EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS

MARCELO MONTENEGRO*
Universidade Estadual de Campinas, Departamento de Matemática
Caixa Postal 6065, Cep. 13081-970, Campinas, SP, Brasil
e-mail: `msm@ime.unicamp.br`

MARCOS MONTENEGRO*
Universidade Federal do Ceará, Departamento de Matemática
Campus do Pici, Bl. 914, Cep. 60455-760, CE, Brasil
e-mail: `montene@mat.ufc.br`

Abstract

In this paper we study the problem

$$\begin{cases} -\Delta_p u = f(x,u,\nabla u) & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset I\!R^N$ is a smooth bounded domain, $N \geq 2$ and $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ defines the $p-$Laplacian. We provide some necessary and sufficient conditions on f under which the problem admits a weak solution. For the case $p = 2$ we obtain more general conditions on f. The main ingredients are Degree Theory and super-subsolution method.

*This research was partially supported by FUNCAP, Ceará

1 INTRODUCTION

During the past thirty years many mathematicians have been investigating quasi-linear elliptic problems of the type

$$\begin{cases} -\mathcal{L}u &= f(x,u,\nabla u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $I\!R^N$, $N \geq 2$, $\mathcal{L}$ is a uniformly elliptic operator of second order verifying the strong maximum principle and $f : \Omega \times I\!R \times I\!R^N \to I\!R$ is a Carathéodory function satisfying appropriate growth conditions. These studies have contributed for a better understanding of several questions related to (1.1) as, for example, regularity, existence and nonexistence of solution. In the literature, problem (1.1) is known to be critical in the gradient if f has quadratic growth in the gradient. This is a reasonable assumption when we are seeking for a solution. In fact, Serrin [14] has proved that if f have a superquadratic growth in the gradient, then given any smooth domain there is a smooth data f for which (1.1) does not admit solution. On the other hand, several authors have obtained results on existence of solution when f has critical growth in the gradient. For instance, see [1], [3], [5], [8], [9], [12], [13] and [16]. Part of these works are concerned to the question of existence when f interacts in some sense with the first eigenvalue of $\mathcal{L}$, see especifically [5], [9] and [16]. The basic tools that have been used are a priori estimates, Degree Theory and super-subsolution method.
Now let us set the problem

$$\begin{cases} -\Delta_p u &= f(x,u,\nabla u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is as in (1.1) and $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$ denotes the $p-$Laplacian with $1 < p < \infty$.
Problems of form (1.2) arise naturally as stationary states of certain models in Fluids Mechanics. Therefore, it is important to obtain information about the existence and nonexistence of solutions for this problem. Similarly, the problem (1.2) is said to be critical in the gradient if f has a growth of power p in the gradient. In this case, some authors have presented various results on existence of weak solution for (1.2),

see [4] and [6]. However, it has not been analysed what occurs when f interacts with the spectrum of the $p-$Laplacian. In the present work, we obtain several results on existence of weak solutions under assumptions that relate f and the first eigenvalue λ_1 of the $p-$Laplacian. Some of these results have not been considered even when $p = 2$. Getting a complete understanding, we provide necessary and sufficient conditions for existence of positive weak solutions. For example, we show that the problem

$$\begin{cases} -\Delta_p u &= \dfrac{a}{1+ku} \mid \nabla u \mid^p + b(1+ku)^{p-1} \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $a, b > 0$ and $k \geq 0$ are constants, admits a positive weak solution if and only if $(k(p-1) + a)^{p-1} b < (p-1)^{p-1}\lambda_1$. Hence,

$$\begin{cases} -\Delta_p u &= a \mid \nabla u \mid^p + b \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

has a positive weak solution if and only if $a^{p-1} b < (p-1)^{p-1}\lambda_1$. Besides, we give sufficient conditions under which the problem

$$\begin{cases} -\Delta_p u &= \dfrac{a(u)}{1+ku} \mid \nabla u \mid^p + bu^{p-1} \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

possesses a positive weak solution. When $p = 2$, we discuss the existence of weak solutions under more general hypotheses on f. In particular, we investigate equations of the type

$$\begin{cases} -\Delta u &= a(x,u) \mid \nabla u \mid^2 + b(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The outline of the paper is as follows. In section 2 we provide necessary and sufficient non-ressonance conditions for the existence of positive weak solutions for (1.2). In section 3 we show some results on existence of positive weak solutions for problems subject to other non-ressonance conditions. Finally, in section 4 we analyse the problem (1.2) in the case $p = 2$. In particular, we give more general sufficient conditions than those of the section 2. Our arguments are based on Díaz and Saa inequality [7], Degree Theory and super-subsolution method.

2 EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS

Throughout the paper Ω denotes a bounded domain of class C^2 belonging to $I\!R^N$, $N \geq 2$ and $1 < p < \infty$ is a real number.
In this section we search for necessary and sufficient conditions for existence of positive weak solutions for the problem

$$\begin{cases} -\Delta_p u &= a(u) \mid \nabla u \mid^p + b(x,u) & \text{in} \quad \Omega \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $a : I\!R_+ \to I\!R_+$ is a continuous function, $I\!R_+ = [0, +\infty)$ and $b : \Omega \times I\!R_+ \to I\!R_+$ is a Carathéodory function, that is, for a.e. $x \in \Omega$ the function $b(x, .)$ is continuous and for every $s \in I\!R$ the function $b(., s)$ is measurable. As a consequence we obtain the existence of nonegative weak solutions for Dirichlet problems of the form (1.2) subject to certain non-ressonance conditions.
Before stating our first result, let us consider the problem

$$\begin{cases} -\Delta_p v &= g(x,v) & \text{in} \quad \Omega \\ v &= 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $g : \Omega \times I\!R_+ \to I\!R_+$ is a Carathéodory function verifying

$g(x, s) \leq c(s^{p-1} + 1)$ for every $x \in \Omega$ a.e., $s \geq 0$ and some constant $c > 0$.

The following result is taken from [2]:

Lemma 2.1 Suppose

$$\limsup_{s \to +\infty} \frac{g(x,s)}{s^{p-1}} < \lambda_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.}$$

Then (2.2) admits, at least, one nonegative weak solution in $C^1(\overline{\Omega})$.

We say that $u \in C^1(\overline{\Omega})$ is a weak solution (supersolution, subsolution) of (1.2) if $f(\cdot, u(\cdot), \nabla u(\cdot)) \in L^1_{loc}(\Omega)$, $u = 0$ on $\partial\Omega$ and

$$\int_\Omega \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla \phi \; dx \;=\; (\geq, \; \leq) \; \int_\Omega f(x, u, \nabla u) \phi \; dx$$

for every $\phi \in C^1_0(\Omega)$ with $\phi \geq 0$ in Ω.

Let $H : I\!R_+ \to I\!R_+$ be defined by

$$H(t) = \int_0^t \exp\left(\frac{1}{p-1} \int_0^s a(r) dr \right) ds \; .$$

Assume that the pair (a, b) satisfies

(H.1) $\quad \limsup_{t \to +\infty} \dfrac{H'(t)^{p-1} b(x,t)}{H(t)^{p-1}} \; < \; \lambda_1, \;$ uniformly for $x \in \Omega$ a.e.,

(H.2) there are constants $c, k > 0$ such that $H'(t)^{p-1} b(x,t) \; \leq \; cH(t)^{p-1}$ for $x \in \Omega$ a.e. and $t \geq k$,

(H.3) $\quad \sup_{t \in [0,k]} b(\cdot, t) \; \in \; L^\infty(\Omega)$.

Proposition 2.1 If **(H.1)**, **(H.2)** and **(H.3)** are fulfilled, then (2.1) admits, at least, one nonegative weak solution.

Proof: Since $H'(t) \geq 1$ for $t \geq 0$, then

$$H(t) \to +\infty \quad \text{as} \quad t \to +\infty \; . \tag{2.3}$$

Define $g(x, s) \; = \; (H'(H^{-1}(s)))^{p-1} \, b(x, H^{-1}(s))$. From (2.3) and **(H.1)**, it follows that

$$\limsup_{s \to +\infty} \frac{g(x,s)}{s^{p-1}} \; < \; \lambda_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.}$$

Futhermore, by **(H.2)** and **(H.3)**, there exists $c > 0$ such that

$$g(x,s) \; \leq \; c(s^{p-1} \; + \; 1) \quad \text{for every} \quad x \in \Omega \quad \text{a.e and} \quad s \geq 0 \; .$$

By Lemma 2.1, (2.2) admits a nonegative weak solution $v \in C^1(\overline{\Omega})$. So, defining $u = H^{-1}(v)$, we conclude that u is a nonegative weak solution of (2.1). ∎

Let $a, \; c : I\!\!R_+ \to I\!\!R_+$ be continuous functions and $b, \; d : \Omega \times I\!\!R_+ \to I\!\!R_+$ be Carathéodory functions such that $a(t_1) \; \leq \; c(t_2)$ and $b(x, t_1) \; \leq \; d(x, t_2)$ for every $x \in \Omega$ a.e. and $t_1, t_2 \geq 0$.

Suppose

(H.4) $\; a(t) \mid \xi \mid^p \; + b(x,t) \; \leq \; f(x,t,\xi) \; \leq \; c(t) \mid \xi \mid^p \; + d(x,t)$ for every $x \in \Omega$ a.e. , $t \geq 0$ and $\xi \in I\!\!R^N$.

Theorem 2.1 Assume **(H.4)** with (a, b) and (c, d) satisfying **(H.1)**, **(H.2)** and **(H.3)**. Then (1.2) admits, at least, one nonegative weak solution.

Proof: Let $\overline{u}$, $\underline{u} \in C^1(\overline{\Omega})$ be nonnegative weak solutions of problems

$$\begin{cases} -\Delta_p \underline{u} & = & a(\underline{u}) \mid \nabla \underline{u} \mid^p + b(x, \underline{u}) & \text{in} & \Omega \\ \underline{u} & = & 0 & \text{on} & \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta_p \overline{u} & = & c(\overline{u}) \mid \nabla \overline{u} \mid^p + d(x, \overline{u}) & \text{in} & \Omega \\ \overline{u} & = & 0 & \text{on} & \partial\Omega \end{cases}$$

By **(H.4)** and the comparison principle, we conclude that $\underline{u}$ is a subsolution of (1.2), $\overline{u}$ is a supersolution of (1.2) and $\underline{u} \leq \overline{u}$ in Ω. Therefore, by $C^{1,\alpha}(\overline{\Omega})$ estimates in [11] and monotonic iteration, one concludes that there exists $u \in C^1(\overline{\Omega})$, which is a weak solution of (1.2) with $\underline{u} \leq u \leq \overline{u}$. ■

Now let us give necessary conditions for the existence of positive weak solution for (2.1).

Assume

(H.5) $H'(t)^{p-1} b(x,t) \geq \lambda_1 H(t)^{p-1}$ for $x \in \Omega$ a.e. and $t > 0$,

(H.6) $H'(t)^{p-1} b(x,t) > \lambda_1 H(t)^{p-1}$ for x in a subset of positive measure of Ω and $t > 0$.

Proposition 2.2 If **(H.5)** and **(H.6)** are satisfied, then (2.1) does not admit positive weak solution.

Proof: Suppose, by contradiction, that (2.1) admits a positive weak solution $u \in C^1(\overline{\Omega})$. Define $v = H(u)$. Then v is a positive weak solution of (2.2), where $g(x,s)$ is as in the proof of Proposition 2.1. In addition, by **(H.5)** and **(H.6)**, one gets

$$g(\cdot, v(\cdot)) \geq \lambda_1 v^{p-1}(\cdot) \text{ in } \Omega \text{ a.e.} \tag{2.4}$$
$$g(\cdot, v(\cdot)) > \lambda_1 v^{p-1}(\cdot) \text{ in a subset of positive measure of } \Omega . \tag{2.5}$$

Let φ_1 be a positive eigenfunction associated to λ_1 such that $\varphi_1 > v$ in Ω. Since $\dfrac{\varphi_1^p}{v^{p-1}}, \dfrac{v^p}{\varphi_1^{p-1}} \in W_0^{1,p}(\Omega)$, we can write

$$I = \int_\Omega \mid \nabla\varphi_1 \mid^{p-2} \nabla\varphi_1 . \nabla \left(\frac{\varphi_1^p - v^p}{\varphi_1^{p-1}} \right) dx + \int_\Omega \mid \nabla v \mid^{p-2} \nabla v . \nabla \left(\frac{v^p - \varphi_1^p}{v^{p-1}} \right) dx .$$

From Díaz and Saa inequality [7], we know that $I \geq 0$. On the other hand, from (2.4) and (2.5), we obtain

$$\begin{aligned} I &= \int_\Omega \lambda_1(\varphi_1^p - v^p)dx + \int_\Omega \frac{g(x,v)}{v^{p-1}}(v^p - \varphi_1^p) \\ &= \int_\Omega (\lambda_1 - \frac{g(x,v)}{v^{p-1}})(\varphi_1^p - v^p)dx < 0, \end{aligned}$$

a contradiction. ∎

Example 2.1 Let us consider the problem

$$\begin{cases} -\Delta_p u = a \mid \nabla u \mid^p + b & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.6}$$

where $a, b > 0$ are constants. Then, by Propositions 2.1 and 2.2, (2.6) admits a positive weak solution if and only if

$$a^{p-1}b < (p-1)^{p-1}\lambda_1 .$$

Example 2.2 More generally, set the following problem

$$\begin{cases} -\Delta_p u = \dfrac{a}{1+ku} \mid \nabla u \mid^p + b(1+ku)^{p-1} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.7}$$

where $a, b > 0$ and $k \geq 0$ are constants. Then, by Propositions 2.1 and 2.2, (2.7) possesses a positive weak solution if and only if

$$(k(p-1) + a)^{p-1}b < (p-1)^{p-1}\lambda_1 .$$

Example 2.3 Consider the following problem

$$\begin{cases} -\Delta_p u = \dfrac{a}{(1+u)^\alpha} \mid \nabla u \mid^p + b(1+u)^{p-1} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.8}$$

where $a, b > 0$ and $\alpha > 1$ are constants. Then, by an extension of L'Hospital's rule and Propositions 2.1 and 2.2, it is easy to conclude that (2.8) has a positive weak solution if and only if $b < \lambda_1$.

3 EXISTENCE OF POSITIVE SOLUTIONS UNDER OTHER NON-RESSONANCE CONDITIONS

In this section we need the following result which is a consequence of the Degree Theory in cones:

Lemma 3.1 Suppose

$$(i)\ \liminf_{s\to+\infty} \frac{g(x,s)}{s^{p-1}} > \lambda_1, \text{ uniformly for } x\in\Omega \text{ a.e.,}$$
$$(ii)\ \limsup_{s\to0^+} \frac{g(x,s)}{s^{p-1}} < \lambda_1, \text{ uniformly for } x\in\Omega \text{ a.e.}$$

Then (2.2) admits, at least, one positive weak solution in $C^1(\overline{\Omega})$.

Proof: At first, one shows that there exists $c>0$ such that $\|u\|_{W_0^{1,p}(\Omega)} \le c$ for every nonegative weak solution u of (2.2). This is done by contradiction, supposing that there is a sequence of nonegative weak solutions u_n such that $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$. Dividing the equation by its norms and taking the limit, one arrives at the following relation

$$\begin{cases} -\Delta_p v &= g(x)v^{p-1} & \text{in} & \Omega \\ v &\ge 0 & \text{in} & \Omega \\ v &= 0 & \text{on} & \partial\Omega, \end{cases}$$

where v is, up to a subsequence, the weak limit of $\frac{u_n}{\|u_n\|_{W_0^{1,p}(\Omega)}}$, $g\in L^\infty(\Omega)$ and, by (i), $g(x)>\lambda_1$ for $x\in\Omega$ a.e. Using Díaz and Saa inequality [7], it is easy to verify that $v\equiv 0$ in Ω, otherwise it must change sign. On the other hand, taking u_n as a test function in the equation satisfied by u_n, dividing by its norms to the power of p and letting $n\to\infty$, one gets a contradiction. The next step is considering the cone $C=\{u\in W_0^{1,p}(\Omega) : u\ge 0 \text{ in } \Omega\}$ and defining the homotopy $H:[0,+\infty)\times C\to C$ by $v=H(t,u)$, where

$$\begin{cases} -\Delta_p v &= g(x,u) + t & \text{in} & \Omega \\ v &= 0 & \text{on} & \partial\Omega \end{cases}$$

Let $T(u)=H(0,u)$. Using (ii), one concludes that there exists $r>0$ such that $u\ne t\,T(u)$ for every $t\in[0,1]$ and $u\in C$ with $\|u\|_{W_0^{1,p}(\Omega)}=r$. In addition, suppose that there are $t \in [0,+\infty)$ and $u \in C$ satisfying $u = H(t,u)$. Using (i), one obtains

$$-\Delta_p u \geq \lambda u^{p-1} + t - c \quad \text{in} \quad \Omega$$

with $\lambda > \lambda_1$. Assuming $t > c$ and again applying the Díaz and Saa inequality [7], one concludes that $u \equiv 0$ in Ω, contradicting $t > c$. Hence, $t \leq c$. Therefore, there exists $M > r$ such that $\|u\|_{W_0^{1,p}(\Omega)} < M$ for every $u \in C$ satisfying $u = H(t,u)$ for some $t \in [0,+\infty)$. Now applying the result about expansion of cones due to Krasnoselskii [10] we are done. ∎

Now, assume

$$\textbf{(H.7)} \quad \liminf_{t \to +\infty} \frac{H'(t)^{p-1} b(x,t)}{H(t)^{p-1}} > \lambda_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.,}$$

$$\textbf{(H.8)} \quad \limsup_{t \to 0^+} \frac{H'(t)^{p-1} b(x,t)}{H(t)^{p-1}} < \lambda_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.}$$

Proposition 3.1 If **(H.2)**, **(H.3)**, **(H.7)** and **(H.8)** are fulfilled, then (2.1) admits, at least, one positive weak solution.

Proof: Let $g : \Omega \times I\!R_+ \to I\!R_+$ be as in the proof of Proposition 2.1. By **(H.7)** and **(H.8)**, we get

$$\liminf_{s \to +\infty} \frac{g(x,s)}{s^{p-1}} > \lambda_1 \quad \text{and} \quad \limsup_{s \to 0^+} \frac{g(x,s)}{s^{p-1}} < \lambda_1,$$

both uniformly for $x \in \Omega$ a.e. Futhermore, by **(H.2)** and **(H.3)**, there exists a constant $c > 0$ such that

$$g(x,s) \leq c(s^{p-1} + 1) \quad \text{for } x \in \Omega \text{ a.e and } s \geq 0 .$$

Therefore, by Lemma 3.1, (2.2) admits a positive weak solution $v \in C^1(\overline{\Omega})$. So, defining $u = H^{-1}(v)$, we conclude that u is a positive weak solution of (2.1). ∎

Note that **(H.8)** implies $u \equiv 0$ in Ω is also a solution of (2.1). By virtue of Proposition 3.1 and with the aid of the auxiliar problem (2.1), a similar procedure used in the proof of Theorem 2.1. allow us to sate the following result.

Theorem 3.1 Assume **(H.4)** with (a,b) and (c,d) satisfying **(H.2)**, **(H.3)**, **(H.7)** and **(H.8)**. Then (1.2) admits, at least, one positive weak solution.

Example 3.1 Let us consider the equation

$$\begin{cases} -\Delta_p u = \dfrac{a(u)}{1+ku} \mid \nabla u \mid^p + bu^{p-1} & \text{in} \quad \Omega \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where $a : I\!R_+ \to I\!R_+$ is a bounded continuous function and $b, k > 0$ are constants. Clearly, $u \equiv 0$ in Ω is a trivial solution of (3.1). However, if

$$\liminf_{t\to+\infty} a(t) = a,$$

$$b < \lambda_1$$

and

$$\left(\frac{a}{k(p-1)} + 1\right)^{p-1} b > \lambda_1,$$

then, by Proposition 3.1, (3.1) admits a positive weak solution. In particular, if

$$\left(1 - \frac{1}{p}\right)^{p-1} \lambda_1 < b < \lambda_1,$$

then the problem

$$\begin{cases} -\Delta_p u = \dfrac{1}{1+u} \mid \nabla u \mid^p + bu^{p-1} & \text{in} \quad \Omega \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a positive weak solution.

4 THE CASE $p = 2$

In this section we generalize Proposition 2.1 and Theorem 2.1 for $p = 2$. Consider the problem

$$
\begin{cases} -\Delta v \ + \ \displaystyle\sum_{i=1}^{N} B_i(x,v) D_i v \ + \ C(x,v)v \ = \ g(x,v) & \text{in} \quad \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad v \ = \ 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}
$$

where $B_i,\ C,\ g : \Omega \times I\!\!R \to I\!\!R$ are Carathéodory functions.

The next lemma follows as a particular case of a result in [15].

Lemma 4.1 Assume

(i) there is a constant $c > 0$ such that $|B_i(x,s)|, |C(x,s)| \le c$ for $x \in \Omega$ a.e. and $s \in I\!\!R$
and define

$$
\mu_1 \ = \liminf_{\substack{\|v\|_{L^2(\Omega)} \to +\infty \\ v \ \in \ W_0^{1,2}(\Omega)}} \quad \frac{J(v,v)}{\| \ v \ \|^2_{L^2(\Omega)}},
$$

where

$$
J(v,v) = \int_\Omega |\ \nabla v\ |^2 \ dx \ + \sum_{i=1}^{N} \int_\Omega B_i(x,v) D_i v \, v \, dx \ + \int_\Omega C(x,v) v^2 dx \ .
$$

Then $\mu_1 \in (-\infty, +\infty)$. In addition, if

(ii) $\displaystyle\limsup_{|s| \to +\infty} \frac{g(x,s)}{s} \ < \ \mu_1,$ uniformly for $x \in \Omega$ a.e.,

(iii) there is a constant $c_1 > 0$ such that $|g(x,s)| \le c_1(|s|+1)$ for $x \in \Omega$ a.e. and $s \in I\!\!R$

then (4.1) admits, at least, one weak solution in $C^1(\overline{\Omega})$.

Remark 4.1 There is $c_0 > 0$ such that if B_i and C verify (i) of Lemma 4.1 with c_0 in place of c, then $J \ge 0$, $\mu_1 > 0$ and the full elliptic operator of the left-hand side of (4.1) satisfies the strong maximum principle. In this case, if $g : \Omega \times I\!\!R_+ \to I\!\!R_+$, then Lemma 4.1 provides the existence of a nonegative weak solution.

Now let us discuss the following problem

$$\begin{cases} -\Delta u &= a(x,u) \mid \nabla u \mid^2 + b(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.2}$$

where $a : \overline{\Omega} \times I\!R_+ \to I\!R_+,\ D_i a,\ D_{ii} a : \overline{\Omega} \times I\!R_+ \to I\!R$ are continuous functions and $b : \Omega \times I\!R_+ \to I\!R_+$ is a Carathéodory function, where $D_i a(x,t) = D_{x_i} a(x,t)$ and $D_{ii} a(x,t) = D_{x_i x_i} a(x,t)$.

Assume the assumption below

(H.9) there is a constant $k > 0$ such that $|\int_0^t D_i a(x,r)dr|,\ |\int_0^t D_{ii} a(x,r)dr| \ \leq\ k$ for every $(x,t) \in \Omega \times I\!R_+$.

Define $H : \overline{\Omega} \times I\!R_+ \to I\!R_+$ by

$$H(x,t) = \int_0^t \exp\left(\int_0^s a(x,r)dr\right) ds\ .$$

Since $H \in C^2(\overline{\Omega} \times I\!R_+)$ and $D_t H(x,t) \geq 1$ for $x \in \Omega$ and $t \geq 0$, by the Implicit Function Theorem or, more precisely, Submersion Local Form Theorem, there exists $K \in C^2(\overline{\Omega} \times I\!R_+)$ such that $s = H(x, K(x,s))$. Let

$$J(v,v) = \int_\Omega \mid \nabla v \mid^2 dx + \sum_{i=1}^N \int_\Omega B_i(x,v) D_i v\, v\, dx + \int_\Omega C(x,v) v^2 dx, \text{ for every } v \in W_0^{1,2}(\Omega),$$

where

$$\begin{aligned} B_i(x,s) &= \frac{2H_{x_i t}(x,K(x,s))}{H_t(x,K(x,s))}, \\ C(x,s) &= \sum_{i=1}^N \left(\frac{H_{x_i x_i}(x,K(x,s))}{s} - \frac{2H_{x_i t}(x,K(x,s)) H_{x_i}(x,K(x,s))}{H_t(x,K(x,s)) s} \right) . \end{aligned}$$

By **(H.9)**, B_i and C verify (i) of Lemma 4.1., so μ_1 is finite. Suppose $J \geq 0$, $\mu_1 > 0$ and the full elliptic operator of the left-hand side of (4.1) verifies the strong maximum principle, see Remark 4.1.

Assume that b satisfies

(H.10) $\displaystyle \limsup_{t\to+\infty} \frac{H'(x,t)b(x,t)}{H(x,t)} \ <\ \mu_1,$ uniformly for $x \in \Omega$ a.e.,

(H.11) there are constants $c, k > 0$ such that $H'(x,t)b(x,t) \ \leq\ cH(x,t)$ for

$x \in \Omega$ and $t \geq k$.

Proposition 4.1 If **(H.3)**, **(H.9)**, **(H.10)** and **(H.11)** are fulfilled, then (4.2) admits, at least, one nonegative weak solution.

Proof: Define $g(x,s) = H_t(x, K(x,s))b(x, K(x,s))$. From **(H.3)**, **(H.9)**, **(H.10)** and **(H.11)**, one has

$$\limsup_{s\to+\infty} \frac{g(x,s)}{s} < \mu_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.},$$
$$g(x,s) \leq c(s+1), \text{ for all } x \in \Omega \text{ a.e. and } s \geq 0 .$$

By Lemma 4.1 and Remark 4.1, (4.1) admits a nonegative weak solution $v \in C^1(\overline{\Omega}) \cap W^{2,p}(\Omega)$. Now, denoting $u = K(x,v)$ we get $u \in C^1(\overline{\Omega}) \cap W^{2,p}(\Omega)$, let us verify that u satisfy (4.2). Indeed, since $v(x) = H(x, u(x))$ we get

$$v_{x_i} = H_{x_i} + H_t\, u_{x_i},$$
$$v_{x_i x_i} = H_{x_i x_i} + 2H_{x_i t}u_{x_i} + H_{tt}u_{x_i}^2 + H_t u_{x_i x_i} .$$

Then

$$\Delta v = \sum_{i=1}^{N}(H_{x_i x_i} + 2H_{x_i t}u_{x_i}) + H_{tt} \mid \nabla u \mid^2 + H_t \Delta u .$$

On the other hand, inserting v in the expression of B_i and C yields

$$\sum_{i=1}^{N} B_i(x,v)D_i v \; + \; C(x,v)v = \sum_{i=1}^{N}(H_{x_i x_i} + 2H_{x_i t}u_{x_i}) .$$

Consequently

$$-\Delta v + \sum_{i=1}^{N} B_i(x,v)D_i v \; + \; C(x,v)v = -H_{tt}(x,u) \mid \nabla u \mid^2 - H_t(x,u)\Delta u .$$

Since $g(x,v) = H_t(x,u)b(x,u)$, it remains to note that $H_{tt} = a\ H_t$. ∎

Finally, let $a,\ c : \Omega \times I\!R_+ \to I\!R_+$ be continuous functions and $b,\ d : \Omega \times I\!R_+ \to I\!R_+$ Carathéodory functions such that $a(x,t_1) \leq c(x,t_2)$ and $b(x,t_1) \leq d(x,t_2)$ for every $x \in \Omega$ a.e. and $t_1, t_2 \geq 0$.

Suppose

(H.12) $a(x,t) \mid \xi \mid^2 + b(x,t) \leq f(x,t,\xi) \leq c(x,t) \mid \xi \mid^2 + d(x,t)$,
for every $x \in \Omega$ a.e. and $t \geq 0$.

An analogous reasoning used in the proof of Theorem 2.1 produces the following result:

Theorem 4.1 Suppose **(H.12)** with (a,b) and (c,d) verifying **(H.3)**, **(H.9)**, **(H.10)** and **(H.11)**. Then (1.2) admits, at least, one nonegative weak solution.

When $a(x,t)=a(t)$, one has $B_i \equiv 0 \equiv C$. Consequently, $\mu_1=\lambda_1$. Hence, when $p=2$, Proposition 4.1 and Theorem 4.1 generalize Proposition 2.1 and Theorem 2.1, respectively. We finish giving an example where the hypotheses of Proposition 2.1 are not necessarily satisfied, however, Proposition 4.1 applies.

Example 4.1 Let us set the following problem

$$\left\{\begin{array}{rcll} -\Delta u &=& \dfrac{a(\varepsilon x)}{(1+u)^\alpha} \mid \nabla u \mid^2 + b(x)(1+u) & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{4.3}$$

where $\alpha>0$, $a: I\!R^N \to I\!R_+$ is a function of class C^2 such that $D_i a$ and $D_{ii}a$ are bounded functions in $I\!R^N$ and $b:\Omega \to I\!R_+$ is a bounded measurable function. If $a(\cdot)\equiv a$ in $I\!R^N$ and $b(x)<\lambda_1$ uniformly for $x\in\Omega$ a.e., then, by Proposition 2.1, (4.3) admits a positive weak solution. On the other hand, if a is not constant, there exists $\varepsilon_0>0$ such that the conditions given in Remark 4.1 hold for every $0<\varepsilon<\varepsilon_0$. Hence, if $0<\varepsilon<\varepsilon_0$ and

$$b(x) \ < \ \mu_1, \quad \text{uniformly} \quad \text{for } \ x\in\Omega \ \text{ a.e.}, \tag{4.4}$$

then problem (4.3) possesses a positive weak solution. In particular, there is $\varepsilon_0>0$ such that if $0<\varepsilon<\varepsilon_0$ and (4.4) holds, then the problem

$$\left\{\begin{array}{rcll} -\Delta u &=& \dfrac{cos^2(\varepsilon \sum_{i=1}^N x_i)}{(1+u)^\alpha} \mid \nabla u \mid^2 + b(x)(1+u) & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \partial\Omega, \end{array}\right.$$

has a positive weak solution.

References

[1] Amann, H. and Crandall, M. - *On some existence theorems for semilinear elliptic equations, Indiana Univ. Math. J.,* ***27****, 1978, pp. 779-790.*

[2] Boccardo, L., Dràbek, P., Giachetti, D. and Kucera, M. - *Generalization of Fredholm alternative for nonlinear differential operators, Nonlinear Anal. TMA,* ***10****, 1986, pp. 1083-1103.*

[3] Boccardo, L., Murat, F. and Puel, J. P. - *Existence de solutions fables pour des équations elliptiques quasi-linéaire à croissance quadratique, Nonlinear Partial Differential Equations and their Applications (London) (H. Brézis and J. L. Lions, eds.), Pitman Research Notes, London, 1983, pp. 19-73.*

[4] Boccardo, L., Murat, F and Puel, J. P. - L^{∞} *Estimate for some nonlinear elliptic partial differential equations and application to an existence result, SIAM J. Math. Anal.,* ***23****, 1992, pp. 326-333.*

[5] Các, N. P. - *Some remarks on a quasilinear elliptic boundary value problem, Nonlinear Anal. TMA,* ***8****, 1984, pp. 697-709.*

[6] Cho, K. and Choe, H. J. - *Nonlinear degenerate elliptic partial differential equations with critical growth conditions on the gradient, Proc. Amer. Math. Soc.,* ***123****, 1995, pp. 3789-3796.*

[7] Díaz, J. I. and Saa, J. E. - *Uniqueness of nonegative solutions for elliptic nonlinear diffusion equations with a general perturbation term, Proc. of the VIII CEDYA, Santander, 1985.*

[8] Ferone, V. and Posteraro, M. R. - *On a class of quasilinear elliptic equations with quadratic growth in the gradient, Nonlinear Anal. TMA,* ***20****, 1993, pp. 703-711.*

[9] Kazdan, J. and Kramer, R. - *Invariant criteria for existence of solutions to second-order quasilinear elliptic equations, Comm. on Pure Appl. Math.,* ***31****, 1978, pp. 619-645.*

[10] Krasnoselskii, M. A. - *Fixed points of cone-compressing or cone-extending operators, Sov. Math. Dokl.,* ***1****, 1960, pp. 1285-1288.*

[11] Lieberman, G. M. - *Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. TMA,* ***12****, 1988, pp. 1203-1219.*

[12] Lions, P. L. - *Résolution de problèmes elliptiques quasilinéaires, Arch. Rat. Mech. Analysis,* ***74****, 1980, pp. 335-353.*

[13] Pohozaev, S. - *On equations of the form* $\Delta u = f(x, u, Du)$*, Math. USSR Sb.,* ***41****, 1982, pp. 269-280.*

[14] Serrin, J. - *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, Phil. Trans. Royal Soc. London,* ***264****, 1969, pp. 413-496.*

[15] Shapiro, V. L. - *Quasilinear ellipticity and the 1st eigenvalue, Comm. in Part. Diff. Eq.,* ***16****, 1991, 1819-1855.*

[16] Xavier, J. B. M. - *A priori estimates and multiplicity results for equations of the form* $-\Delta u = f(x, u, \nabla u)$*, Dynamic Systems and Appl.,* ***3****, 1994, pp. 489-500.*