EXISTENCE OF SOLITARY WAVE SOLUTIONS FOR AN INTERACTION EQUATION OF SHORT AND LONG DISPERSIVE WAVES, PART I.

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Abstract. In this work we study the existence and some properties of solitary wave solutions for an interaction equation between a long internal wave and a short surface wave in a two layer fluid. We obtain the existence of solitary wave solutions using the concentration compactness method developed by P. L. Lions. We also show that solutions of the minimization problem are analytic and they are translations of the symmetric decreasing rearrangement of themself.

1. INTRODUCTION

In this first work we will study the existence and some properties of solitary wave solutions for an interaction equation between a long internal wave and a short surface wave in a two layer fluid when the fluid depth of the lower layer is sufficiently large in comparison with the wavelength of the internal wave. The fluids are assumed with different densities, inviscid and incompressible, and their motions to be two-dimensional and irrotational. If the short wave term is denoted by $u = u(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ and the long wave term by $v = v(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, the phenomena of interaction is described by the following nonlinear coupled system (see [FO]),

$$\begin{cases} iu_t + u_{xx} = \alpha v u, \\ v_t + \gamma D v_x = \beta(|u|^2)_x, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where α, β are positive constants, $\gamma \in \mathbb{R}$ and $D = \mathcal{H}\partial_x$, is a linear differential operator representing the dispersion of the internal wave. Here $\mathcal{H}$ denotes the Hilbert transform defined by

$$\mathcal{H}f(x) = \, p.v. \frac{1}{\pi} \int \frac{f(y)}{y-x} \, dy$$

therefore, D is the multiplier with Fourier operator defined as $\widehat{Dv}(\xi) = |\xi|\widehat{v}(\xi)$.

The system (1.1) has been considered under various settings. For example, Funakoshi and Oikawa ([FO]) have computed numerical solitary wave solutions. Bekiranov, Ogawa and Ponce ([BOP]) proved well-posedness theory of (1.1) in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ based on the techniques introduced by Bourgain ([Bu1], [Bu2]). More precisely, if $|\gamma| < 1$ and $s \geqq 0$, then for any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ there exists $T > 0$ such that the initial value problem (1.1) admits a unique solution $(u(t), v(t)) \in C([0,T); H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R}))$. Moreover, for $T > 0$ the map $(u_0, v_0) \to (u(t), v(t))$ is Lipschitz continuous from $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ to $C([0,T); H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R}))$. For the case $|\gamma| = 1$, we get the same results as above, but for $s > 0$. We note that $T = \infty$ if $s \geqq 1$, as a consequence of the relations (1.2) and (1.3) below.

We note that for any $s \geqq 0$, the solution of Schrödinger part u preserves its $L^2(\mathbb{R})$ – norm, *i.e.,* if

$$H(u) = \int_{\mathbb{R}} |u(x)|^2 \, dx \equiv \|u\|^2 \tag{1.2}$$

then for any $0 < t < T$, $H(u(t)) = H(u_0)$. Moreover, we have the conservations of momentum, and energy:

$$\begin{aligned} G(u,v) &\equiv Im \int_{\mathbb{R}} u(x)\overline{u_x(x)} \, dx - \|v\|^2, \qquad \text{for } s \geqq \frac{1}{2}, \\ E(u,v) &\equiv \|u_x\|^2 + \alpha \int_{\mathbb{R}} v(x)|u(x)|^2 \, dx - \frac{\alpha\gamma}{2\beta}\|D^{1/2}v\|^2, \qquad \text{for } s \geqq 1, \end{aligned} \tag{1.3}$$

The purpose in this paper is to show the existence of solitary wave solutions for (1.1) of the form

$$\begin{cases} u(x,t) = e^{i\omega t}\phi_0(x-ct), \\ v(x,t) = \psi(x-ct), \end{cases} \tag{1.4}$$

where $\phi_0 : \mathbb{R} \to \mathbb{C}$, $\psi : \mathbb{R} \to \mathbb{R}$, are smooth functions such that for each $n \in \mathbb{N}$, $|\phi_0^{(n)}(\xi)| \to 0$, and $\psi^{(n)}(\xi) \to 0$, as $|\xi| \to \infty$, $c > 0$ and $\omega \in \mathbb{R}$. Substituting (u, v)

as above in (1.1) we obtain the coupled system of equations

$$\begin{cases} \phi_0'' - \omega\phi_0 - ic\phi_0' = \alpha\,\psi\phi_0 \\ \gamma\mathcal{H}\psi' - c\psi = \beta\,|\phi_0|^2 \end{cases} \tag{1.5}$$

where "$'$" $= \frac{d}{d\xi}$, $\xi = x - ct$. Now, if we consider $\phi_0(\xi) = e^{ic\xi/2}\phi(\xi)$, for ϕ real-valued, and replace it in (1.5) we finally obtain the pseudo-differential system

$$\begin{cases} \phi'' - \sigma\phi = \alpha\,\psi\phi \\ \gamma\mathcal{H}\psi' - c\psi = \beta\phi^2, \end{cases} \tag{1.6}$$

where $\sigma = \omega - \frac{c^2}{4}$. We show the existence of smooth real solutions (ϕ, ψ) of (1.6) using the Concentration Compactness Method developed by P. L. Lions ([CL], [L1], [L2]). More precisely, we consider the family of minimization problems

$$I_\lambda = \inf\{V(f,g) : (f,g) \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R}) \text{ and } F(f,g) = \lambda\}, \tag{1.7}$$

where $\lambda > 0$ and the functionals V and F are defined as,

$$V(f,g) = \frac{1}{2}\int_{\mathbb{R}} [(f'(x))^2 - \gamma(D^{\frac{1}{2}}g(x))^2 + \sigma f^2(x) + cg^2(x)]dx \tag{1.8}$$

and

$$F(f,g) = \int_{\mathbb{R}} f^2(x)g(x)dx.$$

Now, if we denote the set of minimizers for I_λ by G_λ, namely,

$$G_\lambda = \{(f,g) : (f,g) \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R}),\ V(f,g) = I_\lambda \text{ and } F(f,g) = \lambda\} \tag{1.8a}$$

then, Theorem 2.5 below shows that for each $\gamma < 0$, $\sigma > 0$ and $c > 0$ we have that $G_\lambda \neq \emptyset$, and therefore each element of G_λ, after multiplication by a constant, is a solution of (1.6). In other words, we characterize solutions of system (1.6) as the Euler -Lagrange equation for the constrained minimization problem (1.7).

The variational formulation of solutions for (1.6) combined with the theory of symmetric decreasing rearrangements (see Appendix) allows us to show the existence of solitary wave solutions (ϕ, ψ), such that ϕ and $-\psi$ are even and decreasing positive functions (see Theorem 3.2 below). Moreover, using the recent theory for analyticity of solitary waves setting in Li and Bona ([LB]) we show the analyticity and the strictly decreasing of ϕ and $-\psi$ (see, Theorem 3.3 and Theorem 3.4 below).

Another consequence of our approach is related with the orbital stability theory in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ of the solitary waves defined in (1.4) for the associated initial value problem (1.1). In fact, in the study of the stability of solitary wave solutions for general model of evolution equations (see for example, [A],[B], [Bo], [GSS1], [GSS2], [W]), is necessary to determine the spectral structure of a linear operator associated with the solitary waves. To be more precise, in our case is necessary to show that the operator $\mathcal{L}$ defined as

$$\mathcal{L} = \begin{pmatrix} -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi & \alpha\phi \\ \alpha\phi & -\gamma D + c \end{pmatrix}$$

has exactly one negative eigenvalue of multiplicity one, that zero is a single eigenvalue with eigenfunction (ϕ', ψ'), and that the rest of the spectrum is positive. The existence of a unique negative single eigenvalue for $\mathcal{L}$ is showed in Theorem 2.7 below. To this end the variational characterization of the solitary waves is strongly employed. Other properties required for $\mathcal{L}$ as well as the stability theory will be considered elsewhere.

In comparison with system (1.1), we consider it for the case $\gamma = 0$, namely,

$$\begin{cases} iu_t + u_{xx} = \alpha v u, \\ v_t = \beta(|u|^2)_x. \end{cases} \tag{1.9}$$

This is the most typical case in the theory of wave interaction and it occurs when the fluid depth is sufficiently small in comparison with the wavelength of the internal wave. System (1.9) has been considered under various settings, see for example, Benney ([B1],[B2]), Bekiranov-Ogawa-Ponce ([BOP]), Grimshaw ([G]), Laurençot ([L]), Ma ([Ma]) and Tsutsumi-Hatano ([TH]). In the particular case of the existence and stability theory of solitary wave solutions, the results are more definitive, in the sense that solitary waves for (1.9) ($\gamma = 0$ in (1.6)) are unique (up to translations) and may be computed explicitly as

$$\begin{cases} \phi(\xi) = \sqrt{\frac{2c\sigma}{\alpha\beta}}\,\mathrm{sech}(\sqrt{\sigma}\xi) \\ \psi(\xi) = -\frac{\beta}{c}\phi^2(\xi), \end{cases} \tag{1.10}$$

and the orbital stability of (ϕ, ψ) in (1.10) with regard to the associated initial value problem (1.9) was showed by Laurençot ([L]) in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Thus, it suggests that the analytical solutions for (1.6) are of exponential type and that for γ small enough, these are orbitally stable. In fact, using the implicit function theorem and perturbation theory of linear operators, we will show in a future work that the last conjecture is true, but unfortunately explicit solutions have been arduous to find.

The plan of this paper is as follows. In section 2, we use the Concentration Compactness Method to prove existence of solitary wave solutions of equation (1.1) and we obtain some properties of the solutions of system (1.6). In section 3, we show the evenness and analyticity of solitary wave solutions. In the Appendix, we briefly review some of the main facts about the theory of symmetric decreasing rearrangements of functions necessary in the development of our work.

Notations. Throughout this paper we will denote by $\widehat{f}$ the Fourier transform of f, defined as $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\, dx$. $|f|_{L^p}$ denotes the $L^p(\mathbb{R})$ norm of f, $1 \leqq p \leqq \infty$. In particular, $|\cdot|_{L^2} = \|\cdot\|$ and $|\cdot|_{L^\infty} = |\cdot|_\infty$. We denote by $H^s(\mathbb{R})$ the Sobolev space of all f (tempered distributions) for which the norm $\|f\|_s^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s|\widehat{f}(\xi)|^2\, d\xi$ is finite. The product norm in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$ is denoted by $\|\cdot\|_{s\times r}$. $J^s = (1-\partial_x^2)^{s/2}$ and $D^s = (-\partial_x^2)^{s/2}$ are the Bessel and Riesz potential of order $-s$, respectively, and are defined by $\widehat{J^s f}(\xi) = (1+\xi^2)^{s/2}\widehat{f}(\xi)$ and $\widehat{D^s f}(\xi) = |\xi|^s\widehat{f}(\xi)$. $\mathcal{S}(\mathbb{R})$ is the Schwartz class in $\mathbb{R}$, and $[T, W] = TW - WT$ is the commutator of the operators T and W. In particular, $[\mathcal{H}, f]g = \mathcal{H}(fg) - f\mathcal{H}g$ in which f is regarded as a multiplication operator.

2. EXISTENCE OF SOLITARY WAVE SOLUTIONS

In this section we give a proof of existence of solitary wave solutions for system (1.6) using the Concentration Compactness Method introduced by P.L. Lions (see [L1], [L2]). We call $\{(f_n, g_n)\}_{n\geqq 1}$ in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ a minimizing sequence for I_λ if it satisfies

$$\begin{cases} F(f_n, g_n) = \lambda, \quad \text{for all } n, \qquad \text{and} \\ \lim\limits_{n\to\infty} V(f_n, g_n) = I_\lambda. \end{cases}$$

Considering $\gamma < 0$, $\sigma > 0$ and $c > 0$, we have the following Lemmas.

Lemma 2.1. *For all $\lambda > 0$ we have that I_λ defined in (1.7) satisfies*

$$0 < I_\lambda < \infty,$$

and each minimizing sequence is bounded.

Proof. Since $F(f, g) = \lambda$ and $|f|_\infty \le \|f\|_1$, from Cauchy-Schwartz inequality it follows,

$$\lambda = \int_{\mathbb{R}} f^2(x)g(x)dx \le |f|_\infty \|f\|\|g\| \le \|f\|_1^2 \|g\|_{\frac{1}{2}} \le \|(f,g)\|^3_{1\times\frac{1}{2}}.$$

Now,

$$\begin{aligned} V(f,g) &= \frac{1}{2}\|f'\|^2 + \frac{\sigma}{2}\|f\|^2 - \frac{\gamma}{2}\|D^{\frac{1}{2}}g\|^2 + \frac{c}{2}\|g\|^2 \geq \frac{1}{2}\min\{1,\sigma\}\|f\|_1^2 + \frac{1}{2}\min\{-\gamma, c\}\|g\|_{\frac{1}{2}}^2 \\ &\equiv C_1\|f\|_1^2 + C_2\|g\|_{\frac{1}{2}}^2 \geq \min\{C_1, C_2\}\|(f,g)\|_{1\times\frac{1}{2}}^2 \\ &= C_3\|(f,g)\|_{1\times\frac{1}{2}}^2. \end{aligned}$$

Then,

$$V(f,g) \geq C\lambda^{\frac{2}{3}} > 0$$

and therefore $I_\lambda > 0$.

Now, let $\{(f_n, g_n)\}_{n \geqq 1}$ be a minimizing sequence for (1.7), then for n large

$$C_3\|(f_n, g_n)\|_{1\times\frac{1}{2}}^2 \leq V(f_n, g_n) < I_\lambda + 1$$

therefore, there exists $M > 0$ such that $\|(f_n, g_n)\|_{1\times\frac{1}{2}} \leq M$ for all n. ■

In order to show the existence of solutions of equation (1.6) we will prove that a minimizing sequence for problem (1.7) converges (modulo translations) to a function in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ satisfying the constraint $F(f,g) = \lambda$. To do this, the next result is essential in our work.

Lemma 2.2. *Let $\{\rho_n\}_{n\geq 1}$ be a sequence of non-negative functions in $L^1(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \rho_n(x)dx = \lambda$ for all n and some $\lambda > 0$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\geq 1}$ satisfying one of the following three conditions:*

(1) (Compactness) there are $y_k \in \mathbb{R}$ for $k = 1, 2, ...$, such that $\rho_{n_k}(\cdot + y_k)$ is tight, i.e. for any $\epsilon > 0$, there is $R > 0$ large enough such that

$$\int_{|x-y_k|\leq R} \rho_{n_k}(x)\,dx \geq \lambda - \epsilon;$$

(2) (Vanishing) for any $R > 0$, $\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq R} \rho_{n_k}(x)dx = 0$;

(3) (Dichotomy) there exists $\tilde{\alpha} \in (0, \lambda)$ such that for any $\epsilon > 0$, there exists $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L^1(\mathbb{R})$, $\rho_k^1, \rho_k^2 \geq 0$ such that for $k \geq k_0$,

$$\begin{cases} |\rho_{n_k} - (\rho_k^1 + \rho_k^2)|_{L^1} \leq \epsilon, \\ |\int_{\mathbb{R}} \rho_k^1\,dx - \tilde{\alpha}| \leq \epsilon, \quad |\int_{\mathbb{R}} \rho_k^2\,dx - (\lambda - \tilde{\alpha})| \leq \epsilon, \\ supp\ \rho_k^1 \cap supp\ \rho_k^2 = \emptyset, \ \ dist(supp\ \rho_k^1,\ supp\ \rho_k^2) \to \infty \text{ as } k \to \infty. \end{cases}$$

Remark: In Lemma 2.2 above, the condition $\int_{\mathbb{R}} \rho_n(x)\,dx = \lambda$ can be replaced by $\int_{\mathbb{R}} \rho_n(x)\,dx = \lambda_n$ where $\lambda_n \to \lambda > 0$. It is enough to replace ρ_n by ρ_n/λ_n and apply the Lemma.

Proof. See Lemma I.1 in Lions [L1]. ■

The following Lemma is a consequence of Lemma I.1 in Lions [L2], but we repeat its proof here for the reader's convenience.

Lemma 2.3. *Let $\{(f_n, g_n)\}_{n\geqq 1}$ be a bounded sequence in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$. Assume that for some $R > 0$,*

$$Q_n(R) \equiv \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |f_n|^2 dx \to 0$$

as $n \to \infty$. Then,

$$\int_{\mathbb{R}} f_n^2(x) g_n(x)\,dx \to 0 \qquad as \;\; n \to \infty.$$

Proof. Initially from Hölder's inequality and from the embedding $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$, $p \geq 2$, we have, for $C > 0$

$$\int_{\mathbb{R}} |f_n|^2 |g_n|\,dx \leq \Big(\int_{\mathbb{R}} |f_n|^3\,dx\Big)^{\frac{2}{3}} \Big(\int_{\mathbb{R}} |g_n|^3\,dx\Big)^{\frac{1}{3}} \leq C\|g_n\|_{\frac{1}{2}} |f_n|^2_{L^3}. \tag{2.1}$$

Now, from the embedding of $W^{1,1}(\Omega)$ into $L^{\frac{3}{2}}(\Omega)$, there is a constant $C_1 = C_1(R) > 0$ such that

$$\begin{aligned}\int_{y-R}^{y+R} |f_n|^3 dx &\leq C_1\Big(\int_{y-R}^{y+R} |f_n|^2 + |(f_n^2)'|\,dx\Big)^{\frac{3}{2}} \\ &\leqq C_1\Big(Q_n(R) + 2M[Q_n(R]^{\frac{1}{2}}\Big)^{\frac{1}{2}}\Big(2\int_{y-R}^{y+R} |f_n|^2\,dx + \int_{y-R}^{y+R} |f_n'|^2\,dx\Big) \\ &\leq C_2\delta_n^{\frac{1}{2}}\Big(\int_{y-R}^{y+R} |f_n|^2 + |f_n'|^2\,dx\Big)\end{aligned} \tag{2.2}$$

with $\delta_n \to 0$ as $n \to \infty$. Then covering $\mathbb{R}$ by intervals of radius R in such way that any point of $\mathbb{R}$ is contained in at most 2 intervals, we deduce from (2.2) that

$$\int_{\mathbb{R}} |f_n|^3\,dx \leq C_2\|f_n\|_1^2\delta_n^{\frac{1}{2}} \leq C_3\delta_n^{\frac{1}{2}} \quad \Longrightarrow \lim_{n\to\infty} \int_{\mathbb{R}} |f_n|^3\,dx = 0.$$

Hence, from (2.1), (2.2) and Lemma 2.1, the Lemma follows. ■

Lemma 2.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with $\varphi' \in L^\infty(\mathbb{R})$. Then the operator $[J^{\frac{1}{2}}, \varphi]$ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ with*

$$\|[J^{\frac{1}{2}}, \varphi]f\| \leq C|\varphi'|_\infty \|f\| \tag{2.3}$$

where C is a positive constant.

Proof. We note initially that for I the identity operator on $L^2(\mathbb{R})$,

$$[J^{\frac{1}{2}}, \varphi] = [I + (J^{\frac{1}{2}} - I), \varphi] = [J^{\frac{1}{2}} - I, \varphi] \equiv [S, \varphi]$$

where S has symbol $s(\xi) = (1+\xi^2)^{\frac{1}{4}} - 1$, that is, $\widehat{Sf}(\xi) = s(\xi)\hat{f}(\xi)$. Thus

$$S = \frac{d}{dx}W = W\frac{d}{dx}$$

where W is a Fourier multiplier operator with symbol $w(\xi) = \frac{s(\xi)}{i\xi}$, for $\xi \neq 0$ and $w(0) = 0$. Since $s(0) = 0$ and s is differentiable in $\xi = 0$ we have that w is continuous in $\xi = 0$ and therefore bounded in $\mathbb{R}$. Moreover w is C^∞ in $\mathbb{R}$. Then, W is a bounded operator in $L^2(\mathbb{R})$ and for each $j \geq 0$, we have

$$\sup_{\xi \in \mathbb{R}} |\xi|^j \left| \frac{d^j}{d\xi^j} w(\xi) \right| < \infty.$$

Therefore, from Theorem 35 in Coifman-Meyer [CM], we have

$$\|[W, \varphi]f'\| \leq C|\varphi'|_\infty \|f\| \tag{2.3a}$$

with $C > 0$ independent of φ and f. Hence (2.3) follows from the relation

$$\begin{aligned}\|[S, \varphi]f\| &= \|W\frac{d}{dx}(\varphi f) - \varphi W \frac{d}{dx} f\| \\ &\leq \|W(\varphi' f)\| + \|[W, \varphi]f'\|,\end{aligned}$$

together with the fact that W is bounded in $L^2(\mathbb{R})$ and (2.3a). ■

Now we establish the main result of this section,

Theorem 2.5. *Let $\alpha, \beta, \sigma, c > 0$, $\gamma < 0$, and let λ be any positive number. Then the set G_λ defined in (1.8a) is nonempty. Hence, there is a solution of problem (1.7), and therefore there exists a non-trivial solution of problem (1.6). Thus the equation (1.1) has solitary-wave solutions corresponding to phase ω and wave speed c.*

Proof. Let $\{(f_n, g_n)\}_{n\geqq 1}$ be a minimizing sequence for problem (1.7) and consider the function

$$\rho_n(x) = |f_n(x)|^2 + |f_n'(x)|^2 + |J^{\frac{1}{2}} g_n(x)|^2.$$

Let $\mu_n = \int_{\mathbb{R}} \rho_n(x)\,dx$. Since $\mu_n = \|f_n\|_1^2 + \|g_n\|_{\frac{1}{2}}^2 \equiv \|(f_n, g_n)\|_{1\times\frac{1}{2}}^2$, we have that μ_n is bounded and $\mu_n \geq \lambda^{\frac{2}{3}}$ (see proof of Lemma 2.1). We suppose that $\mu_n \to \mu$ as $n \to \infty$. In order to prove Theorem 2.5 we apply Lemma 2.2 to the sequence $\{\rho_n\}_{n\geqq 1}$, after ruling out the possibilities of Vanishing and Dichotomy. Suppose there is a subsequence $\{\rho_{n_k}\}_{k\geq 1}$ of $\{\rho_n\}_{n\geqq 1}$ which satisfies either Vanishing or Dichotomy. If Vanishing occurs, then for any $R > 0$

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} \rho_{n_k}(x)\,dx = 0,$$

then

$$\lim_{k\to\infty} Q_{n_k}(R) = 0,$$

so, by Lemma 2.3

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_{n_k}^2(x) g_{n_k}(x) dx = 0.$$

But this is a contradiction, since $F(f_{n_k}, g_{n_k}) = \lambda > 0$.

If Dichotomy occurs, there is $\overline{\mu} \in (0, \mu)$ such that for every $\epsilon > 0$ there exist $k_0 \geq 1$ and ρ_k^1, $\rho_k^2 \geq 0$ such that for $k \geq k_0$,

$$\begin{cases} |\rho_{n_k} - (\rho_k^1 + \rho_k^2)|_{L^1} \leq \epsilon \\ |\int_{\mathbb{R}} \rho_k^1\,dx - \overline{\mu}| \leq \epsilon, \quad |\int_{\mathbb{R}} \rho_k^2\,dx - (\mu - \overline{\mu})| < \epsilon, \\ \mathit{supp}\ \rho_k^1 \cap \mathit{supp}\ \rho_k^2 = \emptyset \quad \mathit{dist}(\mathit{supp}\ \rho_k^1,\ \mathit{supp}\ \rho_k^2) \to \infty \ \ \mathit{as}\ k \to \infty. \end{cases} \tag{2.4}$$

Moreover, we may assume that the supports of ρ_k^1 and ρ_k^2 are separated since follows

$$\begin{cases} \mathit{supp}\ \rho_k^1 \subset (y_k - R_0, y_k + R_0) \\ \mathit{supp}\ \rho_k^2 \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, +\infty) \end{cases} \tag{2.5}$$

for some fixed $R_0 > 0$, a sequence $\{y_k\}_{k\geqq 1} \subset \mathbb{R}$ and $R_k \to \infty$, as $k \to \infty$ (see Lions [L1]).

Now denoting by $\mathbf{h}_{n_k} = (f_{n_k}, g_{n_k})$ we obtain splitting functions $\mathbf{h}^1_{n_k}$ and $\mathbf{h}^2_{n_k}$ of $\mathbf{h}_{n_k}$. Let $\varphi,\ \zeta \in C^\infty(\mathbb{R})$ with $0 \le \varphi,\ \zeta \le 1$ be such that $\zeta(x) = 1$, for $|x| \le 1$ and $\zeta(x) = 0$ for $|x| \ge 2$, $\varphi(x) = 0$ for $|x| \le 1$ and $\varphi(x) = 1$ for $|x| \ge 2$.

Denote by $\zeta_k(x) = \zeta(\frac{x-y_k}{R_1})$ and $\varphi_k(x) = \varphi(\frac{x-y_k}{R_k})$, for $x \in \mathbb{R}$, where $R_1 > R_0$ is chosen large enough such that

$$\Big| \int_{\mathbb{R}} |\zeta_k f_{n_k}|^2 + |(\zeta_k f_{n_k})'|^2 + |J^{\frac{1}{2}}(\zeta_k g_{n_k})|^2 - \rho^1_k \, dx \Big| \le \epsilon \tag{2.6}$$

and

$$\Big| \int_{\mathbb{R}} |\varphi_k f_{n_k}|^2 + |(\varphi_k f_{n_k})'|^2 + |J^{\frac{1}{2}}(\varphi_k g_{n_k})|^2 - \rho^2_k \, dx \Big| \le \epsilon. \tag{2.7}$$

To see that this is possible, from (2.4) and (2.5), we have initially that

$$\begin{aligned} &\int_{|x-y_k|\le R_0} |\rho_{n_k} - \rho^1_k| \, dx \le \epsilon, \quad \int_{|x-y_k|\ge 2R_k} |\rho_{n_k} - \rho^2_k| \, dx \le \epsilon \\ &\int_{R_0 \le |x-y_k| \le 2R_k} \rho_{n_k} \, dx \le \epsilon. \end{aligned} \tag{2.8}$$

Now, since

$$\|J^{\frac{1}{2}}(\zeta_k g_{n_k})\|^2 = \|\zeta_k J^{\frac{1}{2}} g_{n_k}\|^2 + \|[J^{\frac{1}{2}}, \zeta_k] g_{n_k}\|^2 + 2 \int_{\mathbb{R}} \zeta_k J^{\frac{1}{2}}(g_{n_k}) [J^{\frac{1}{2}}, \zeta_k] g_{n_k} \, dx$$

we have from Lemma 2.4, Lemma 2.1 and Cauchy-Schwarz inequality that the last two terms in the right hand side of last equality can be estimated as

$$\begin{aligned} \|[J^{\frac{1}{2}}, \zeta_k] g_{n_k}\|^2 + 2\Big| \int_{\mathbb{R}} \zeta_k J^{\frac{1}{2}}(g_{n_k}) [J^{\frac{1}{2}}, \zeta_k] g_{n_k} \, dx \Big| & \\ \leqq C(|\zeta_k'|^2_\infty \|g_{n_k}\|^2) + 2|\zeta_k'|_\infty \|J^{\frac{1}{2}} g_{n_k}\| \|g_{n_k}\| & \\ \leqq C_1\Big(\frac{|\zeta'|^2_\infty}{R_1^2} + \frac{|\zeta'|_\infty}{R_1} \Big) \le \frac{C_2}{R_1}, & \end{aligned}$$

for large R_1. It follows then from (2.8) that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} \rho_k^1\,dx - \int_{\mathbb{R}} |\zeta_k f_{n_k}|^2 + |(\zeta_k f_{n_k})'|^2 + |J^{\frac12}(\zeta_k g_{n_k})|^2\,dx\Big| \\
&\qquad = \Big|\int_{\mathbb{R}} \rho_k^1\,dx - \int_{\mathbb{R}} |\zeta_k f_{n_k}|^2 + |(\zeta_k f_{n_k})'|^2 + |\zeta_k J^{\frac12} g_{n_k}|^2\,dx \\
&\qquad - \int_{\mathbb{R}} |[J^{\frac12},\zeta_k]g_{n_k}|^2 + 2\zeta_k J^{\frac12}(g_{n_k})[J^{\frac12},\zeta_k]g_{n_k}\,dx\Big| \\
&\qquad \le \Big|\int_{\mathbb{R}} |\zeta_k f_{n_k}|^2 + |(\zeta_k f_{n_k})'|^2 + |\zeta_k J^{\frac12} g_{n_k}|^2 - \rho_k^1\,dx\Big| + \frac{C_2}{R_1} \\
&\qquad \le |\int_{|x-y_k|\le R_0} |f_{n_k}|^2 + |f_{n_k}'|^2 + |J^{\frac12} g_{n_k}|^2 - \rho_k^1\,dx\Big| \\
&\qquad + \int_{R_0\le|x-y_k|\le 2R_1} |\zeta_k f_{n_k}|^2 + |(\zeta_k f_{n_k})'|^2 + |\zeta_k J^{\frac12} g_{n_k}|^2\,dx + \frac{C_2}{R_1} \\
&\qquad \le \int_{|x-y_k|\le R_0} |\rho_{n_k} - \rho_k^1|\,dx + 2(|\zeta_k|_\infty^2 + |\zeta_k'|_\infty^2)\int_{R_0\le|x-y_k|\le 2R_1} \rho_{n_k}\,dx + \frac{C_2}{R_1} \\
&\qquad \le \epsilon + 2\Big(|\zeta|_\infty^2 + \frac{|\zeta'|_\infty^2}{R_1^2}\Big)\epsilon + \frac{C_2}{R_1} = O(\epsilon)
\end{aligned}$$

if R_1 is large enough. Therefore we obtain (2.6). Similarly we have that (2.7) is satisfied.

Thus if we set,

$$\begin{cases} \mathbf{h}_k^1 = \zeta_k \mathbf{h}_{n_k}, \\ \mathbf{h}_k^2 = \varphi_k \mathbf{h}_{n_k} \end{cases}$$

and we define,

$$\mathbf{w}_k = \mathbf{h}_{n_k} - (\mathbf{h}_k^1 + \mathbf{h}_k^2)$$

we have that $\mathbf{h}_k^1, \mathbf{h}_k^2,\ \mathbf{w}_k \in H^1(\mathbb{R})\times H^{\frac12}(\mathbb{R})$. Now, since $\int_{\mathbb{R}} |\zeta_k^3 f_{n_k}^2 g_{n_k}|dx$ is bounded, there exists a subsequence of $\{\mathbf{h}_k^1\}_{k\geqq 1}$, still denoted by $\{\mathbf{h}_k^1\}$, for which there is $k_0 > 0$ and $\theta \in \mathbb{R}$ such that for all $k \ge k_0$

$$\Big|\int_{\mathbb{R}} \zeta_k^3 f_{n_k}^2 g_{n_k}\,dx - \theta\Big| \le \epsilon. \tag{2.9}$$

Moreover, since for $\chi_k = 1 - \zeta_k - \varphi_k$

$$\|\chi_k f_k\|_1^2 \le 2\Big(|\chi_k|_\infty^2 + |\chi_k'|_\infty^2\Big)\int_{R_1\le|x-y_k|\le 2R_k} \rho_{n_k}(x)\,dx = O(\epsilon)$$

and

$$\|\chi_k g_{n_k}\|_{\frac{1}{2}}^2 \leq 2\|[J^{\frac{1}{2}},\chi_k]g_{n_k}\|^2 + 2\int_{R_1\leq|x-y_k|\leq 2R_k} \chi_k^2(x)(J^{\frac{1}{2}}g_{n_k}(x))^2\,dx$$
$$\leq C|\chi_k'|_\infty^2\|g_{n_k}\|^2 + 2|\chi_k|_\infty^2\int_{R_1\leq|x-y_k|\leq 2R_k}\rho_{n_k}\,dx$$
$$\leq C_1\Big(\frac{|\zeta'|_\infty^2}{R_1^2}+\frac{|\varphi'|_\infty^2}{R_k^2}\Big)+C_2\epsilon = O(\epsilon),$$

we have that $\|\mathbf{w}_k\|_{1\times\frac{1}{2}} = O(\epsilon)$. Therefore, it follows from (2.9) that for $k\geq k_0$

$$\Big|\int_{\mathbb{R}}\varphi_k^3 f_{n_k}^3 g_{n_k}\,dx - (\lambda-\theta)\Big| < \epsilon, \tag{2.10}$$

in fact, from Cauchy-Schwarz inequality and Sobolev embedding, we have for $k \geqq k_0$

$$\Big|\int_{\mathbb{R}}\varphi_k^3 f_{n_k}^3 g_{n_k}\,dx - (\lambda-\theta)\Big| \leq \Big|\int_{\mathbb{R}} f_{n_k}^2 g_{n_k}(\varphi_k^3+\zeta_k^3-1)\,dx\Big| + \epsilon$$
$$\leq \int_{\mathbb{R}}|\chi_k|^3|f_{n_k}^2 g_{n_k}|\,dx + 3\int_{\mathbb{R}}|\chi_k|^2|f_{n_k}|^2|g_{n_k}|\,dx + 3\int_{\mathbb{R}}|\chi_k||f_{n_k}|^2|g_{n_k}|\,dx+\epsilon$$
$$\leq |\chi_k f_{n_k}|_{L^4}^2\|\chi_k g_{n_k}\| + 3|\chi_k f_{n_k}|_{L^4}^2\|g_{n_k}\| + 3\|\chi_k g_{n_k}\||f_{n_k}|_{L^4}^2+\epsilon$$
$$\leq C_1\|\mathbf{w}_k\|_{1\times\frac{1}{2}}^3 + C_2\|\mathbf{w}_k\|_{1\times\frac{1}{2}}^2 + C_3\|\mathbf{w}_k\|_{1\times\frac{1}{2}}+\epsilon = O(\epsilon).$$

Now, denoting $\mathbf{w}_k = (a_k, b_k)$ we have from (1.8) that

$$V(\mathbf{h}_{n_k}) = V(\mathbf{w}_{n_k}+\mathbf{h}_k^1+\mathbf{h}_k^2) = V(\mathbf{w}_{n_k}) + V(\mathbf{h}_k^1) + V(\mathbf{h}_k^2) + J_1 + J_2 \tag{2.11}$$

where,

$$J_1 = \int_{\mathbb{R}} a_k'(\zeta_k f_{n_k})' + (a_k' + (\zeta_k f_{n_k})')(\varphi_k f_{n_k})' + \sigma a_k\zeta_k f_{n_k} + \sigma(a_k+\zeta_k f_{n_k})\varphi_k f_{n_k}\,dx$$

and

$$J_2 = -\int_{\mathbb{R}}\gamma D^{\frac{1}{2}}(b_k)D^{\frac{1}{2}}(\zeta_k g_{n_k}) + \gamma(D^{\frac{1}{2}}b_k + D^{\frac{1}{2}}(\zeta_k g_{n_k}))D^{\frac{1}{2}}(\varphi_k g_{n_k})\,dx$$
$$-\int_{\mathbb{R}} c(b_k+\zeta_k g_{n_k})\varphi_k g_{n_k} + cb_k\zeta_k g_{n_k}\,dx.$$

Next we estimate each term integral in J_i. In fact, from Cauchy-Schwarz inequality it follows that

$$J_1 \leq C\|\mathbf{w}_k\|_{1\times\frac{1}{2}}\|\mathbf{h}_k\|_{1\times\frac{1}{2}} = O(\epsilon) \tag{2.12}$$

where $C = C(\zeta,\varphi) > 0$. With regard to J_2 we have first that

$$\begin{aligned}\int_{\mathbb{R}} D^{\frac{1}{2}} b_k D^{\frac{1}{2}}(\zeta_k g_{n_k})\,dx &\leq \|D^{\frac{1}{2}}b_k\|\Big(\|(D^{\frac{1}{2}} - J^{\frac{1}{2}})\zeta_k g_{n_k}\| + \|J^{\frac{1}{2}}(\zeta_k g_{n_k})\|\Big)\\ &\leq C\|\mathbf{w}_k\|_{1\times\frac{1}{2}}\Big(\|\zeta_k g_{n_k}\| + \|[J^{\frac{1}{2}},\zeta_k]g_{n_k}\| + \|\zeta_k J^{\frac{1}{2}} g_{n_k}\|\Big)\\ &\leq C\|\mathbf{w}_k\|_{1\times\frac{1}{2}}\Big(\|g_{n_k}\| + |\zeta_k'|_\infty\|g_{n_k}\| + \|J^{\frac{1}{2}}g_{n_k}\|\Big) = O(\epsilon),\end{aligned} \tag{2.13}$$

where we used that $(J^{\frac{1}{2}} - D^{\frac{1}{2}}) \in B(L^2(\mathbb{R}), L^2(\mathbb{R}))$ and Lemma 2.4. Similarly we have

$$\int_{\mathbb{R}} D^{\frac{1}{2}}(b_k)D^{\frac{1}{2}}(\varphi_k g_{n_k})\,dx = O(\epsilon). \tag{2.14}$$

Now, let n_k be fixed and consider $\{p_j\}_{j\geqq 1} \subset \mathcal{S}(\mathbb{R})$ such that, $p_j \to g_{n_k}$, as $j \to \infty$, in $H^{\frac{1}{2}}(\mathbb{R})$. Then, from Lemma 2.4, $\zeta_k p_j \to \zeta_k g_{n_k}$ and $\varphi_k p_j \to \varphi_k g_{n_k}$, as $j \to \infty$, in $H^{\frac{1}{2}}(\mathbb{R})$. Thus,

$$\int_{\mathbb{R}} D^{\frac{1}{2}}(\zeta_k p_j)D^{\frac{1}{2}}(\varphi_k p_j)\,dx \longrightarrow \int_{\mathbb{R}} D^{\frac{1}{2}}(\zeta_k g_{n_k})D^{\frac{1}{2}}(\varphi_k g_{n_k})\,dx, \qquad \text{as } j \to \infty.$$

Now, since $D = \mathcal{H}\partial_x$ and the supports of ζ_k, φ_k are disjoints, we have that

$$\begin{aligned}\int_{\mathbb{R}} D^{\frac{1}{2}}(\zeta_k p_j)D^{\frac{1}{2}}(\varphi_k p_j)\,dx = \int_{\mathbb{R}} \varphi_k p_j D(\zeta_k p_j)\,dx &= \int_{\mathbb{R}} \varphi_k p_j([\mathcal{H},\zeta_k]p_j' + \mathcal{H}(\zeta_k' p_j))\,dx\\ &\leqq \|\varphi_k p_j\|\Big(\|[\mathcal{H},\zeta_k]p_j'\| + \|\zeta_k' p_j\|\Big)\\ &\leqq C\frac{|\zeta'|_\infty}{R_1}\|p_j\|^2,\end{aligned}$$

where in the last inequality we used the Calderon Commutator Theorem ([C]). Therefore, letting $j \to \infty$ in the last inequality, we have finally that

$$\int_{\mathbb{R}} D^{\frac{1}{2}}(\zeta_k g_{n_k})D^{\frac{1}{2}}(\varphi_k g_{n_k})\,dx \leqq C\frac{|\zeta'|_\infty}{R_1}\|g_{n_k}\|^2 = O(\epsilon). \tag{2.15}$$

Thus from (2.13), (2.14), (2.15) and Lemma 2.1 it follows that

$$J_2 \leqq C_1\epsilon + c\int_{\mathbb{R}} |b_k(\varphi_k+\zeta_k)g_{n_k}|\,dx \le C_1\epsilon + C_2\|\mathbf{w}_k\|\|g_{n_k}\| = O(\epsilon). \tag{2.16}$$

Therefore, since $V(\mathbf{w}_k)\le C\|\mathbf{w}_k\|^2_{1\times\frac{1}{2}}$, it follows from (2.11), (2.12) and (2.16) that

$$V(\mathbf{h}_{n_k}) = V(\mathbf{h}_k^1) + V(\mathbf{h}_k^2) + O(\epsilon),$$

thus

$$I_\lambda = \liminf_n V(\mathbf{h}_{n_k}) \ge \liminf_n V(\mathbf{h}_k^1) + \liminf_n V(\mathbf{h}_k^2) + O(\epsilon). \tag{2.17}$$

Now, if

$$\int_{\mathbb{R}} \zeta_k^3 f_{n_k}^2 g_{n_k}\,dx \to \theta = 0,$$

then from (2.10) for ϵ small we have that if k is large enough, $F(\mathbf{h}_k^2) > \frac{\lambda}{2}$. Let k be fixed and consider $d_k > 0$ such that

$$\int_{\mathbb{R}} d_k^3\varphi_k^3 f_{n_k}^3 g_{n_k}\,dx = \lambda,$$

therefore $F(d_k\mathbf{h}_k^2)=\lambda$. Moreover,

$$|d_k - 1| = \frac{|\lambda^{\frac{1}{3}} - F(\mathbf{h}_k^2)^{\frac{1}{3}}|}{F(\mathbf{h}_k^2)^{\frac{1}{3}}} \le \frac{2}{\lambda}|\lambda^{\frac{1}{3}} - F(\mathbf{h}_k^2)^{\frac{1}{3}}| \le C_1\epsilon$$

with C_1 independent of $\mathbf{h}_k^2$ and ϵ. Hence, $b_k\to 1$, as $k\to\infty$, and

$$I_\lambda \le V(d_k\mathbf{h}_k^2) = d_k^2 V(\mathbf{h}_k^2) = V(\mathbf{h}_k^2) + O(\epsilon). \tag{2.18}$$

Now, from (2.6) and (2.4) it follows that,

$$\begin{aligned}\liminf V(\mathbf{h}_k^1) &\ge C_2\liminf \|\mathbf{h}_k^1\|^2_{1\times\frac{1}{2}} = C_2\liminf\left(\|\zeta_k f_{n_k}\|_1^2 + \|\zeta_k g_{n_k}\|^2_{\frac{1}{2}}\right)\\ &\ge C_2\liminf |\rho_k^1|_{L^1} + O(\epsilon) \ge C_2\overline{\mu} + O(\epsilon),\end{aligned}$$

and therefore, from (2.17) and (2.18)

$$I_\lambda \ge C_2\overline{\mu} + I_\lambda + O(\epsilon).$$

Finally, letting $\epsilon \to 0$ in the last relation leads to the contradiction $I_\lambda \geq C_2\overline{\mu} + I_\lambda$.

If, on the other hand, $\int_{\mathbb{R}} \zeta_k^3 f_{n_k}^3 g_{n_k}\, dx \to \theta \neq 0$, we can assume without loss of generality that $0 < \theta < \lambda$, and using the same last procedure, together with (2.4) and (2.7), we can prove that

$$I_\lambda \geq I_\theta + I_{\lambda-\theta} + O(\epsilon)$$

and let $\epsilon \to 0$ to obtain

$$I_\lambda \geq I_\theta + I_{\lambda-\theta}.$$

But, for $\tau > 0$, $I_{\tau\lambda} = \tau^{\frac{2}{3}} I_\lambda$. Therefore, if we write $\theta = \tau\lambda$, we have then

$$I_\lambda \geq I_{\tau\lambda} + I_{(1-\tau)\lambda} = (\tau^{\frac{2}{3}} + (1-\tau)^{\frac{2}{3}}) I_\lambda > I_\lambda$$

another contradiction. Thus the case of Dichotomy cannot occur.

Since Vanishing and Dichotomy have been ruled out, it follows from Lemma 2.2 that there is a sequence $\{y_k\}_{k\geqq 1} \subset \mathbb{R}$ suth that for any $\epsilon > 0$, there are $R > 0$ large and $k_0 > 0$ such that for $k \geq k_0$

$$\int_{|x-y_k|\leq R} \rho_{n_k}(x)\, dx \geq \mu - \epsilon, \qquad \int_{|x-y_k|\geq R} \rho_{n_k}(x)\, dx \leq \epsilon,$$

and

$$\begin{aligned} \Big| \int_{|x-y_k|\geq R} f_{n_k}^2 g_{n_k}\, dx \Big| &\leq |g_{n_k}|_{L^3} \Big(\int_{|x-y_k|\geq R} |f_{n_k}|^3\, dx \Big)^{\frac{2}{3}} \\ &\leq C \|g_{n_k}\|_{\frac{1}{2}} |f_{n_k}|_\infty^{\frac{2}{3}} \Big(\int_{|x-y_k|\geq R} \rho_{n_k}(x)\, dx \Big)^{\frac{2}{3}} = O(\epsilon). \end{aligned}$$

Then it follows that

$$\Big| \int_{|x-y_k|\leq R} f_{n_k}^2 g_{n_k}\, dx - \lambda \Big| \leq \epsilon. \tag{2.19}$$

Letting $\widetilde{\mathbf{h}}_{n_k}(x) = (\widetilde{f}_{n_k}(x), \widetilde{g}_{n_k}(x)) \equiv (f_{n_k}(x-y_k), g_{n_k}(x-y_k))$, we have that $\{\widetilde{\mathbf{h}}_{n_k}\}_{k\geqq 1}$ is bounded in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ and therefore $\{\widetilde{\mathbf{h}}_{n_k}\}_{k\geqq 1}$ (or a subsequence) converges weakly in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ to a vector-function $\widetilde{\mathbf{h}} = (f_0, g_0)$. It follows then from (2.19) that for $k \geqq k_0$,

$$\lambda \geq \int_{-R}^{R} \widetilde{f}_{n_k}^2(x) \widetilde{g}_{n_k}(x)\, dx \geq \lambda - \epsilon. \tag{2.20}$$

Now, since $H^1([-R,R])$ and $H^{\frac{1}{2}}([-R,R])$ are compactly embedded in $L^2([-R,R])$, we have from Cauchy-Schwarz inequality that

$$\begin{aligned}&\Big|\int_{-R}^{R}\widetilde{f}_{n_k}^2(x)\widetilde{g}_{n_k}(x)\,dx-\int_{-R}^{R}f_0^2(x)g_0(x)\,dx\Big|\\&\qquad\le|\widetilde{f}_{n_k}+f_0|_\infty\|\widetilde{g}_{n_k}\|\|\widetilde{f}_{n_k}-f_0\|_{L^2(-R,R)}+\|\widetilde{f}_{n_k}\|_1^2\|\widetilde{g}_{n_k}-g_0\|_{L^2(-R,R)}\\&\qquad\le C\Big(\|\widetilde{f}_{n_k}-f_0\|_{L^2(-R,R)}+\|\widetilde{g}_{n_k}-g_0\|_{L^2(-R,R)}\Big)\to 0,\quad\text{as}\ \ k\to\infty,\end{aligned}$$

and therefore, from (2.20)

$$\lambda\ge\int_{-R}^{R}f_0^2(x)g_0(x)\,dx\ge\lambda-\epsilon.$$

Thus for $\epsilon=\frac{1}{j}$, $j\in\mathbb{N}$, there exists $R_j>j$ such that

$$\lambda\ge\int_{-R_j}^{R_j}f_0^2(x)g_0(x)\,dx\ge\lambda-\frac{1}{j}$$

and consequently as $j\to\infty$, we have finally that $F(f_0,g_0)=\lambda$. Furthermore, from the weak lower semicontinuity of V and invariance of V by translations, we have

$$I_\lambda=\liminf_{n\to\infty}V(\widetilde{f}_{n_k},\widetilde{g}_{n_k})\ge V(f_0,g_0)\ge I_\lambda.$$

Thus the vector-function $\widetilde{\mathbf{h}}=(f_0,g_0)$ solves the variational problem (1.7) and therefore there exists $K>0$ (Lagrange multiplier) such that,

$$\begin{cases}-f_0''+\sigma f_0=Kf_0g_0\\-\gamma Dg_0+cg_0=Kf_0^2.\end{cases}\tag{2.21}$$

Therefore, considering $(\phi,\psi)=(-K\sqrt{\frac{2}{\alpha\beta}}f_0,-\frac{2K}{\alpha}g_0)$ it follows from (2.21) that (ϕ,ψ) solves the problem (1.6). Thus, Theorem 2.5 is proved. ■

Remark: We note from (2.21) that $(\phi^+,\psi)=(K\sqrt{\frac{2}{\alpha\beta}}f_0,-\frac{2K}{\alpha}g_0)$ is also a solution of (1.6).

Corollary 2.6. *If (f_0, g_0) is a minimum for problem I_1, then (f_0, g_0) is a minimum for the unconstrained functional*

$$T(f,g) = \frac{V(f,g)}{(F(f,g))^{2/3}}. \tag{2.22}$$

Proof. It follows immediatly from Theorem 2.5. ■

Corollary 2.6 allows to obtain some properties on the structure of the spectrum of the linear operator, $\mathcal{L}$, associated with (1.6) for $\alpha = 2\beta$, defined by

$$\mathcal{L} = \begin{pmatrix} -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi & \alpha\phi \\ \alpha\phi & -\gamma D + c \end{pmatrix}, \tag{2.23}$$

more precisely we have,

Theorem 2.7. *The operator $\mathcal{L}$ given by (2.23) has exactly one negative eigenvalue of multiplicity one.*

Proof. Let $\widetilde{\mathbf{f}} = (f_0, g_0)$ be a minimum for problem I_1, then from Corollary 2.6 the second variation of T at $\widetilde{\mathbf{f}}$ is nonnegative, i.e. for all $\widetilde{\mathbf{h}} \in C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$ we have that

$$\mathcal{P}(\widetilde{\mathbf{h}}) = \frac{d^2}{d\epsilon^2} T(\widetilde{\mathbf{f}} + \epsilon\widetilde{\mathbf{h}})\Big|_{\epsilon=0} \geqq 0.$$

Hence, denoting by $(\phi, \psi) = -\frac{2K}{\alpha}(f_0, g_0)$, where $K > 0$ is the Lagrange multiplier associated with the constrained problem I_1, we have explicitly that

$$\mathcal{P}(\widetilde{\mathbf{h}}) = \mathcal{L}\widetilde{\mathbf{h}} + \frac{1}{3K^3} < \begin{pmatrix} \alpha\phi\psi \\ \frac{\alpha}{2}\phi^2 \end{pmatrix}, \widetilde{\mathbf{h}} > \begin{pmatrix} \alpha\phi\psi \\ \frac{\alpha}{2}\phi^2 \end{pmatrix}.$$

Therefore, $\mathcal{P}$ is a nonnegative operator. Since $\mathcal{P} = \mathcal{L} + \mathcal{R}$, where $\mathcal{R}$ is a rank-one operator, it follows from the min-max principle (see [RS]) that $\mathcal{L}$ has at most one negative eigenvalue. In order to show that there is exactly one negative eigenvalue, again by min-max principle, it is sufficient to find one direction $\widetilde{\mathbf{h}}$ such that $< \mathcal{L}\widetilde{\mathbf{h}}, \widetilde{\mathbf{h}} >< 0$. In fact, for $\widetilde{\mathbf{h}} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ it follows from (1.6) that

$$\begin{aligned} < \mathcal{L}\widetilde{\mathbf{h}}, \widetilde{\mathbf{h}} > &= \int_{\mathbb{R}} \phi(-\Delta + \sigma + \alpha\psi)\phi\, dx + \int_{\mathbb{R}} \psi(-\gamma D + c)\psi\, dx + 2\alpha \int_{\mathbb{R}} \phi^2\psi\, dx \\ &= \frac{3\alpha}{2} \int_{\mathbb{R}} \phi^2\psi\, dx < 0 \end{aligned}$$

where in the last inequality we use that $\int_{\mathbb{R}} f_0^2 g_0\, dx > 0$. This shows the Theorem. ■

3. EVENNESS AND ANALYTICITY OF SOLITARY WAVE SOLUTIONS

In this section we establish some properties of the solutions of system (1.6) for $\gamma < 0$, and $\sigma, \alpha, \beta, c > 0$. The first property is concerning the sign of ψ and it can be deduced studying the kernel associated with the second equation in (1.6). Indeed, we observe that if ψ satisfies

$$\gamma \mathcal{H}\psi' - c\psi = \beta\phi^2,$$

then, for $\mu = \frac{-c}{\gamma} > 0$, the Fourier transform implies that

$$\widehat{\psi}(\xi) = \frac{\beta}{\gamma}\frac{1}{|\xi| + \mu}\,\widehat{\phi^2}(\xi) = \frac{\beta}{\gamma}\widehat{K_\mu}(\xi)\,\widehat{\phi^2}(\xi) = \frac{\beta}{\gamma}\widehat{K_\mu * \phi^2}(\xi), \tag{3.1}$$

where the kernel K_μ is the following even function,

$$K_\mu(x) = \frac{1}{\pi}\int_0^\infty \frac{e^{-x\tau}}{\mu^2 + \tau^2}\,d\tau, \qquad \text{for } x > 0.$$

Note that $K_\mu \in C^\infty(\mathbb{R} - \{0\}) \cap L^1(\mathbb{R})$. Therefore, from (3.1) ψ satisfies the following convolution equation

$$\psi(x) = \frac{\beta}{\gamma}\,K_\mu * \phi^2(x). \tag{3.2}$$

Since $K_\mu(x)$ is a positive kernel, it follows immediately from (3.2) that

$$\psi(x) < 0 \qquad \text{for all } \ x \in \mathbb{R}. \tag{3.3}$$

provided that (ϕ, ψ) is a nontrivial solution of (1.6).

Now we show that the sign of ϕ can also be determined explicitly as follows.

Theorem 3.1. *If (f_0, g_0) is a minimum for problem I_λ then so is $(|f_0|, g_0)$. Therefore there exists a constant $K > 0$ such that, $(\phi, \psi) = (\pm K\sqrt{\frac{2}{\alpha\beta}}f_0, -K\frac{2}{\alpha}g_0)$ is a solution of (1.6). Moreover, we have that $\phi(x) > 0$ for all $x \in \mathbb{R}$ or $\phi(x) < 0$ for all $x \in \mathbb{R}$.*

Proof. A bootstrapping argument shows that if $(f_0, g_0) \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ is a solution of (2.21) then $(f_0, g_0) \in H^\infty(\mathbb{R}) \times H^\infty(\mathbb{R})$. Now, from Theorem A.2 (Appendix) it follows that $|f_0| \in H^1(\mathbb{R})$ and $\int_{\mathbb{R}} |\frac{d}{dx}f_0(x)\|^2\,dx \geqq \int_{\mathbb{R}} |\frac{d}{dx}|f_0|(x)\|^2\,dx$, and so

$$I_\lambda = V(f_0, g_0) \geqq V(|f_0|, g_0).$$

Since $F(|f_0|, g_0) = F(f_0, g_0) = \lambda$, then we obtain that $(|f_0|, g_0) \in G_\lambda$. From (2.21) it follows that $K = \frac{2}{3\lambda}V(f_0, g_0) = \frac{2}{3\lambda}V(|f_0|, g_0)$, then for $\mathcal{T} \equiv -\frac{d^2}{dx^2} - 2Kg_0$ we have

$$\begin{cases} \mathcal{T}(|f_0|) = -\sigma|f_0| \\ \mathcal{T}(f_0) = -\sigma f_0. \end{cases}$$

Since $|f_0(x)| > 0$ for all $x \in \mathbb{R}$, it follows from Sturm-Liouville Theory that $-\sigma$ is the least eigenvalue of $\mathcal{T}$ and therefore is simple. Hence there exists a $\rho \in \mathbb{R} - \{0\}$ such that $f_0 = \rho|f_0|$ and the proof is complete. ■

The following result asserts that there is at least one solitary wave solution of system (1.6), (ϕ, ψ) , such that $\phi, -\psi$ are even and decreasing functions. To see this, we use the theory of symmetric decreasing rearrangements of functions on $\mathbb{R}$ (see Riesz [R], Brascamp, Lieb and Luttinger [BLL], Hardy, Littlewood and Polya [HLP], Kawohl [K], and Friedman and McLeod [FM]). Initially, we see that if $(f, g) \in G_\lambda$ and $(f_1, g_1) \in G_\Delta$ for $\lambda, \Delta > 0$, then

$$V(f_1, g_1) = \left(\frac{\Delta}{\lambda}\right)^{2/3} V(f, g). \tag{3.4}$$

Indeed, for $a = \left(\frac{\Delta}{\lambda}\right)^{1/3}$ the relations

$$\begin{cases} F(af, ag) = a^3 F(f, g) = \frac{\Delta}{\lambda}\lambda = \Delta \\ V(af, ag) = a^2 V(f, g) \geqq V(f_1, g_1), \end{cases}$$

and

$$\begin{cases} F(a^{-1}f_1, a^{-1}g_1) = a^{-3}F(f, g) = a^{-3}\Delta = \lambda \\ V(a^{-1}f_1, a^{-1}g_1) = a^{-2}V(f_1, g_1) \geqq V(f, g), \end{cases}$$

shows (3.4).

Theorem 3.2. *Let $\gamma < 0$, and $\sigma, \alpha, \beta, c > 0$. Then there is a solution (ϕ, ψ) of (1.6) such that ϕ and $-\psi$ are even decreasing positive functions.*

Proof. Let $(f_0, g_0) \in G_\lambda$. From Theorem 3.1 we can assume $f_0(x) > 0$ for all $x \in \mathbb{R}$. Let (f_0^*, g_0^*) be the symmetric decreasing rearrangements of f_0 and g_0 respectively and $\Delta = \int_{\mathbb{R}} (f_0^*)^2 g_0^* \, dx$. Then from Theorem A.1 (Appendix) we have that

$$\lambda = \int_{\mathbb{R}} f_0^2 g_0 \, dx \leqq \int_{\mathbb{R}} (f_0^2)^* g_0^* \, dx = \int_{\mathbb{R}} (f_0^2)^* g_0^* \, dx = \Delta.$$

Now, we see that $\lambda = \Delta$. Let $(f_1, g_1) \in G_\Delta$, then from (3.4) and Theorems A.1 and A.2 we have that

$$\begin{aligned} V(f_0^*, g_0^*) &\leqq V(f_0, g_0) = \Big(\frac{\lambda}{\Delta}\Big)^{2/3} V(f_1, g_1) \\ &\leqq V(f_1, g_1) \leqq V(f_0^*, g_0^*) \end{aligned} \tag{3.5}$$

where the last inequality holds since $F(f_0^*, g_0^*) = \Delta$. Then from (3.5) we obtain

$$\begin{cases} \Big(\frac{\lambda}{\Delta}\Big)^{2/3} V(f_1, g_1) = V(f_1, g_1) \\ V(f_0, g_0) = V(f_0^*, g_0^*). \end{cases}$$

Hence $\lambda = \Delta$ and $(f_0^*, g_0^*) \in G_\lambda$. Thus there exists $K = \frac{2}{3\lambda} V(f_0, g_0) = \frac{2}{3\lambda} V(f_0^*, g_0^*) > 0$ such that

$$(\phi, \psi) = (K\sqrt{\frac{2}{\alpha\beta}} f_0^*, -\frac{2K}{\alpha} g_0^*)$$

is solution of (1.6). This shows the Theorem. ■

We continue the analysis of the solitary-wave solutions of the equation (1.6). Let (ϕ, ψ) be a solution of (1.6). Interest now focuses on the analyticity and the strictly decreasing of ϕ and $-\psi$. The idea is based on the recent theory for analyticity of solitary-wave solutions of model equations for long waves worked out by Li and Bona ([LB]). This theory consist in demostrating that the Fourier transform of $\widehat{\phi}$ and $\widehat{\psi}$ has exponential decay at $\pm\infty$. Then, the Paley-Wiener Theorem assures that ϕ and ψ has an analytic extension to a strip in the complex plane which is symmetric about the real axis. In our situation, it is shown that this strip can be choosen equal for ϕ and ψ.

Theorem 3.3. *For (ϕ, ψ) a solitary-wave solution of (1.6), we have:*
(i) there is a constant $\sigma_0 > 0$ such that

$$\sup_{\xi \in \mathbb{R}} e^{\nu|\xi|} |\widehat{\phi}(\xi)| < \infty, \qquad \sup_{\xi \in \mathbb{R}} e^{\nu|\xi|} |\widehat{\psi}(\xi)| < \infty \tag{3.6}$$

for any ν with $0 < \nu < \sigma_0$.
(ii) there are functions $\Phi(z)$ and $\Psi(z)$ that are defined and analytic on the open strip $\{z \in \mathbb{C} : |Im(z)| < \sigma_0\}$ such that

$$\Phi(x) = \phi(x), \qquad \Psi(x) = \psi(x)$$

for all $x \in \mathbb{R}$.

Proof. (i) If (ϕ, ψ) is a solution of (1.6) then from (3.2) the function

$$\widehat{\varphi}(\xi) = \sqrt{\frac{\alpha}{\sigma}}\,\widehat{\phi}(\xi) \tag{3.7}$$

solves the equation

$$(\eta\xi^2 + 1)\widehat{\varphi}(\xi) = \frac{\beta}{|\gamma|}(\widehat{K}_\mu \cdot \widehat{\varphi^2}) * \widehat{\varphi}(\xi) \tag{3.8}$$

where $\eta = \frac{1}{\sigma}$. Now, when $0 \leqq k \leqq 1$, (3.8) can be used to conclude that

$$\begin{aligned} |\xi^k \widehat{\varphi}(\xi)| &= \frac{\beta}{|\gamma|}\frac{|\xi|^k}{1+\eta\xi^2}\Big|\int_{\mathbb{R}} \widehat{\varphi}(\xi - x)\frac{1}{|x|+\mu}\widehat{\varphi^2}(x)\,dx\Big| \\ &\leqq \frac{\beta}{|\gamma|}\frac{|\xi|^k}{1+\eta\xi^2}\,(|\widehat{\varphi}| * |\widehat{\varphi}| * |\widehat{\varphi}|)(\xi) \\ &\leqq \frac{\beta}{|\gamma|}\frac{1}{\eta^{k/2}}(k+1)^{k-1}(|\widehat{\varphi}| * |\widehat{\varphi}| * |\widehat{\varphi}|)(\xi). \end{aligned}$$

Therefore, using induction and Young's inequality we obtain

$$|\xi^k \widehat{\varphi}(\xi)| \leqq \frac{\beta}{|\gamma|}\frac{\|\widehat{\varphi}\|^2}{\eta^{k/2}}(k+1)^{k-1}\,|\widehat{\varphi}|_{L^1}^{3([k/2]2+1)-2}, \tag{3.9}$$

for any $\xi \in \mathbb{R}$ and any integer $k \geqq 0$ (see Theorem 2 in [LB]). ($[k/2]$ denotes the greatest integer less than or equal to $k/2$). Now, consider the sequence

$$a_k = \frac{1}{k!\eta^{k/2}}(k+1)^{k-1}|\widehat{\varphi}(\xi)|_{L^1}^{3(k+1)} \tag{3.10}$$

for $k = 0, 1, 2, ...$ Since the ratio $\frac{a_{k+1}}{a_k}$ takes the form

$$\frac{a_{k+1}}{a_k} = \frac{1}{\eta^{1/2}}(1 + \frac{1}{k+1})^k|\widehat{\varphi}(\xi)|_{L^1}^3,$$

then $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{e|\widehat{\varphi}(\xi)|_{L^1}^3}{\eta^{1/2}}$. Hence the power series $\sum_{k=0}^\infty a_k\nu^k$ converges for $|\nu| < \frac{\eta^{1/2}}{e|\widehat{\varphi}(\xi)|_{L^1}^3} \equiv \sigma_0$. Then, from (3.9) and (3.10), for any $\xi \in \mathbb{R}$, we have

$$\begin{aligned} e^{\nu|\xi|}|\widehat{\varphi}(\xi)| &= \sum_{k=0}^\infty \frac{\nu^k|\xi|^k}{k!}|\widehat{\varphi}(\xi)| \\ &\leqq \frac{\beta}{|\gamma|}\frac{\|\widehat{\varphi}(\xi)\|^2}{|\widehat{\varphi}(\xi)|_{L^1}^2}\sum_{k=0}^\infty \frac{\nu^k}{k!\eta^{k/2}}(k+1)^{k-1}|\widehat{\varphi}(\xi)|_{L^1}^{3(k+1)} \\ &= \frac{\beta}{|\gamma|}\frac{\|\widehat{\varphi}(\xi)\|^2}{|\widehat{\varphi}(\xi)|_{L^1}^2}\sum_{k=0}^\infty a_k\nu^k < \infty, \end{aligned} \tag{3.11}$$

provided that $|\nu| < \sigma_0$. Thus the function $e^{\nu|\xi|}|\widehat{\phi}(\xi)|$ is uniformly bounded for such ν's.

Now, consider $0 < \nu < \sigma_0$ and let $\nu < \nu_1 < \sigma_0$. Then, from (3.2) and (3.11) we have

$$\begin{aligned} e^{\nu|\xi|}|\widehat{\psi}(\xi)| &= \frac{\beta}{|\gamma|}\frac{e^{\nu|\xi|}}{1+\mu|\xi|}\,|\widehat{\phi^2}(\xi)| \leqq \frac{\beta}{|\gamma|}e^{\nu|\xi|}\,|\widehat{\phi^2}(\xi)| \leqq \frac{\beta}{|\gamma|}\int_{\mathbb{R}} e^{\nu|\xi-x|}e^{\nu|x|}|\widehat{\phi}(\xi-x)||\widehat{\phi}(x)|\,dx \\ &\leqq \frac{\beta}{|\gamma|}\sup_{x\in\mathbb{R}}\left(e^{\nu|x|}|\widehat{\phi}(x)|\right)\int_{\mathbb{R}} e^{\nu|\xi-x|}|\widehat{\phi}(\xi-x)|\,dx \\ &= \frac{\beta}{|\gamma|}\sup_{x\in\mathbb{R}}\left(e^{\nu|x|}|\widehat{\phi}(x)|\right)\int_{\mathbb{R}} e^{-(\nu_1-\nu)|\xi-x|}e^{\nu_1|\xi-x|}|\widehat{\phi}(\xi-x)|\,dx \\ &\leqq \frac{\beta}{|\gamma|}\sup_{x\in\mathbb{R}}\left(e^{\nu|x|}|\widehat{\phi}(x)|\right)\sup_{x\in\mathbb{R}}\left(e^{\nu_1|x|}|\widehat{\phi}(x)|\right)\int_{\mathbb{R}} e^{-(\nu_1-\nu)|x|}\,dx < \infty, \end{aligned}$$

which completes the proof of (3.6).

(ii) Let ν lie in the open interval $(0,\sigma_0)$. Choose $\nu_1 > 0$ satisfying $0 < \nu < \nu_1 < \sigma_0$. Then for $\widehat{\varphi}_1(\xi) \equiv \widehat{\phi}(\xi)$ and $\widehat{\varphi}_2(\xi) \equiv \widehat{\psi}(\xi)$ we have that

$$\int_{\mathbb{R}} e^{2\nu|\xi|}|\widehat{\varphi}_i(\xi)|^2 d\xi \leqq \sup_{\xi\in\mathbb{R}}\left(e^{\nu_1|\xi|}|\widehat{\varphi}_i(\xi)|\right)^2\int_{\mathbb{R}} e^{-2(\nu_1-\nu)|\xi|}\,d\xi < \infty.$$

Therefore from Paley-Wiener Theorem ([PW], [RS1]) the functions

$$\Phi(z) = \int_{\mathbb{R}} \widehat{\phi}(\xi)e^{iz\xi}d\xi, \qquad \Psi(z) = \int_{\mathbb{R}} \widehat{\psi}(\xi)e^{iz\xi}d\xi$$

are well-defined and analytic on $\{z\in\mathbb{C} : |Im(z)| < \sigma_0\}$. Finally, Plancherel's Theorem implies that $\Phi(x) = \phi(x)$ and $\Psi(x) = \psi(x)$ for any $x\in\mathbb{R}$. ■

An immediate consequence of the analyticity established in Theorem 3.3 is the existence of solitary-wave solutions of (1.6) that are strictly decreasing. Indeed, we have two different proofs of this fact.

Theorem 3.4. *Let $\gamma < 0$, and $\sigma, \alpha, \beta, c > 0$. Then there exists a solution of (1.6), (ϕ,ψ), such that ϕ and $-\psi$ are even and strictly decreasing positive functions.*

Proof. Let (ϕ,ψ) be a solution of (1.6) given for Theorem 3.2. Then ϕ and $-\psi$ are even and decreasing positive functions.

Now, with regard to the strict decreasing we have:

First proof: Let $0 < \nu < \sup_{x\in\mathbb{R}}\phi(x)$. Then from Theorem 3.4 we have that in any bounded set $B\subset\mathbb{R}$, there is at most a finite number of points $x\in B$ such that $\phi(x)=\nu$. Therefore, since ϕ is decreasing, the set $\{x\in\mathbb{R}^+ : \phi(x)=\nu\}$ is a point-set; hence ϕ is strictly decreasing. Similar result is obtained for $-\psi$.

Second proof: Since $K'_\mu \in L^1(\mathbb{R})$ it follows from (3.2) that

$$\frac{\gamma}{\beta}\psi'(x) = K'_\mu * \phi^2(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} K'_\mu(x-y)\phi^2(y)\,dy. \tag{3.12}$$

Since K'_μ is odd, we obtain for $x>0$

$$\int_{x+\epsilon}^{\infty} K'_\mu(x-y)\phi^2(2x-y)\,dy = \int_{-\infty}^{x-\epsilon} K'_\mu(x-(2x-\xi))\phi^2(\xi)\,d\xi = -\int_{-\infty}^{x-\epsilon} K'_\mu(x-\xi)\phi^2(\xi)\,d\xi,$$

therefore, from (3.12) it follows that

$$\frac{\gamma}{\beta}\psi'(x) = \lim_{\epsilon\to 0}\int_{x+\epsilon}^{\infty} K'_\mu(x-y)[\phi^2(y)-\phi^2(2x-y)]\,dy. \tag{3.13}$$

Finally, since ϕ is even and decreasing positive function, we have that if $x>0$ and $y>x$ the function $y\to\phi^2(2x-y)$ has its graph as the reflection of the graph of $\phi^2|_{(-\infty,x)}$ with respect to the vertical straight line crossing by x. Therefore, $\phi^2(y)\leqq\phi^2(2x-y)$ for all $y>x$ and so there exists an interval $I\subset(x,\infty)$ such that $\phi^2(y)+\delta<\phi^2(2x-y)$ for all $y\in I$ and some $\delta>0$. Thus from (3.13) we have that for $x>0$

$$\frac{\gamma}{\beta}\psi'(x)<0.$$

Finally, from the first equation in (1.6) it follows that $\phi'(x)<0$ for all $x>0$. This shows the Theorem. ∎

Finally, combining properties of the theory of symmetric decreasing rearrangements and analyticity, we obtain that for each element (f,g) of G_λ, their components are even, up translations, and strictly decreasing positive functions.

Theorem 3.5. *Let $\gamma<0$, and $\sigma,c>0$. Then for each $(f,g)\in G_\lambda$ we have that*

$$f(x)=f^*(x+r),\qquad g(x)=g^*(x+r)\qquad \text{for all } x\in\mathbb{R}$$

and some $r\in\mathbb{R}$, where f^* and g^* are the symmetric decreasing rearrangements of f and g respectively.

Proof. Let $(f,g)\in G_\lambda$. Then f and g satisfy the equation

$$\begin{cases} -f''+\sigma f = 2Kfg \\ -\gamma Dg + cg = Kf^2 \end{cases} \tag{3.14}$$

for $K > 0$. Therefore it follows that $V_1(f) \equiv \frac{1}{2}\int_{\mathbb{R}}[(f'(x))^2 + \sigma f^2(x)]dx = K\lambda$ and $V_2(g) \equiv \frac{1}{2}\int_{\mathbb{R}}[-\gamma(D^{\frac{1}{2}}g(x))^2 + cg^2(x)]dx = \frac{K}{2}\lambda$. Hence,

$$V(f,g) = V_1(f) + V_2(g) = \frac{3}{2}V_1(f).$$

Now, from the proof of Theorem 3.2 we have that $(f^*, g^*) \in G_\lambda$. Hence (f^*, g^*) satisfies (3.14) and consequently we have that $V(f^*, g^*) = \frac{3}{2}V_1(f^*)$. Using the facts that $V(f,g) = V(f^*, g^*)$ and $\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |f^*(x)|^2 dx$, we conclude that

$$\int_{\mathbb{R}} \left|\frac{d}{dx}f(x)\right|^2 dx = \int_{\mathbb{R}} \left|\frac{d}{dx}f^*(x)\right|^2 dx.$$

Then, since $f \in H^\infty(\mathbb{R})$ and it is analytic in $\mathbb{R}$ (Theorem 3.3), it follows from Corollary A.4 (Appendix) that

$$f(x) = f^*(x - r)$$

for all $x \in \mathbb{R}$ and some $r \in \mathbb{R}$. Finally, since $(f_r(\cdot), g_r(\cdot)) = (f(\cdot + r), g(\cdot + r))$ is solution of (3.14) we obtain that g_r is even and from the second proof of Theorem 3.4, $g_r'(x) < 0$ for $x > 0$, and thus

$$g(x) = g^*(x - r).$$

This shows the Theorem. ■

APPENDIX: PROPERTIES ON SYMMETRIC DECREASING REARRANGEMENTS

In this Appendix, we collect some facts about the symmetric decreasing rearrangement of a function in $\mathbb{R}$. The results mentioned here in one or higher dimensions can be found in the papers of Riesz [R], Brascamp, Lieb and Luttinger [BLL] and in the books of Hardy, Littlewood and Polya [HLP] and Kawohl [K].

Let

$$S = \{f : \mathbb{R} \to [0,\infty] \,|\, f(x) \leqq f(y) \text{ if } |x| \geqq |y|\,\}$$

be the even decreasing functions. Let f be a nonnegative measurable function on $\mathbb{R}$, let $\Omega_y(f) = \{x|\, f(x) \geqq y\}$ and $M_y(f) = m(\Omega_y(f))$, where m denotes the Lebesgue measure. Assume that $M_a(f) < \infty$ for some $a < \infty$. If f^* is another function on $\mathbb{R}$ with the same properties as f and, additionally,

$$\begin{cases} (a) & f^*(x) = f^*(-x), \qquad \text{for all } x \\ (b) & 0 < x < y \implies f^*(x) \geqq f^*(y), \\ (c) & M_y(f^*) = M_y(f), \qquad \text{for all } y > 0, \end{cases}$$

then f^* is called a *symmetric decreasing rearrangement* of f.

Remark: (1) if g and h are two symmetric decreasing rearrangements of f, then

$$g(x) = h(x) \qquad a.e.$$

(2) If χ is the characteristic function of a measurable set in $\mathbb{R}$, we define χ^* by

$$\chi^*(x) = \begin{cases} 1 & \text{if } 2|x| \leqq \int \chi \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\chi^* \in S$ and if $\chi \in L^1(\mathbb{R})$ then $|\chi|_{L^1} = |\chi^*|_{L^1}$. For a general function, $f : \mathbb{R} \to [0, \infty]$, let

$$\chi_y(x) = \begin{cases} 1 & \text{if } x \in \Omega_y(f) \\ 0 & \text{otherwise.} \end{cases}$$

Then is easy to see that $f(x) = \int_0^\infty \chi_y(x)\, dy$. This implies that

$$f^*(x) = \int_0^\infty \chi_y^*(x)\, dy, \tag{A1}$$

is a symmetric decreasing rearrangement of f. Also we note (see Kawohl [K]) that if $M_a(f) < \infty$ for all $a > 0$ and we define

$$\Omega_y^*(f) = \begin{cases} \{x \in \mathbb{R} |\ |x| \leqq \frac{1}{2} m(\Omega_y(f))\} & \text{if } \Omega_y(f) \neq \emptyset \\ \emptyset & \text{if } \Omega_y(f) = \emptyset, \end{cases}$$

called the *symmetric rearrangement* of $\Omega_y(f)$, then

$$f^*(x) \equiv \sup \{y \in \mathbb{R} |\ x \in \Omega_y^*(f)\ \} \tag{A2}$$

is a symmetric decreasing rearrangement of f. Consequently, if f is continuous then by Remark (1) f^* can be found using (A1) or (A2). The fact that $M_a(f) < \infty$ implies that $f^*(x) < \infty$, for all $x \neq 0$.

(3) If $f : \mathbb{R} \to \mathbb{C}$ we define

$$f^* = |f|^*.$$

Therefore, for all y, $m(\Omega_y(f^*)) = m(\Omega_y(|f|))$.

(4) Since in the following Theorems we deal with integrals, by Remark (1), f^* is unique for our purposes.

Now we collect in a single Theorem various properties of the symmetric decreasing rearrangement.

Theorem A.1.
(i) For all $p \in [1,\infty]$, $|f^*|_{L^p} = |f|_{L^p}$.
(ii) For all $r > 0$, $(f^r)^* = (f^*)^r$ *pointwise.*
(iii) If $f, g \in L^2(\mathbb{R})$

$$\Big|\int_{\mathbb{R}} f(x)g(x)\,dx\Big| \leqq \int_{\mathbb{R}} f^*(x)g^*(x)\,dx.$$

(iv) **The Inequaltity of Riesz:** *Let* $f_1, ..., f_n$ *be measurable functions on* $\mathbb{R}$ *such that* $m(\Omega_y(f_i)) < \infty$ *for all* $y > 0$ *and all* $1 \leqq i \leqq n$. *Then*

$$|(f_1 * f_2 * ... * f_n)(0)| \leqq [(f_1^*) * (f_2^*) * ... * (f_n^*)](0)$$

in the sense that if the right-hand side is finite, then the left-hand side exists and the inequality holds.

Proof. It is easy to check (i) and (ii). For (iii) see Kawohl ([K]) and (iv) see Riesz ([R]) and Brascamp, Lieb and Luttinger ([BLL]). ■

The following result is essential in this work, and the idea of the proof is based on Lemma 3.5 in Albert, Bona and Saut [ABS].

Theorem A.2.
(i) If $f \in H^1(\mathbb{R})$ *then* $|f|, f^* \in H^1(\mathbb{R})$ *and*

$$\int_{\mathbb{R}} \Big|\frac{d}{dx}f(x)\Big|^2 dx \geqq \int_{\mathbb{R}} \Big|\frac{d}{dx}|f|(x)\Big|^2 dx, \qquad \int_{\mathbb{R}} \Big|\frac{d}{dx}f(x)\Big|^2 dx \geqq \int_{\mathbb{R}} \Big|\frac{d}{dx}f^*(x)\Big|^2 dx.$$

(ii) If $f \in H^{\frac{1}{2}}(\mathbb{R})$ *then* $|f|, f^* \in H^{\frac{1}{2}}(\mathbb{R})$ *and*

$$\int_{\mathbb{R}} |D^{\frac{1}{2}}f(x)|^2\,dx \geqq \int_{\mathbb{R}} |D^{\frac{1}{2}}|f|(x)|^2\,dx, \qquad \int_{\mathbb{R}} |D^{\frac{1}{2}}f(x)|^2\,dx \geqq \int_{\mathbb{R}} |D^{\frac{1}{2}}f^*(x)|^2\,dx$$

Proof. The affirmation (ii) is Lemma 3.4 and lemma 3.5 in [ABS]. Now, for (i), we first show that $|f| \in H^1(\mathbb{R})$. Let $\nu > 0$, and define the function $N_\nu(x)$ by

$$\widehat{N_\nu}(\xi) = \frac{1}{\nu + \xi^2}\ .$$

Then $N_\nu(x) > 0$ for all $x \in \mathbb{R}$ and $N_\nu \in S \cap L^p(\mathbb{R})$ for every $p \in [1,\infty]$. In fact, the Residue Theorem shows that $N_\nu(x) = \frac{e^{-\sqrt{\nu}|x|}}{2\sqrt{\nu}}$. Now, if $g = |f|$ then $N_\nu * g(x) \geqq N_\nu * f(x)$ for all $x \in \mathbb{R}$ and every $\nu > 0$. Therefore,

$$\int_{\mathbb{R}} \frac{1}{\nu+\xi^2}|\widehat{g}(\xi)|^2\,d\xi = \int_{\mathbb{R}} g(x)(N_\nu * g)(x)\,dx \geqq \int_{\mathbb{R}} f(x)(N_\nu * f)(x)\,dx$$
$$= \int_{\mathbb{R}} \frac{1}{\nu+\xi^2}|\widehat{f}(\xi)|^2\,d\xi.$$

Since $\int_{\mathbb{R}} |\widehat{g}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi$ by Parseval's identity, it follows that

$$\int_{\mathbb{R}} \nu\Big[1 - \frac{\nu}{\nu+\xi^2}\Big] |\widehat{f}(\xi)|^2 \, d\xi \geqq \int_{\mathbb{R}} \nu\Big[1 - \frac{\nu}{\nu+\xi^2}\Big] |\widehat{g}(\xi)|^2 \, d\xi.$$

Since $\lim_{\nu\to\infty} \nu\Big[1 - \frac{\nu}{\nu+\xi^2}\Big] = \xi^2$, taking the limit as $\nu \to \infty$ on both sides of the preceding inequality and using the Monotone Convergence Theorem gives

$$\int_{\mathbb{R}} |\xi|^2 |\widehat{f}(\xi)|^2 \, d\xi \geqq \int_{\mathbb{R}} |\xi|^2 |\widehat{g}(\xi)|^2 \, d\xi,$$

which together with (i) in Theorem A.1 shows that $|f| \in H^1(\mathbb{R})$.

For the other affirmation we note initially that since $N_\nu^* = N_\nu$ then for $g = |f|^* = f^*$ the inequaltiy of Riesz gives

$$\int_{\mathbb{R}} \frac{1}{\nu+\xi^2} |\widehat{g}(\xi)|^2 \, d\xi = \int_{\mathbb{R}} g(x)(N_\nu * g)(x) \, dx \geqq \int_{\mathbb{R}} f(x)(N_\nu * f)(x) \, dx.$$

Also, by Parseval's identity $\int_{\mathbb{R}} |\widehat{f^*}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi$. The result then follows exactly as in the preceding proof . ∎

The next result is discussed in Kawohl ([K]) and Friedman and McLeod ([FM]), here we establish it in the case of $\mathbb{R}$.

Theorem A.3. *Let f be Lipschitz continuous on $\mathbb{R}$, with $\frac{df}{dx} \in L^p(\mathbb{R})$ for some $1 < p < \infty$. Let*

$$m = \text{ess inf} f, \qquad M = \text{ess sup} f.$$

If $m(\{x : f(x) = t\}) = 0$ for all $m < t < M$ and $f \in C^1$ in the set $\{x : m < f(x) < M\}$ then the strict equality in

$$\int_{\mathbb{R}} \Big|\frac{d}{dx} f(x)\Big|^p \, dx \geqq \int_{\mathbb{R}} \Big|\frac{d}{dx} f^*(x)\Big|^p \, dx$$

holds only if $f = f^*$ modulo translations.

An immediate consequence of Theorem A.3 is the following result used in our work.

Corollary A.4. *Let $f \in H^s(\mathbb{R})$ for some $s > 3/2$. If f is analytic on $\mathbb{R}$ then the strict equality in*

$$\int_{\mathbb{R}} \Big|\frac{d}{dx} f(x)\Big|^2 \, dx \geqq \int_{\mathbb{R}} \Big|\frac{d}{dx} f^*(x)\Big|^2 \, dx$$

holds only if $f = f^*$ modulo translations.

Remark: We call a function f analytic on $\mathbb{R}$ if there exist a constant $\nu > 0$ and a function $F(z)$ defined and analytic on the open strip $\{z \in \mathbb{C} : \ |Im(z)| < \nu \}$ such that $F(x) = f(x)$ for all $x \in \mathbb{R}$.

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