j + \beta_j \mu_{xj} \\ \mu_{xj} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_j = \mathbf{B}_j \boldsymbol{\Omega}_j \mathbf{B}_j' = \begin{pmatrix} \beta_j^2 \sigma_{xj}^2 + \sigma_{ej}^2 & \beta_j \sigma_{xj}^2 \\ \beta_j \sigma_{xj}^2 & \sigma_{xj}^2 + \sigma_{uj}^2 \end{pmatrix},$$

as can be seen, for example, in Wong (1991). Note that $\boldsymbol{\mu}_j = \boldsymbol{\mu}(\boldsymbol{\theta}_j)$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_j(\boldsymbol{\theta}_j)$, where

$$\boldsymbol{\theta}_j = (\alpha_j, \mu_{xj}, \sigma_{xj}^2, \sigma_{ej}^2, \sigma_{uj}^2, \beta_j)', \tag{1.6}$$

$j = 1, \ldots, k$, so that the model parameter is given by $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \ldots, \boldsymbol{\theta}_k')'$. The simple case with $k = 1$ is considered, for example, in Fuller (1987), Wong (1989), Bolfarine and Cordani (1993) and Arellano-Valle and Bolfarine (1995), among others.

The main object of this paper is to consider inference for model parameters under the elliptical models and testing hypotheses of interest using the likelihood ratio statistics, where Bartlett type corrections are considered for the testing statistics. As in Arellano-Valle and Bolfarine (1995, 1996), the approach followed is based on the direct evaluation of the expected value of the likelihood ratio statistics and differs from the approch considered in Lawley (1956), which is based on asymptotic expansions of the likelihood function. The approach allows the generalization of some results in Wong (1991) to a class of distributions more general than the normal distributions which is the elliptical family of distributions (see Fang et al., 1990). In the literature it is typicall to consider two versions of the elliptical model, known as the dependent and independent models.

In dependent elliptical models, it is considered that the observed data follows jointly an elliptical distribution as considered, for example, in Zellner (1976), Anderson et al. (1986) and Arellano-Valle and Bolfarine (1996), among others. In these works it is shown that the inference under the normal model typically also holds under dependent elliptical models, so that such procedures are robust with respect to this type of nonnormality. In particular, it is shown that, under dependent elliptical models, the maximum likelihood estimators (MLE) for location (scale) parameters are the same (proportional) that the corresponding estimators by using the normal model.

In independent elliptical models, it is considered that the observations are independent and follows elliptical marginal distributions. This is the approach followed, for example, in Tyler (1983), Lange et al. (1989) and Kano, Berkane and Bentler (1993). Under this approach, inference is typically based on asymptotic results and one important aspect of these models is that they can be robust with respect outlying observations. In the context of hypothesis testing, it is well known that some tests which rely on asymptotic distributions such as the likelihood ratio, score and Wald statistics present some limitations for smal sample size. However, when the regularity conditions are valid the aproximation of the distributions of such statistics by the reference chisquare distribution can be improved in models with continuous sample distributions. This improved is typically achieved by multiplying the statistics by Bartlett type (Cordeiro, 1983) correction factors. The correction factors are typically fucntions of the cumulants of the logarithm of the likelihood function and its derivation typically involve great amount of algebraic manipulations.

The paper is organized as follows. Section 2 presents a description of some extensions of the normal model by considering elliptical distributions. Dependent and independent versions of the elliptical model are considered. In each case, the maximum likelihood estimators are derived and some relations of the estimators with the normal case are considered. Section 3 is devoted to the derivation of the Fisher information matrix for each model. Using the information matrices, orthogonal parametrizations are also obtained in each case. By using properties of the Wishart distribution, the distribution of the sample variances are obtained which are used in deriving Bartlett correction for the likelihood ratio statistics. Section 4 is devoted to the derivations of the likelihood ratio statistics and its Bartlett corrected versions.

2. Elliptical measurement error models

In this paper, it is considered that, in (1.3), $\mathbf{r}_{1j}, \dots, \mathbf{r}_{n_j j}$ are uncorrelated, $j = 1, \dots, k$, and that $\mathbf{r}_{ij} \sim El_3(\boldsymbol{\eta}_j, \boldsymbol{\Omega}_j; \phi)$, $i = 1, \dots, n_j$, with $\boldsymbol{\eta}_j$ and $\boldsymbol{\Omega}_j$ being as in (1.4). Thus, we are considering that $\mathbf{r}_{ij}$ is elliptically distributed with parameters $\boldsymbol{\eta}_j$ and $\boldsymbol{\Omega}_j$ and characteristic function of the form $\exp(i\mathbf{t}'\boldsymbol{\eta}_j)\phi(\mathbf{t}'\boldsymbol{\Omega}_j\mathbf{t})$, $\mathbf{t} \in \mathcal{R}^3$, so that $E[\mathbf{r}_{ij}] = \boldsymbol{\eta}_j$ and $Var[\mathbf{r}_{ij}] = \delta\boldsymbol{\Omega}_j$ when they exists, where $\delta = -2\phi'(0)$ ($\delta = 1$ for the normal model). We assume in the sequel that $\mathbf{r}_{ij}$ has density function, which has the form $|\boldsymbol{\Omega}_j|^{-1/2}h_{(3)}((\mathbf{r}_{ij} - \boldsymbol{\eta}_j)'\boldsymbol{\Omega}_j^{-1}(\mathbf{r}_{ij} - \boldsymbol{\eta}_j))$, where $h_{(p)}$ denotes a p-dimensional spherical density generator (see Fang et al., 1990). Now, from the assumptions on the $\mathbf{r}_{ij}$ and properties of the elliptical distributions, it follows that $\mathbf{Z}_{ij} \sim El_2(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j; \phi)$, $j = 1, \dots, k$, where $\boldsymbol{\mu}_j$ and $\boldsymbol{\Sigma}_j$ are as in (1.5). Notice that the above results represents the distribution of the observed data associated with the j-th population. In order to specify the joint distribution associated with the k populations, we consider three different specifications for the joint distribution of the random vectors $\mathbf{r}_{ij}$, $i = 1, \dots, n_j$, $j = 1, \dots, k$, within the class of the elliptical distributions. Thus, denoting by

$$\mathbf{r}_{(j)} = (\mathbf{r}'_{1j}, \dots, \mathbf{r}'_{n_j j})', \ j = 1, \dots, k \quad \text{and} \quad \mathbf{r}_* = (\mathbf{r}'_{(1)}, \dots, \mathbf{r}'_{(k)})', \tag{2.1}$$

the following situations are considered:

(A) $\mathbf{r}_* \sim El_{3n}(\boldsymbol{\eta}_*, \boldsymbol{\Omega}_*; \phi)$,

(B) $\mathbf{r}_* \sim El_{3n}^{(I)}(\boldsymbol{\eta}_*, \boldsymbol{\Omega}_*; \phi) \longleftrightarrow \mathbf{r}_{(j)} \overset{ind}{\sim} El_{3n_j}(\boldsymbol{\eta}_{(j)}, \boldsymbol{\Omega}_{(j)}; \phi), \ j = 1 \dots, k$,

(C) $\mathbf{r}_* \sim El_{3n}^{(II)}(\boldsymbol{\eta}_*, \boldsymbol{\Omega}_*; \phi) \longleftrightarrow \mathbf{r}_{(j)} \sim El_{3n_j}^{(I)}(\boldsymbol{\eta}_{(j)}, \boldsymbol{\Omega}_{(j)}; \phi), \ j = 1 \dots, k$, where $n = n_1 + \ldots + n_k$, $\boldsymbol{\eta}_* = (\boldsymbol{\eta}'_{(1)}, \ldots, \boldsymbol{\eta}'_{(k)})'$ and $\boldsymbol{\Omega}_* = \text{diag}(\boldsymbol{\Omega}_{(1)}, \ldots, \boldsymbol{\Omega}_{(k)})$, with $\boldsymbol{\eta}_{(j)} = \mathbf{1}_{n_j} \otimes \boldsymbol{\eta}_j$ and $\boldsymbol{\Omega}_{(j)} = \mathbf{I}_{n_j} \otimes \boldsymbol{\Omega}_j$, $j = 1, \ldots, k$, and where $\boldsymbol{\eta}_j$ and $\boldsymbol{\Omega}_j$ are as in(1.3). Moreover, $\otimes$ denotes the usual Kronecker product, $\text{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_p)$ denotes a diagonal matrix, where the elements $\mathbf{A}_1, \ldots, \mathbf{A}_p$ are matrices of appropriate dimensions, and $\mathbf{1}_m$ denotes the m-dimensional vector of ones. We call attention to the fact that the random vectors $\mathbf{r}_{(1)}, \ldots, \mathbf{r}_{(k)}$ are not independent under situation (A), while they are independent under situations (B) and (C). Moreover, for each $j = 1, \ldots, k$, the random vectors $r_{j1}, \ldots, r_{n_j j}$ are not independent under situation (A) and (B), while they are independent under situation (C). It is also worth noticing that under normality the three specifications coincide. Moreover, for $k = 1$ (one population), specifications (A) and (B) coincide with the dependent elliptical model and specification (C) becomes the independent elliptical model, which are defined in Arellano-Valle and Bolfarine (1996).

In matrix notation, we can represent the model corresponding to the observations from the j-th population as

$$\mathbf{Z}_{(j)} = \mathbf{1}_{n_j} \otimes \mathbf{a}_j + (\mathbf{I}_{n_j} \otimes \mathbf{B}_j)\mathbf{r}_{(j)},$$

$j = 1, \ldots, k$, where $\mathbf{r}_{(j)}$ is as in (2.1) and $\mathbf{a}_j$ and $\mathbf{B}_j$ as in (1.3). Similarly, we can represent the model corresponding to the observations from the k populations of size $n = n_1 + \ldots n_k$ as

$$\mathbf{Z}_* = \mathbf{a}_* + \mathbf{B}_*\mathbf{r}_*,$$

where $\mathbf{Z}_* = (\mathbf{Z}'_{(1)}, \ldots, \mathbf{Z}'_{(k)})'$, $\mathbf{a}_* = (\mathbf{1}'_{n_1} \otimes \mathbf{a}'_1, \ldots, \mathbf{1}'_{n_k} \otimes \mathbf{a}'_k)'$, $\mathbf{B}_* = \text{Diag}(\mathbf{I}_{n_1} \otimes \mathbf{B}_1, \ldots, \mathbf{I}_{n_k} \otimes \mathbf{B}_k)$, and $\mathbf{r}_*$ being as in (2.1). Thus, from the properties of the elliptical family of distributions it follows for the different specifications that:

$$\begin{array}{lll}
\text{(A)} & \mathbf{Z}_* \sim El_{2n}(\boldsymbol{\mu}_*, \boldsymbol{\Sigma}_*; \phi), & \\
\text{(B)} & \mathbf{Z}_* \sim El^{(I)}_{2n_j}(\boldsymbol{\mu}_*, \boldsymbol{\Sigma}_*; \phi) \;\longleftrightarrow\; & \mathbf{Z}_{(j)} \overset{ind}{\sim} El_{2n_j}(\boldsymbol{\mu}_{(j)}, \boldsymbol{\Sigma}_{(j)}; \phi), \\
\text{(C)} & \mathbf{Z}_* \sim El^{(II)}_{2n}(\boldsymbol{\mu}_*, \boldsymbol{\Sigma}_*; \phi) \;\longleftrightarrow\; & \mathbf{Z}_{(j)} \sim El^{(I)}_{2n_j}(\boldsymbol{\mu}_{(j)}, \boldsymbol{\Sigma}_{(j)}; \phi),
\end{array}$$

where $n = n_1 + \ldots + n_k$, $\boldsymbol{\mu}_* = (\boldsymbol{\mu}'_{(1)}, \ldots, \boldsymbol{\mu}'_{(k)})'$ and $\boldsymbol{\Sigma}_* = \text{diag}(\boldsymbol{\Sigma}_{(1)}, \ldots, \boldsymbol{\Sigma}_{(k)})$, with $\boldsymbol{\mu}_{(j)} = \mathbf{1}_{n_j} \otimes \boldsymbol{\mu}_j$ and $\boldsymbol{\Sigma}_{(j)} = \mathbf{I}_{n_j} \otimes \boldsymbol{\Sigma}_j$, $j = 1, \ldots, k$, where $\boldsymbol{\mu}_j = \boldsymbol{\mu}(\boldsymbol{\theta}_j)$ and $\boldsymbol{\Sigma}_j = \boldsymbol{\Sigma}(\boldsymbol{\theta}_j)$ are as in (1.5), so that $\boldsymbol{\mu}_* = \boldsymbol{\mu}_*(\boldsymbol{\theta})$ and $\boldsymbol{\Sigma}_* = \boldsymbol{\Sigma}_*(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\boldsymbol{\theta}'_1, \ldots, \boldsymbol{\theta}'_k)'$ wiht $\boldsymbol{\theta}_j$ being as in (1.6). Now, under the existence of a density function, it follows that the likelihood function under elliptical models with specfications (A), (B) and (C) are given, respectively, by:

$$\begin{array}{llcl}
\text{(A)} & f(\mathbf{z}_*|\boldsymbol{\theta}) & = & |\boldsymbol{\Sigma}_*|^{-1/2} h_{(2n)}((\mathbf{z}_* - \boldsymbol{\mu}_*)'\boldsymbol{\Sigma}_*^{-1}(\mathbf{z}_* - \boldsymbol{\mu}_*)), \\
\text{(B)} & f(\mathbf{z}_*|\boldsymbol{\theta}) & = & \prod_{j=1}^{k} |\boldsymbol{\Sigma}_{(j)}|^{-1/2} h_{(2n_j)}((\mathbf{z}_{(j)} - \boldsymbol{\mu}_{(j)})'\boldsymbol{\Sigma}_{(j)}^{-1}(\mathbf{z}_{(j)} - \boldsymbol{\mu}_{(j)})), \\
\text{(C)} & f(\mathbf{z}_*|\boldsymbol{\theta}) & = & \prod_{j=1}^{k} \prod_{i=1}^{n_j} |\boldsymbol{\Sigma}_j|^{-1/2} h_{(2)}((\mathbf{z}_{ij} - \boldsymbol{\mu}_j)'\boldsymbol{\Sigma}_j^{-1}(\mathbf{z}_{ij} - \boldsymbol{\mu}_j)),
\end{array}$$

where $h_{(p)}$ is an spherical density in $\Re^p$, which is independent of $\boldsymbol{\theta}$.

In the sequel, we assume that the relation (see (1.5))

$$\boldsymbol{\theta}_j \to (\boldsymbol{\mu}(\boldsymbol{\theta}_j), \boldsymbol{\Sigma}(\boldsymbol{\theta}_j)), \tag{2.2}$$

$j = 1, \ldots, k$, is one to one, which implies that in (A), (B) and (C) the elliptical model is identifiable (Arellano-Valle and Bolfarine, 1996). However, as the normal model, we need an additional condition on $\boldsymbol{\theta}_j$, $j = 1, \ldots, k$, in order to make the relation (2.2) one to one under the elliptical models. Under the normality assumption, Wong (1991) consider that the ratio $\lambda_{e_j} = \sigma^2_{e_j}/\sigma^2_{uj}$ is known, while Bolfarine and Cordani (1993) consider that the ratio $\lambda_{xj} = \sigma^2_{xj}/\sigma^2_{uj}$ is known, with $k = 1$. Considering the normal model with $k = 1$, these cases are unified in Arellano-Valle and Bolfarine (1995) and extend to elliptical models in Arellano-Valle and Bolfarine (1996). Thus, noting that the indentification of an elliptical model under the three specifications depends on the identification for each one of the subpopulations, which is identifiable for both cases, when the ratio λ_{ej} is known or when λ_{xj} is known, $j = 1, \ldots, k$, it follows that these conditions implies also the identifiability of the elliptical model under the three specifications.

It is assumed also that the density generator functions $h_{(2n)}$, $h_{(2n_j)}$ and $h_{(2)}$, under specifications (A), (B) and (C), respectively, are decreasing and continuosly diferentiable in the interval $(0, \infty)$.

Under the above assumptions, the following results are considered, which are relate with the maximun likelihood estimators (MLE) under the elliptical models (A), (B) and (C).

Proposition 2.1 *Let* $(\hat{\boldsymbol{\mu}}_j, \hat{\boldsymbol{\Sigma}}_j)$ *be the MLE of* $(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$, $j = 1, \ldots, k$, *under the model (A) or (B) or (C). Then, if the relation (2.2) is one to one, the MLE, say* $\hat{\boldsymbol{\theta}}_j$, *of* $\boldsymbol{\theta}_j$, $j = 1, \ldots, k$, *are the solution to the equations*

$$\boldsymbol{\mu}(\hat{\boldsymbol{\theta}}_j) = \hat{\boldsymbol{\mu}}_j \quad \text{and} \quad \boldsymbol{\Sigma}(\hat{\boldsymbol{\theta}}_j) = \hat{\boldsymbol{\Sigma}}_j, \tag{2.5}$$

$j = 1, \ldots, k$.

Proof: Follows from the invariance property of the MLE.

Proposition 2.2 *Let consider the elliptical models (A) and (B). Then, the MLE of* $\boldsymbol{\mu}_j$ *and* $\boldsymbol{\Sigma}_j$ *are given by*

$$\hat{\boldsymbol{\mu}}_j = \bar{\mathbf{Z}}_j \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_j = c_j \mathbf{S}_j, \tag{2.3}$$

$j = 1, \ldots, k$, *where*

$$\bar{\mathbf{Z}}_j = \begin{pmatrix} \bar{\mathbf{Y}}_j \\ \bar{\mathbf{X}}_j \end{pmatrix} \quad \text{and} \quad \mathbf{S}_j = \begin{pmatrix} S^{(j)}_{YY} & S^{(j)}_{YX} \\ S^{(j)}_{XY} & S^{(j)}_{XX} \end{pmatrix}$$

are the sample mean vector and sample covariance matrix, respectively, corresponding to the j-th population,

$$c_j = \begin{cases} \frac{2n}{u^*_{(2n)}}, & \textit{in case (A)}, \\ \frac{2n_j}{u^*_{(2n_j)}}, & \textit{in case (B)}, \end{cases}$$

with $u^*_{(p)}$ being the maximum of the function $u^{p/2}h_{(p)}(u)$, $u > 0$.

Proof: From Proposition 1 in Anderson et al. (1986) it follows that

$$\hat{\boldsymbol{\mu}}_j = \hat{\boldsymbol{\mu}}_{jN} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_j = c_j \hat{\boldsymbol{\Sigma}}_{jN}, \tag{2.4}$$

where $(\hat{\boldsymbol{\mu}}_{jN}, \hat{\boldsymbol{\Sigma}}_{jN})$ is the MLE of $(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$, under the normal model. Since $(\hat{\boldsymbol{\mu}}_{jN}, \hat{\boldsymbol{\Sigma}}_{jN}) = (\bar{\mathbf{Z}}_j, \mathbf{S}_j)$, $j = 1, \ldots, k$, we have the proof.

Note that in the special case where $h_{(p)}(u) = k(p, \nu)\{\nu + u\}^{-\frac{\nu+p}{2}}$, corresponding to the Student-t distribution with ν degrees of freedom, we have that $u^*_{(p)} = p$, for all $\nu > 0$.

Proposition 2.3 *Consider now model (C). Then, the MLE of $\boldsymbol{\mu}_j$ and $\boldsymbol{\Sigma}_j$ are given by the solution to the equations*

$$\sum_{i=1}^{n_j} w_{(2)}(d_{ij})(\mathbf{Z}_{ij} - \hat{\boldsymbol{\mu}}_j) = \mathbf{0} \quad \text{and} \quad \sum_{i=1}^{n_j} w_{(2)}(d_{ij})(\mathbf{Z}_{ij} - \hat{\boldsymbol{\mu}}_j)(\mathbf{Z}_{ij} - \hat{\boldsymbol{\mu}}_j)' = n_j \hat{\boldsymbol{\Sigma}}_j, \tag{2.5}$$

$j = 1, \ldots, k$, where $d_{ij} = (\mathbf{Z}_{ij} - \hat{\boldsymbol{\mu}}_j)\hat{\boldsymbol{\Sigma}}_j^{-1}(\mathbf{Z}_{ij} - \hat{\boldsymbol{\mu}}_j)'$ and $w_{(2)}(u) = -2h'_{(2)}(u)/h_{(2)}(u)$.

Proof: Is direct from the derivate of the log-likelihood function of the model (C) with respect to $(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$, $j = 1, \ldots, k$, (see Section 3).

Note that under specification (C) no closed form are available for the MLE. Thus, in this case the MLEs have to be computed numerically from the equantions (2.5).

One important result relates inference in the elliptical context with results under normality. As such, if $\mathbf{X} \sim El_p(\mathbf{0}, \mathbf{I}_p; \phi)$ with $P(\mathbf{X} = \mathbf{0}) = \mathbf{0}$ and $\delta(\mathbf{X})$ is an statistics such that $\delta(a\mathbf{X}) \stackrel{d}{=} \delta(\mathbf{X})$, for all $a > 0$, then (see Fang et al., 1990) $\delta(\mathbf{X}) \stackrel{d}{=} \delta(\mathbf{Z})$, where $\mathbf{Z} \sim N_p(\mathbf{0}, \mathbf{I}_p)$ and $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$ mean that both $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution.

3. Information matrix and orthogonal parameterization

In this section we consider the information matrix under the three specifications considered in the previous section. By making used these results, orthogonal parameterizations (Cox and Reid, 1987) and the MLE these orthogonal parameters are also considered. The notation and some results given in Arellano-Valle and Bolfarine (1995, 1996) are used.

Let $L = L(\boldsymbol{\theta})$ be the log-likelihood function, where $\boldsymbol{\theta}$ is as in the previous section. Then, under these three specifications L can be written as

$$L = \begin{cases} \sum_{j=1}^{k}(-\frac{n_j}{2})\log|\boldsymbol{\Sigma}_j| + \log f(||\mathbf{T}_*||^2), & \text{in case (A)}, \\ \sum_{j=1}^{k}(-\frac{n_j}{2})\log|\boldsymbol{\Sigma}_j| + \sum_{j=1}^{k}\log g(||\mathbf{T}_{(j)}||^2), & \text{in case (B), ,} \\ \sum_{j=1}^{k}(-\frac{n_j}{2})\log|\boldsymbol{\Sigma}_j| + \sum_{j=1}^{k}\sum_{i=1}^{n_j}\log h(||\mathbf{T}_{ij}||^2), & \text{in case (C)}, \end{cases}$$

where $\mathbf{T}_* = \boldsymbol{\Sigma}_*^{-1/2}(\mathbf{Z}_* - \boldsymbol{\mu}_*)$, $\mathbf{T}_{(j)} = \boldsymbol{\Sigma}_{(j)}^{-1/2}(\mathbf{Z}_{(j)} - \boldsymbol{\mu}_{(j)})$ and $\mathbf{T}_{ij} = \boldsymbol{\Sigma}_j^{-1/2}(\mathbf{Z}_{ij} - \boldsymbol{\mu}_j)$, with

$$||\mathbf{T}_*||^2 = \sum_{j=1}^{k}||\mathbf{T}_{(j)}||^2 = \sum_{j=1}^{k}\sum_{i=1}^{n_j}||\mathbf{T}_{ij}||^2.$$

Thus, denoting by θ_{ij}, $i = 1, \ldots, 5$ the i-th component of $\boldsymbol{\theta}_j$ in (1.6), it follows that

$$\frac{\partial L}{\partial \theta_{lj}} = \begin{cases} -\frac{n_j}{2}tr(\boldsymbol{\Sigma}_j^{-1}\frac{\partial \boldsymbol{\Sigma}_j}{\partial \theta_{lj}}) + W_{(2n)}(||\mathbf{T}_*||^2)\frac{\partial||\mathbf{T}_*||^2}{\partial \theta_{lj}}, & \text{in case (A)}, \\ -\frac{n_j}{2}tr(\boldsymbol{\Sigma}_j^{-1}\frac{\partial \boldsymbol{\Sigma}_j}{\partial \theta_{lj}}) + W_{(2n_j)}(||\mathbf{T}_{(j)}||^2)\frac{\partial||\mathbf{T}_{(j)}||^2}{\partial \theta_{lj}}, & \text{in case (B)}, \\ -\frac{n_j}{2}tr(\boldsymbol{\Sigma}_j^{-1}\frac{\partial \boldsymbol{\Sigma}}{\partial \theta_{lj}}) + \sum_{i=1}^{n_j}W_{(2)}(||\mathbf{T}_{ij}||^2)\frac{\partial||\mathbf{T}_{ij}||^2}{\partial \theta_{lj}}, & \text{in case (C)}, \end{cases}$$

where

$$W_{(p)}(u) = \frac{\partial \log h_{(p)}(u)}{\partial u} = \frac{h'(u)}{h(u)} \quad \text{and} \quad \frac{\partial||\mathbf{T}_*||^2}{\partial \theta_{lj}} = \frac{\partial||\mathbf{T}_{(j)}||^2}{\partial \theta_{lj}} = \sum_{i=1}^{n_j}\frac{\partial||\mathbf{T}_{ij}||^2}{\partial \theta_{lj}}$$

with

$$\frac{\partial||\mathbf{T}_{ij}||^2}{\partial \theta_{lj}} = -2\frac{\partial \boldsymbol{\mu}_j'}{\theta_{lj}}\boldsymbol{\Sigma}_j^{-1/2}\mathbf{T}_{ij} - \mathbf{T}_{ij}'\boldsymbol{\Sigma}_j^{-1/2}\frac{\partial \boldsymbol{\Sigma}_j}{\partial \theta_{lj}}\boldsymbol{\Sigma}_j^{-1/2}\mathbf{T}_{ij},$$

$l = 1, \ldots, 5$, $j = 1, \ldots, k$. By considering this notation, we obtain the following results.

Proposition 3.1 *Let $\mathbf{K} = \mathbf{K}(\boldsymbol{\theta})$ be the information matrix under the three elliptical models (A), (B) and (C). Then, it follows that*

$$\mathbf{K} = \text{diag}(\mathbf{K}_1, \ldots, \mathbf{K}_k), \tag{3.1}$$

where $\mathbf{K}_j = ((\kappa_{l,m}^{(j)}))$, $l, m = 1, \ldots, 5$, with

$$\kappa_{l,m}^{(j)} = \frac{4n_j}{p}a(2,1)\frac{\partial \boldsymbol{\mu}_j'}{\partial \boldsymbol{\theta}_{lj}}\boldsymbol{\Sigma}_j^{-1}\frac{\partial \boldsymbol{\mu}_j}{\partial \theta_{mj}} + \frac{2n_j}{p(p+2)}a(2,2)tr(\boldsymbol{\Sigma}_j^{-1}\frac{\partial \boldsymbol{\Sigma}_j}{\partial \theta_{lj}}\boldsymbol{\Sigma}_j^{-1}\frac{\partial \boldsymbol{\Sigma}_j}{\partial \theta_{mj}}) \tag{3.2}$$

$$+\frac{pn_j}{2}\{\frac{1}{p(p+2)}a(2,2)-\frac{1}{4}\}tr(\boldsymbol{\Sigma}_j^{-1}\frac{\partial\boldsymbol{\Sigma}_j}{\partial\theta_{lj}})tr(\boldsymbol{\Sigma}_j^{-1}\frac{\partial\boldsymbol{\Sigma}_j}{\partial\theta_{mj}}),$$

where

$$a(r,s)=E[(W_{(p)}(||\mathbf{T}||^2))^r||\mathbf{T}||^{2s}],$$

$r,s=1,2,\ r\geq s,$ *with* $\mathbf{T}\sim El_p(\mathbf{0},\mathbf{I}_p;\phi)$ *and*

$$p=\begin{cases}2n, & \textit{in case (A)},\\ 2n_j, & \textit{in case (B)},\\ 2, & \textit{in case (C)}.\end{cases}$$

Proof: Since, $\kappa_{l,m}^{(j)}=E[(\partial L/\partial\theta_{lj})(\partial L/\partial\theta_{mj})]$, the proof follows from the fact that the distributions of the vectors $\mathbf{T}_*,\mathbf{T}_{(j)},\mathbf{T}_{ij}$ are symmetric and by using standard properties of the elliptical distributions (see Arellano-Valle and Bolfarine, 1996).

To simplify the derivation of some statistical procedures, we consider in the following an orthogonal reparameterization (in the sense of Cox and Reid, 1987), which is such that in (3.2) $\kappa_{l,5}^{(j)}=\kappa_{l,\beta_j}=0,\ l\neq 5$. Under normality, orthogonal parameterization was obtained by Wong (1989,1991) when λ_{ej} is known and by Bolfarine and Cordani (1993) when λ_{xj} is known. A unified treatment and the extension for elliptical models is considered in Arellano-Valle and Bolfarine (1995) and Arellano-Valle and Bolfarine (1996), respectively. Thus, as in Arellano-Valle and Bolfarine (1995, 1996), this parameterization can be written as $\boldsymbol{\phi}_j=(\boldsymbol{\phi}_{Lj}',\boldsymbol{\phi}_{Sj}')'$, where

$$\boldsymbol{\phi}_{Lj}=(\phi_{1j},\phi_{2j})' \quad\text{and}\quad \boldsymbol{\phi}_{Sj}=(\phi_{3j},\phi_{4j},\beta_j)', \tag{3.3}$$

with

$$\phi_{1j}=\alpha_j+\beta_j\mu_{xj},\quad \phi_{2j}=\mu_{xj},\quad \phi_{3j}=\sigma_{uj}^2(\lambda_{xj}\beta_j^2+\lambda_{xj}\lambda_{ej}+\lambda_{ej}),\quad \phi_{4j}=\sigma_{uj}^2\quad\text{and}\quad \phi_{5j}=\beta_j,$$

$j=1,\ldots,k$. Thus, from (1.5) we have that $\boldsymbol{\mu}_j=\boldsymbol{\mu}(\boldsymbol{\phi}_{Lj})=\boldsymbol{\phi}_{Lj}$ and

$$\boldsymbol{\Sigma}_j=\boldsymbol{\Sigma}_j(\boldsymbol{\phi}_{Sj})=\begin{cases}(\lambda_{x_j}+1)^{-1}\begin{pmatrix}\phi_{3j}+(\lambda_{xj}\beta_j)^2\phi_{4j} & (\lambda_{xj}+1)\lambda_{xj}\beta_j\phi_{4j}\\ (\lambda_{xj}+1)\lambda_{x_j}\beta_j\phi_{4j} & (\lambda_{xj}+1)^2\phi_{4j}\end{pmatrix}, & \text{if } \lambda_{xj} \text{ is known},\\ (\beta_j^2+\lambda_{ej})^{-1}\begin{pmatrix}\beta_j^2\phi_{3j}+\lambda_{ej}^2\phi_{4j} & \beta_j(\phi_{3j}-\lambda_{ej}\phi_{4j})\\ \beta_j(\phi_{3j}-\lambda_{ej}\phi_{4j}) & \phi_{3j}+\beta_j^2\phi_{4j}\end{pmatrix}, & \text{if } \lambda_{ej} \text{ is known},\end{cases}$$

so that, under both identifiability conditions,

$$|\boldsymbol{\Sigma}_j|=\phi_{3j}\phi_{4j}. \tag{3.4}$$

Proposition 3.2 *Let* $\mathbf{K} = \mathbf{K}(\boldsymbol{\phi})$ *be the information matrix under the orthogonal parameterization given in (3.3). Then,* $\mathbf{K} = \mathbf{K}(\boldsymbol{\phi})$ *is as in (3.1), with*

$$\mathbf{K}_j = \mathrm{diag}(\mathbf{K}_{Lj}, \mathbf{K}_{Sj}),$$

where $\mathbf{K}_{Lj}$ *and* $\mathbf{K}_{Sj}$ *are the information submatrices corresponding to the parameter vectors* $\boldsymbol{\phi}_{Lj}$ *and* $\boldsymbol{\phi}_{Sj}$*, respectively, and are given by*

$$\mathbf{K}_{Lj} = \frac{4}{p} a(2,1)(\frac{1}{n_j}\boldsymbol{\Sigma}_j)^{-1},$$

and $\mathbf{K}_{Sj} = ((\kappa_{l,m}^{(j)}))$*,* $l, m = 3, 4, 5$*, with*

$$\kappa_{l,m}^{(j)} = \begin{cases} \frac{p}{2}\{\frac{p+4\delta_{lm}}{p^2(p+2)} a(2,2) - \frac{1}{4}\}(\frac{\phi_{lj}\phi_{mj}}{n_j})^{-1}, & l, m = 3, 4 \\ \frac{4}{p(p+2)} a(2,2)(\frac{\sigma_{\beta_j}^2}{n_j})^{-1}, & l, m = 5, \\ 0, & l = 3, 4, \quad m = 5 \text{ or } l = 5,\ m = 3, 4, \end{cases},$$

where $\delta_{ij} = 1$*, if* $i = j$ *and zero otherwise,* $a(r, s)$ *and* p *are as in Proposition 3.1, and*

$$\sigma_{\beta_j}^2 = \begin{cases} \frac{\phi_{3j}}{\lambda_{x_j}^2 \phi_{4j}}, & \text{if } \lambda_{xj} \text{ is known,} \\ (\frac{\beta_j^2 + \lambda_{ej}}{\phi_{3j} - \lambda_{ej}\phi_{4j}})^2 \phi_{3j}\phi_{4j}, & \text{if } \lambda_{ej} \text{ is known.} \end{cases}$$

Proof: See Arellano-Valle and Bolfarine (1996).

In particular, for the Student-t model with ν degrees of freedom and generador density function given by $h_{(p)}(u) = k(p, \nu)\{\nu + u\}^{-(\nu+p)/2)}$, it follows that (see Arellano-Valle and Bolfarine, 1996)

$$a(2,1) = \frac{p(\nu + p)}{2(\nu + p + 2)} \quad \text{and} \quad a(2,2) = \frac{p(p+2)(\nu + p)}{2(\nu + p + 2)}.$$

With these expressions, the information matrices can be easily obtained for the Student-t model under specifications (A), (B) and (C).

Proposition 3.3 *Let* $(\hat{\boldsymbol{\mu}}_j, \hat{\boldsymbol{\Sigma}}_j)$*, be the MLE of* $(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$*,* $j = 1, \ldots, k$*, under an elliptical model. Then, if* λ_{ej} *or* λ_{xj} *is known the MLE of* $\boldsymbol{\phi}_{Lj} = (\phi_{1j}, \phi_{2j})'$ *and* $\boldsymbol{\phi}_{Ej} = (\phi_{3j}, \phi_{4j}, \beta_j)'$ *are given by* $\hat{\boldsymbol{\phi}}_{Lj} = \hat{\boldsymbol{\mu}}_j$*,* $\hat{\phi}_{lj} = \mathbf{a}_{lj}(\hat{\beta}_j)'\hat{\boldsymbol{\Sigma}}_j\mathbf{a}_{lj}(\hat{\beta}_j)$*,* $l = 3, 4$*, and* $\hat{\beta}_j$ *is the solution to the equation* $\mathbf{a}_{3j}(\hat{\beta}_j)'\hat{\boldsymbol{\Sigma}}_j\mathbf{a}_{4j}(\hat{\beta}_j) = 0$*,* $j = 1, \ldots, k$*, where*

$$\mathbf{a}_{3j} = \begin{cases} (\lambda_{xj} + 1)^{-1/2} \begin{pmatrix} \lambda_{xj} + 1 \\ -\lambda_{x_j}\beta_j \end{pmatrix}, & \text{if } \lambda_{xj} \text{ is known,} \\ (\beta_j^2 + \lambda_{ej})^{-1/2} \begin{pmatrix} \beta_j \\ \lambda_{e_j} \end{pmatrix}, & \text{if } \lambda_{ej} \text{ is known,} \end{cases}$$

and

$$\mathbf{a}_{4j} = \begin{cases} (\lambda_{x_j}+1)^{-1/2}\begin{pmatrix}0\\1\end{pmatrix}, & \text{if } \lambda_{xj} \text{ is known,}\\ (\beta_j^2+\lambda_{ej})^{-1/2}\begin{pmatrix}1\\-\beta_j\end{pmatrix}, & \text{if } \lambda_{ej} \text{ is known.}\end{cases}$$

Moreover, under the models (A) and (B) it follows that

$$\hat{\phi}_{1j} = \hat{\phi}_{1j}^N = \bar{\mathbf{Y}}_j, \quad \hat{\phi}_{2j} = \hat{\phi}_{2j}^N = \bar{\mathbf{X}}_j, \quad \hat{\phi}_{lj} = c_j\hat{\phi}_{lj}^N = c_j\mathbf{a}_{lj}(\hat{\beta}_j)'\mathbf{S}_j\mathbf{a}_{lj}(\hat{\beta}_j), \; l = 3,4,$$

and

$$\hat{\beta}_j = \hat{\beta}_j^N = \begin{cases} (\frac{\lambda_{x_j}+1}{\lambda_{x_j}})\frac{S_{YX}^{(j)}}{S_{XX}^{(j)}}, & \text{if } \lambda_{xj} \text{ is known,}\\ \frac{S_{YY}^{(j)}-\lambda_{ej}S_{XX}^{(j)}+\{(S_{YY}^{(j)}-\lambda_{ej}S_{XX}^{(j)})^2+4\lambda_{ej}S_{YX}^{(j)}\}^{1/2}}{2S_{YX}^{(j)}}, & \text{if } \lambda_{ej} \text{ is known,}\end{cases}$$

$j = 1,\ldots,k$, *where* c_j *is as in Proposition 2.2 and* $\hat{\phi}_{lj}^N$, $l = 1,\ldots,4$, *and* $\hat{\beta}_j^N$ *are the likelihood estimators of* ϕ_{lj}, $l = 1,\ldots,4$, *and* β_j, *respectively, under the normal model.*

Proof: Follows from Propositions 2.1 and 2.2 and by using the fact that, under the orthogonal parameterization (3.4), $\text{diag}(\phi_{3j},\phi_{3j}) = \mathbf{A}_j\mathbf{\Sigma}_j\mathbf{A}_j'$, $j = 1,\ldots,k$, where $\mathbf{A}_j' = (\mathbf{a}_{3j},\mathbf{a}_{4j})$, (see Arellano-Valle and Bolfarine, 1995 and 1996).

4. Bartlett corrected statistics

In this section we consider likelihood ratio statistics for testing hypothesis of interest related to the models (A) and (B) considered in the previous sections. Under the null hypothesis we consider Bartlett type corrections for improving the approximation to the chisquare distribution. The likelihood ratio statistics is denoted by G and its expected value under null hypothesis by $E_0[G]$, which will be obtained by computing the expected value of G directly by using the properties of the digamma function and the following result.

Lemma 4.1 *Let* $V \sim Gamma(a,b)$ *be the Gamma distribution with parameters* a *and* b. *Then, it follows that*

$$E[\log(V)] = \Psi(a) - \log(b), \tag{4.1}$$

where $\Psi(.)$ *is the digamma function. Moreover,*

$$\Psi\Big(\frac{n_j-1}{2}\Big) - \log\frac{n_j}{2} = -\frac{1}{n_j}\Big\{2 + \frac{11}{6n_j} + O(n_j^{-2})\Big\} \tag{4.2}$$

and
(4.3)
$$\Psi\Big(\frac{n-k-1}{2}\Big)+\Psi\Big(\frac{n-k}{2}\Big)-2\log(\frac{n}{2})=-\frac{1}{n}\Big\{2k+3+\frac{6(k+1)(k+2)+1}{6n}+O(n^{-2})\Big\},$$
where n_j *is the size of the* j*-th population,* $j=1,\ldots,k$ *and* $n=n_1+\ldots+n_k$.

We will also make use of the orthogonal parameterization $\boldsymbol{\phi}=(\boldsymbol{\phi}_1',\ldots,\boldsymbol{\phi}_k')'$, where $\boldsymbol{\phi}_j$ is as in (3.3), $j=1,\ldots,k$. Thus, the likelihood ratio statistics can be write as

$$G=2\{L(\hat{\boldsymbol{\phi}})-L(\tilde{\boldsymbol{\phi}})\}, \tag{4.4}$$

where $\hat{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{\phi}}$ are the maximum likelihood estimators under the unrestricted are restricted model by considering H_0, respectively. Moreover, the corrected version of G is given by
$$G^*=(1+d)^{-1}G,$$
where d is the Bartlett correction factor, which is defined in Cordeiro (1983).

Proposition 4.1 *Consider the elliptical models (A) and (B) under the identifiability conditions* λ_{ej} *or* λ_{xj} *known,* $j=1,\ldots,k$*. Then, under the null hypothesis* $H_0:\beta_j=\beta_{0j}$*, with* β_{0j} *being known,* $j=1,\ldots,k$*, it follows that* $G=G_N$ *and* $G^*=(1+d_N)^{-1}G_N$*, where* G_N *and* d_N *are the likelihood ratio statistics and the respective Bartlett correction factor under normality, which are given by*

$$G_N=\sum_{j=1}^{k}n_j\log\Big\{\frac{\tilde{\phi}_{3j}^N\tilde{\phi}_{4j}^N}{\hat{\phi}_{3j}^N\hat{\phi}_{4j}^N}\Big\}\quad\text{and}\quad d_N=\sum_{j=1}^{k}\frac{5}{2n_jk}. \tag{4.5}$$

with $\hat{\phi}_{lj}^N$ *and* $\tilde{\phi}_{lj}^N$ *being the unrestricted and resticted maximun likelihood estimators of* ϕ_{lj}, $l=3,4$, *respectively, under the normal model.*

Proof: As in the unrestricted case (see (2.4) in Proposition 2.2), the restricted maximum likelihood estimators of $\boldsymbol{\mu}_j$ and $\boldsymbol{\Sigma}_j$ under the elliptical models (A) and (B) are given by $\tilde{\boldsymbol{\mu}}_j=\bar{\mathbf{Z}}_j$ and $\tilde{\boldsymbol{\Sigma}}_j=c_j\tilde{\boldsymbol{\Sigma}}_{Nj}$, $j=1,\ldots,k$, where $\tilde{\boldsymbol{\mu}}_j$ and $\tilde{\boldsymbol{\Sigma}}_{N_j}$ are the respective restricted maximum likelihood estimator under normality and c_j is as in Proposition 2.2. Considering these results and the fact that $L(\boldsymbol{\phi})=L(\boldsymbol{\mu}(\boldsymbol{\phi}),\boldsymbol{\Sigma}(\boldsymbol{\phi}))$, from (4.4) it follows that in both models, (A) and (B), the likelihood ratio statistics G is such that

$$G=\sum_{j=1}^{k}n_j\log\Big\{\frac{|\tilde{\boldsymbol{\Sigma}}_j|}{|\hat{\boldsymbol{\Sigma}}_j|}\Big\}=\sum_{j=1}^{k}n_j\log\Big\{\frac{|\tilde{\boldsymbol{\Sigma}}_{Nj}|}{|\hat{\boldsymbol{\Sigma}}_{Nj}|}\Big\}=G_N. \tag{4.6}$$

Thus, the expresion given in (4.5) for $G=G_N$ follows by considering (3.4) and the Proposition 3.3. On the other hand, considering (4.1) and (4.2), Arellano-Valle and

Bolfarine (1996) (see also Arellano-Valle and Bolfarine, 1995) show that, under H_0,

$$E_0\Big[n_j \log\Big\{\frac{\tilde{\phi}_{3j}^N \tilde{\phi}_{4j}^N)}{\hat{\phi}_{3j}^N \hat{\phi}_{4j})^N}\Big\}\Big] = n_j\Big\{\Psi\Big(\frac{n_j-1}{2}\Big) - \Psi\Big(\frac{n_j-2}{2}\Big)\Big\} = 1 + \frac{5}{2n_j} + O(n_j^{-2}),$$

from where it follows that

$$E_0[G_N] = k + \sum_{j=1}^{k} \frac{5}{2n_j} + O(n_*^{-2}),$$

with $n_* = \min_{1\le j\le k}\{n_j\}$. Thus, the corrected likelihood ratio statistics is given by $G^* = G_N^* = (1+d_N)^{-1}G_N$, with $d_N = \sum_{j=1}^{k}(5/2n_jk)$.

It is important to note that the above result agrees with the results in Wong (1991), who consider the normal model under the identifiability assumptions $\lambda_{ej} = 1$, $j = 1,\ldots,k$. However, Wong (1991) consider the approach introduced in Lawley (1956).

Notice also that under the indetifiability assumptions λ_{xj}, $j = 1,\ldots,k$, known, it follows that (see Arellano-Valle and Bolfarine, 1995) $\tilde{\phi}_{4j}^N = \hat{\phi}_{4j}^N = S_{XX}^{(j)}/(\lambda_{xj}+1)$, $j = 1,\ldots,k$, so that

$$G_N = \sum_{j=1}^{k} n_j \log\Big\{\frac{\tilde{\phi}_{3j}^N}{\hat{\phi}_{3j}^N}\Big\}. \tag{4.6}$$

The same result follows when we consider the indentifiability assumptions λ_{xj} and $\sigma_{uj}^2 = \sigma_{0j}^2$, known, $j = 1,\ldots,k$, since in such case we have that $\tilde{\phi}_{4j}^N = \hat{\phi}_{4j}^N = \sigma_{0j}^2$, $j = 1,\ldots,k$. From this result and from the fact that, under normality (see Arellano-Valle, 1995),

$$n_j \frac{\hat{\phi}_{3j}^N}{\phi_{3j}} \sim \chi^2_{n_j-2} \quad \text{and} \quad n_j \frac{\tilde{\phi}_{3j}^N}{\phi_{3j}} \sim \chi^2_{n_j-1},$$

$j = 1,\ldots,k$, we obtain the following Corollary.

Corollary 4.1 *Under elliptical models (A) and (B) and the conditions λ_{xj} and $\sigma_{uj}^2 = \sigma_{0j}^2$, known, $j = 1,\ldots,k$, it follows that $G = G_N$ is given by (4.6) and $G^* = (1+d_N)^{-1}G_N$, where d_N and H_0 are as in Proposition 4.1.*

Similar results were obtained by Wong (1991) under normality and the identifiability condition which specifies that $\lambda_{ej} = 1$ and σ_{0j}^2, $j = 1,\ldots,k$, known by using the approch introduced in Lawley (1956).

In the following, we consider hypothesis testing for the null hypotheses

$$H_0 : \mathbf{\Sigma}_j = \mathbf{\Sigma} \tag{4.7}$$

$j = 1, \ldots, k$, and

$$H_0 : \boldsymbol{\mu}_j = \boldsymbol{\mu}, \quad \boldsymbol{\Sigma}_j = \boldsymbol{\Sigma}, \tag{4.8}$$

$j = 1, \ldots, k$, where $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\phi}_L)$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\phi}_S)$, with $\boldsymbol{\phi}_L = (\phi_1, \phi_2)'$ and $\boldsymbol{\phi}_S = (\phi_3, \phi_4, \beta)'$.

Proposition 4.2 *Let consider the null hypothesis given by (4.7). Then, under the elliptical model (A) with the identifiability conditions λ_{ej} or λ_{xj} known, $j = 1, \ldots, k$, it follows that $G^* = (1 + d_N)^{-1} G_N$, where*

$$G_N = \sum_{j=1}^{k} n_j \log\Big\{\frac{\tilde{\phi}_3^N \tilde{\phi}_4^N}{\hat{\phi}_{3j}^N \hat{\phi}_{4j}^N}\Big\} \quad \text{and} \quad d_N = \frac{1}{18(k-1)}\Big\{\sum_{j=1}^{k} \frac{37}{n_j} - \frac{6(k+1)(k+2)+1}{n}\Big\},$$

with $n = n_1 + \ldots + n_k$.

Proof. Under the null hypothesis (4.7), the model (A) yields to restricted maximum likelihood given by $\tilde{\boldsymbol{\mu}}_j = \bar{\mathbf{Z}}_j$ and $\tilde{\boldsymbol{\Sigma}} = (2n/u^*_{(2n)})\tilde{\boldsymbol{\Sigma}}_N$, where

$$\tilde{\boldsymbol{\Sigma}}_N = \frac{1}{n} \sum_{j=1}^{k} n_j \mathbf{S}_{n_j}.$$

Thus, as in (4.6), $G = G_N = \sum_{i=1}^{k} n_j \log\{|\tilde{\boldsymbol{\Sigma}}_N|/|\hat{\boldsymbol{\Sigma}}_{Nj}|\}$, where, from (3.4), $|\tilde{\boldsymbol{\Sigma}}_N| = \tilde{\phi}_3^N \tilde{\phi}_4^N$ and $\hat{\boldsymbol{\Sigma}}_{Nj} = \hat{\phi}_{3j}^N \hat{\phi}_{4j}^N$, $j = 1, \ldots, k$. Moreover, under normality and H_0, we have that $\hat{\boldsymbol{\Sigma}}_j = \mathbf{S}_j \overset{ind.}{\sim} W_2(n_j - 1, n^{-1}\boldsymbol{\Sigma})$, $j = 1, \ldots, k$, so that $\tilde{\boldsymbol{\Sigma}}_N \sim W_2(n-k, n^{-1}\boldsymbol{\Sigma})$, where $W_p(m, \mathbf{M})$ denotes the p-dimensional Wishart distribution with m degrees of fredoom and dispersion matrix $\mathbf{M}$. Thus, from the properties of the Wishart distribution (see Muirhead, 1982) and from (4.1) to (4.3) it follows, after some algebraic manipulations, that

$$E_0[G] = 3(k-1) + \frac{1}{6}\Big\{\sum_{j=1}^{k} \frac{37}{n_j} - \frac{6(k+1)(k+2)+1}{n}\Big\} + O(n_*^{-2}),$$

from where we obtain the corrected statistics $G^* = (1 + d_N)^{-1} G_N$, which is closer to the $\chi^2_{3(k-1)}$–distribution than the distribution of the uncorrected statistics G.

Proposition 4.3 *Lets consider the null hypothesis given by (4.8). Then, under the elliptical model (A) with the identifiability conditions λ_{ej} or λ_{xj} known, $j = 1, \ldots, k$, it follows that $G^* = (1 + d_N)^{-1} G_N$, where G_N is as in Proposition 4.2 and*

$$d_N = \frac{37}{30(k-1)}\Big\{\sum_{j=1}^{k} \frac{1}{n_j} - \frac{1}{n}\Big\}.$$

Proof: Is analogouos to the proof of Proposition 4.2, but in this case the maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ under H_0 are given, respectively, by $\tilde{\boldsymbol{\mu}} = \bar{\mathbf{Z}}$ and $\tilde{\boldsymbol{\Sigma}} = (2n/u^*_{(2n)})\tilde{\boldsymbol{\Sigma}}_N$, where

$$\bar{\mathbf{Z}} = \frac{1}{n}\sum_{j=1}^{n} n_j \bar{\mathbf{Z}}_j \quad \text{and} \quad \tilde{\boldsymbol{\Sigma}}_N = \frac{1}{n}\sum_{j=1}^{k}\sum_{i=1}^{n_j}(\mathbf{Z}_{ij} - \bar{\mathbf{Z}})(\mathbf{Z}_{ij} - \bar{\mathbf{Z}})'.$$

Notice that, under normality, $\tilde{\boldsymbol{\Sigma}}_N \sim W_2(n-1, \frac{1}{n}\boldsymbol{\Sigma})$. Thus, as in the proof of Proposition 4.2, from the properties of the Wishar distribution and the digamma function it follows that

$$E_0[G_N] = 5(k-1) + \frac{37}{6}\Big\{\sum_{j=1}^{k}\frac{1}{n_j} - \frac{1}{n}\Big\} + O(n_*^{-2}).$$

5. Final conclusions

In this paper three different extensions the additive normal models are considering by replacing the usual normal model by the more general class of the elliptical distributions. By using orthogonal parametrizations and properties of the distribution of the sample variances, Bartlett corrected likelihood ratio statistics are obtained for the testing of some null hypothesis. Wong (1991) derived such correted statistics for the case of the normal model by using expansions of the likelihood function and computing cummulants up to 4th order. Our approach is different from the one considered in Wong (1991) and we just have to directly compute expected values of the likelihood ratio statistics and properties of the sample variances under the normal model and using invariance properties of the likelihood ratio statistics as considered in Anderson et al. (1986).

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