# POSITIVITY FOR QUASILINEAR ELLIPTIC OPERATORS

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**Abstract**

We study the positivity of a function satisfying some inequalities involving quasilinear elliptic operators which may grow polinomially, exponentially or logarithmicaly.

## 1. INTRODUCTION

At first, the purpose of this paper is to ensure that $u > 0$ in $\Omega$ whenever $u \in C^1(\Omega)$, $u \geq 0$ in $\Omega$, $u \not\equiv 0$ in $\Omega$ satisfy weakly the following inequality in some arbitray domain $\Omega \subset I\!R^N$, $N \geq 1$, unless otherwise stated:

$$Lu = -\mathrm{div}A(x, u, \nabla u) + B(x, u, \nabla u) \geq 0. \tag{1}$$

So that $u$ cannot vanish identically locally in subdomains of $\Omega$. We recall that a function $u$ satisfies $Lu \geq 0$ weakly in $\Omega$ iff $A(x, u, \nabla u)$ and $B(x, u, \nabla u)$ are locally integrable when $u$ and $\nabla u$ are and

$$\int_\Omega \{A(x, u, \nabla u) \cdot \nabla \phi + B(x, u, \nabla u)\phi\} dx \geq 0 \tag{2}$$

for every $\phi \in C_c^1(\Omega)$ with $\phi \geq 0$ in $\Omega$.

The divergence form operators we are going to deal with have the pricipal part $A : \Omega \times I\!R \times I\!R^N \to I\!R^N$ with a general form:

$$A_j(x, \mu, \eta) \in C^1(\Omega \times I\!R \times (I\!R^N - \{0\})) \cap C^0(\Omega \times I\!R \times I\!R^N) \text{ for } j = 1, \cdots, N, \tag{3}$$

$$\sum_{i,j=1}^N D_{\eta_i} A_j(x, \mu, \eta)\xi_i\xi_j \geq \lambda(x, \mu, \eta) \mid \xi \mid^2, \tag{4}$$

$$\sum_{j=1}^N D_{\eta_j} A_j(x, \mu, \eta) \leq \Lambda(x, \mu, \eta), \tag{5}$$

$$\sum_{j=1}^N D_\mu A_j(x, \mu, \eta)\eta_j \geq 0 \tag{6}$$

and

$$\sum_{j=1}^N D_{x_i} A_j(x, \mu, \eta) \geq 0, \tag{7}$$

for every $(x, \mu, \eta) \in \Omega \times I\!R \times I\!R^N$ , $\xi \in I\!R^N$, $\xi \neq 0$. The functions $\lambda, \Lambda : \Omega \times I\!R \times (I\!R^N - \{0\}) \to I\!R$ are such that

$$\lambda(x, \mu, \eta) \geq \rho(\mid \eta \mid) \tag{8}$$

and

$$\Lambda(x,\mu,\eta) \le \gamma\rho(\mid \eta \mid) \text{ for some constant } \gamma > 0. \tag{9}$$

The function $\rho$ has the following properties:

$$\rho \in C^1(0,+\infty), \tag{10}$$

$$\alpha(t) = \rho(t)t \text{ for } t \ge 0 \text{ is odd, increasing and } \alpha(0) = 0, \tag{11}$$

$$\alpha \in C^1(0,+\infty) \text{ if } \rho'(t)t < 0 \text{ for } t > 0 \tag{12}$$

or

$$\alpha \in C^1[0,+\infty) \text{ if } \rho'(t)t > 0 \text{ for } t > 0, \tag{13}$$

since we are tacitly assuming that

$$\rho \text{ is either increasing or decreasing .} \tag{14}$$

If

$$\text{there exists a constant } \rho_0 > 0 \text{ such that } \rho(t) \equiv \rho_0, \tag{15}$$

assumptions (10)–(14) are unecessary. Properties (3)–(14) reveal that the class of operators we are studying may degenerate at vanishing gradient points of $\Omega$. This means that the first eigenvalue of the matrix $D_{\eta_i}A_j$ may tend to 0 or to $+\infty$ as $\mid \eta \mid \to 0$. Clearly $D_{\eta_i}A_j$ may be assumed symmetric by rewriting it as $(D_{\eta_i}A_j + D_{\eta_j}A_i)/2$. The uniformly elliptic case (15) will be treated together.

The term $B : \Omega \times I\!R \times I\!R^N \to I\!R$ has the form

$$B(x,\mu,\eta) = \rho(\mid \eta \mid)\eta \cdot C(x) + d(x)f(\mu) \tag{16}$$

with the properties

$$C \in (L^\infty(\Omega))^N, \tag{17}$$

$$d \in L^\infty_{loc}(\Omega) \text{ and } d(x) \ge 0 \text{ a.e. in } \Omega, \tag{18}$$

$$\text{there exists an } \varepsilon > 0 \text{ such that } f \in C^0[0,\varepsilon] \text{ is nondecreasing with } f(0) = 0 \tag{19}$$

and

$$\text{for every } \delta > 0 \text{ we have } \int_0^\delta [\chi^{-1}(F(s))]^{-1} ds = \infty, \tag{20}$$

where $F(t) = \int_0^t f(s)ds$ and $\chi(t) = \alpha(t)t - \int_0^t \alpha(s)ds$ for $t \geq 0$. Note that $\chi$ is strictly increasing for $t \geq 0$, since we always may assume that $f(s) > 0$ for $s \in (0, +\infty)$ and $f$ is extended continulosly and nondecreasing for $s \geq \varepsilon$. Another assumption we can make of on $f$ is

$$f(t) = 0 \text{ for } t \text{ in some interval } [0, t_0] \text{ with } t_0 < \varepsilon \text{ and } f \text{ is nondecreasing in } [t_0, \varepsilon]. \tag{21}$$

Let us in turn consider nondivergence form operators satisfying the following inequality in $\Omega$:

$$Qu = -\sum_{i,j=1}^N a_{ij}(x, \nabla u) D_{ij} u + B(x, u, \nabla u) \geq 0, \tag{22}$$

where the functions $a_{ij} : \Omega \times I\!R^N \to I\!R^N$ satisfy

$$a_{ij}(x, \eta) \in C^1(\Omega \times (I\!R^N - \{0\})) \cap C^0(\Omega \times I\!R^N) \text{ for } i, j = 1, \cdots, N, \tag{23}$$

$$\sum_{i,j=1}^N a_{ij}(x, \eta) \xi_i \xi_j \geq \lambda(x, \eta) \mid \xi \mid^2 \tag{24}$$

and

$$\sum_{j=1}^N a_{jj}(x, \eta) \leq \Lambda(x, \eta) \tag{25}$$

for every $(x, \eta) \in \Omega \times I\!R^N$ , $\xi \in I\!R^N$, $\xi \neq 0$. Again, $a_{ij}$ may be assumed symmetric by rewriting it as $(a_{ij} + a_{ji})/2$. As before the functions $\lambda, \Lambda : \Omega \times (I\!R^N - \{0\}) \to I\!R$ are subject to the assumptions (8)–(15), and our operator may be degenerate. The function $B : \Omega \times I\!R \times I\!R^N \to I\!R$ has the same properties (16)–(21). Now we require an improvement in the regularity of $u$ in order to inequality (22) be satisfied classically, so we take $u \in C^2(\Omega)$. This regularity of $u$ has to do with some classical comparison principles we are going to apply. Such comparison principles prevent us form having a dependence on $\mu$ in the functions $a_{ij}$, an example in [4] is provided.

**THEOREM 1**. Let a function $u \in C^1(\Omega)$, $u \geq 0$ in $\Omega$, $u \not\equiv 0$ in $\Omega$ such that $Lu \geq 0$ in $\Omega$ and assume (3)–(20), then $u > 0$ in $\Omega$, if only one of the following extra structural hypotheses holds:

$$(i)\ A = A(x, \mu, \eta),\ \rho \text{ is decreasing or a positive constant, } f \in C^1(0, \varepsilon] \text{ and } d \in C^1(\Omega) \tag{26}$$

$$(ii)\ A = A(x, \eta),\ \rho(t) =\mid t \mid^{p-2} \text{ for } p > 2,\ f \in C^1[0, \varepsilon],\ d \in C^1(\Omega) \text{ and} \tag{27}$$

$$D_\mu B - 1/N(D_\eta B)^q > 0 \text{ for } (x, \mu, \eta) \in \Omega \times I\!R \times I\!R^N \text{ and } 1/p + 1/q = 1$$

$$(iii)\ A = A(x, \mu, \eta) \text{ and } C(x) \equiv 0 \tag{28}$$

$$(iv) \text{ In particular, } A = A(\eta) = \rho(\mid \eta \mid)\eta \text{ and } C(x) \equiv 0.$$

For nondivergence form operators we have the following counterpart.

**THEOREM 2**. Let a function $u \in C^2(\Omega)$, $u \geq 0$ in $\Omega$, $u \not\equiv 0$ in $\Omega$ such that $Qu \geq 0$ in $\Omega$ and assume (8)–(20) and (23)–(25), then $u > 0$ in $\Omega$.

**THEOREM 3**. The above theorems remain true with hypotheses (19)–(20) are replaced by (21).

Let us see a normal boundary derivative version.

**THEOREM 4**. Let $x_0 \in \partial\Omega$ be a point satisfying the interior sphere condition and let $\nu$ be the inward unit normal vector at $x_0$. If $u$ satisfies the hypotheses of the above theorems, $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$, then $\partial u(x_0)/\partial\nu > 0$.

The positivity problem we are studying is related to the nonexistence of a dead core of a solution, that is, a region on the domain $\Omega$ in which $u$ vanishes. Condition (20) is necessary for the nonexistence of such regions. This question have been studied in [2], [7], [9], [10] and [11]. Here a broader class of elliptic operators is considered with the aid of different tools. The second member $B$ is also more general than those considered in the previous cited works, because we may allow a gradient dependence, let us see some examples:

$$-\mathrm{div}((\exp(\mid \nabla u \mid) - 1)\nabla u) \tag{29}$$

$$-\text{div}(\log(|\nabla u| + 1)\nabla u) \tag{30}$$

$$-\text{div}((1 + |\nabla u|^2)^{-1/2}\nabla u) \tag{31}$$

Existence theorems for examples of the growth type (29)–(30) have been studied in [3] and [5] and for the capilarity operator (31) in [8]. We also refer to [4] for lots of examples of such operators. In the uniformly elliptic case, they model many physical and geometrical aspects such as cracking of plates, blast furnaces, precribed mean curvature, gas dynamics and capilarity. The degenerate operators are related to non-Newtonian fluids and flow through porous media, see [2].

Our resuls are carried out by means of a contradiction argument. We build a function $w$ with nonvanishing gradient satisfying the reverse inequality $Lw \leq 0$ and compare with $u$ in some annulus contained in $\Omega$. This underlying heuristical idea is similar to that for proving the Classical Hopf's Maximum Principle [6]. In that case, the comparison function $w$ has a standard expression. In contrast, here we only know the existence of function $w$ with the properties we need. This function $w$ is a solution of a two-point boundary value problem for the ordinary differential equation $-(a(r)\alpha(u'(r)))' + b(r)f(u(r)) = 0$. We use degree theory to solve it. The functions $a$ and $b$ will be timely introduced. The later equation has to do with the radial formula of $L$ in the constructed annulus.

When condition (20) does not hold the function $u$ cannot be positive, provided $A(x, \mu, \eta) = \rho(|\eta|)\eta$, $C(x) \equiv 0$ and $d(x) = 1$. If

$$\text{there exists } \delta > 0 \text{ such that } \int_0^\delta [\chi^{-1}(F(s))]^{-1} ds < \infty, \tag{32}$$

using degree theory we construct a function $v$ vanishing in some portion of $\Omega$ and satifying $Lv = 0$ in $\Omega$. So, assumption (20) is necessary and sufficient for obtaining the positivity of $u$. This led us to call the following result nonunique continuation property.

**THEOREM 5**. If (3)–(19), (28) and (32) are fulfilled with $C(x) \equiv 0$ and $d(x) = 1$, then for every $x_0 \in \partial(I\!R^N \setminus \overline{\Omega}) \subset \partial\Omega$ and all $R > 0$ there exists a function $v \in C^1(\Omega)$ such that $v \geq 0$ in $\Omega$ and $v \not\equiv 0$ in $\Omega$ satisfying $Lv = 0$ in $\Omega$ and $v = 0$ in $\Omega \setminus B_R(x_0)$. Under more general assumptions, replacing (28) by (26) or (27), we have $Lv \geq 0$ in $\Omega$, but the other properties of $v$ are preserved.

In third place, we mention that it is possible to repeat the above study for nonisotropic operators and $u$ satifying the following inequality in $\Omega$:

$$Su = -\sum_{j=1}^{N} D_{\eta_j} A_j(x, u, D_{x_j} u) + \sum_{j=1}^{N} \rho_j(\mid D_{x_j} u \mid) D_{x_j} u \cdot C_j(x) + d(x) f(u) \geq 0. \quad (33)$$

We replace $A_j(x, \mu, \eta)$, $\lambda$, $\Lambda$, $\rho$, $\gamma$, $\alpha$ and $\chi$, respectively, by $A_j(x, \mu, \eta_j)$, $\lambda_j$, $\Lambda_j$, $\rho_j$, $\gamma_j$, $\alpha_j$ and $\chi_j$ for $j = 1, \cdots, N$, with the same properties for each index $j$. Now $D_{\eta_i} A_j(x, u, \eta_j) = 0$ if $i \neq j$. Much of the above program can be repeated separately for each $j$. It is possible to conclude that Theorems 1, 3 and 4 are still valid. The comparison function $w$ we need is the sum from $j = 1$ to $j = N$ of $w_j$, obained in each step $j$. Theorem 5 does not feature a complete generalization, under the analogues of assumptions (26)–(28) we conclude that $Sv \geq 0$ in $\Omega$ and $v$ vanishes in the complementary of some neighbourhood of $x_0$. For that matter we use the inequality $f(\sum_{j=1}^{N} t_j) \geq (1/N) \sum_{j=1}^{N} f(t_j)$ for $t_j \geq 0$.

We accomplish with a different way to handle the following particular result.

**THEOREM 6**. Let $u \in C^2(\Omega)$, where $\Omega \subset I\!R^N$, $N \geq 1$, is a bounded domain. Assume that $Tu = \sum_{j=1}^{N} D_{x_j}(\rho_j(\mid \nabla u \mid) D_{x_j} u) \geq 0$ and the properties on the functions $\rho_j$: $\rho_j \in C^1(0, +\infty)$, $\rho_j(s) > 0$, $(\rho_j(s)s)' \geq 0$ and either $\rho_j'(s) \geq 0$ or $\rho_j'(s) \leq 0$ for every $j = 1, \cdots, N$ and $s > 0$. If u achives its maximum at an nondegenertate point $x_0 \in \Omega$, then $u(x) \equiv u(x_0)$ in $\Omega$.

**REMARK**. A paralell study for functions belonging to some Sobolev or Orlicz-Sobolev space may be developed in the same scheme of [7]. To this aim some additional assumptions on $A$ and $B$ in order to get a $C^1$ regularity are needed.

## 2. PROOF OF THEOREMS

We need a refinament of some well known weak comparison principles. We state them below under conditions (3)–(25) followed by the main steps of its proofs, just to stress what is different from the classical results. The hypothesis on the gradient of the function $w$ in the following lemmas may seem to be somewhat artificial, but this is exactly the situation we are going to face when proving our theorems. This condition is formulated in order to give a positive lower bound, depending on $v$, for

$\lambda(x, t\nabla w + (1-t)\nabla u)$ for $x$ in a bounded subdomain $\Omega_0 \subset \Omega$.

**LEMMA 1**. If $u, w \in C^1(\Omega_0)$ with $\nabla w \neq 0$ in $\overline{\Omega_0}$ satisfy $Lw \leq Lu$ in $\Omega_0$ and $w \leq u$ on $\partial\Omega_0$, then $w \leq u$ in $\Omega_0$, provided (19) holds and $C(x) \equiv 0$ for $x$ in $\Omega$.

**PROOF OF LEMMA 1**. Preliminarily we should note that

$$\sum_{j=1}^{N} A_j(x,\mu,\eta) - A_j(x,\mu,\eta'))(\eta_j - \eta'_j) =$$

$$= \sum_{j=1}^{N} \int_0^1 \frac{d}{dt}[A_j(x,\mu,t\eta + (1-t)\eta')]dt\ (\eta_j - \eta'_j) =$$

$$= \int_0^1 \sum_{i,j=1}^{N} D_{\eta_j} A_j(x,\mu,t\eta + (1-t)\eta')dt\ (\eta_i - \eta'_i)(\eta_j - \eta'_j) \geq$$

$$\geq \int_0^{1/4} \rho(\mid t\eta + (1-t)\eta' \mid) \mid t\eta + (1-t)\eta' \mid^{-1} dt\ \ \mid \eta - \eta' \mid^2 \geq$$

$$\geq 1/4\ \alpha(1/4 \mid \eta - \eta' \mid)(\mid \eta \mid + \mid \eta' \mid)^{-1} \mid \eta - \eta' \mid^2$$

for every $x \in \Omega_0$ and $\eta, \eta' \in I\!R^N$. Since $M = \sup_{\overline{\Omega_0}}(\mid \nabla u \mid + \mid \nabla w \mid) < \infty$ and $1/4 \mid \nabla u - \nabla w \mid \leq \mid t\nabla w + (1-t)\nabla u \mid \leq \mid \nabla u \mid + \mid \nabla w \mid \leq M$ for $t \in [0, 1/4]$. The following integration against the test function $\phi = (w-u)^+$ establishes the result, provided $f$ is extended continuously and nondecreasing to the whole $I\!R$:

$$0 \leq \alpha(M)/4M \int_{\{w>u\}} \alpha(1/4 \mid \nabla w - \nabla u \mid) \mid \nabla w - \nabla u \mid^2 dx \leq -\int_{\{w>u\}} d(x)(f(w) - f(u))(w-u)dx \leq 0. \ \Box$$

**LEMMA 2**. Assume (26). If $u, w \in C^1(\overline{\Omega_0})$ with $\nabla w \neq 0$ in $\overline{\Omega_0}$ satisfy $Lw \leq Lu$ in $\Omega_0$ and $w \leq u$ on $\partial\Omega_0$, then $w \leq u$ in $\Omega_0$.

**PROOF OF LEMMA 2**. Again, with the aid of the mean value theorem and extending $f$ to $I\!R$ so that $f \in C^1(I\!R - \{0\}) \cap C^0(I\!R)$ and $f$ is nondecreasing, we write $z = w - u$,

$$\overline{a}_{ij} = \int_0^1 D_{\eta_j} A_j(x, t\nabla w + (1-t)\nabla u)dt,$$

$$c_j = \int_0^1 D_{\eta_j} B(x, w, t\nabla w + (1-t)\nabla u)dt$$

and

$$d = \int_0^1 D_\mu B(x, tw + (1-t)u, \nabla w)dt.$$

Therefore

$$\begin{cases} \mathcal{L}z = -\sum_{i,j=1}^{N} D_i(\overline{a}_{ij} D_j z) + \sum_{j=1}^{N} c_j D_j z + dz & \leq \; 0 \quad \text{in} \quad \Omega_0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad z & \leq \; 0 \quad \text{on} \quad \partial\Omega_0. \end{cases}$$

Note that $d \geq 0$ and bounded in $\overline{\Omega_0}$ and that we may rewrite $\overline{a}_{ij}$ as $(\overline{a}_{ij} + \overline{a}_{ji})/2$ if necessary. So that $\overline{a}_{ij}$ is a positive definite symmetric matrix. Since $M = \sup_{\overline{\Omega_0}}(\mid \nabla u \mid + \mid \nabla w \mid) < \infty$ and $\mid t\nabla w + (1-t)\nabla u \mid \leq \mid \nabla u \mid + \mid \nabla w \mid \leq M$ the terms $\overline{a}_{ij}$ and $c_i$ are bounded in $\overline{\Omega_0}$. The possible singularity of $D_{\eta_j} A_j(x, \eta)$ and $D_{\eta_j} B(x, \mu, \eta)$ for $\eta = 0$ is not relevant. It remains to verify that $\mathcal{L}$ is uniformly elliptic in $\overline{\Omega_0}$, indeed

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \int_0^1 \rho(\mid t\nabla w + (1-t)\nabla u \mid)dt \ \mid \xi \mid^2 \geq \rho(\mid \nabla w \mid + \mid \nabla u \mid) \ \mid \xi \mid^2 \geq \rho(M) \ \mid \xi \mid^2 .$$

for every $\xi \in I\!R^N$. Now are able to conclude that $z \leq 0$ in $\Omega_0$. $\square$

The following result is taken from [1].

**LEMMA 3**. Assume (27). If $u, w \in C^1(\overline{\Omega_0})$ satisfy $Lw \leq Lu$ in $\Omega_0$ and $w \leq u$ on $\partial\Omega_0$, then $w \leq u$ in $\Omega_0$.

**LEMMA 4**. If $u, w \in C^2(\Omega) \cap C^0(\overline{\Omega_0})$ with $\nabla w \neq 0$ in $\overline{\Omega_0}$ satisfy $Qw \leq Qu$ in $\Omega_0$ and $w \leq u$ on $\partial\Omega_0$, then $w \leq u$ in $\Omega_0$.

**PROOF OF LEMMA 4**. We write $z = w - u$ and

$$\sum_{i,j=1}^{N} a_{ij}(x, \nabla w)D_{ij}w - a_{ij}(x, \nabla w)D_{ij}u + a_{ij}(x, \nabla w)D_{ij}u - a_{ij}(x, \nabla u)D_{ij}u +$$

$$+\varphi(\mid \nabla u \mid)\nabla u \cdot C(x) - \varphi(\mid \nabla w \mid)\nabla w \cdot C(x) \geq d(x)(f(w) - f(u)) \geq 0$$

in $\Omega_0^+ = \{x \in \Omega_0^+ : z(x) > 0\}$. With the same notation of Lemma 2, define $b_j = c_j + \sum_{i=1}^N \overline{a}_{ij} D_{ij} u$ with $\overline{a}_{ij}$ and $c_j$ replaced by its analogues, hence

$$\begin{cases} \mathcal{Q}z = -\sum_{i,j=1}^N a_{ij}(x, \nabla z) D_{ij} z + \sum_{j=1}^N b_j D_j z & \leq 0 \quad \text{in} \quad \Omega_0^+ \\ z & \leq 0 \quad \text{on} \quad \partial\Omega_0 \end{cases}$$

and $\mathcal{Q}$ is uniformly elliptic. If $\Omega_0^+ \neq \emptyset$, then $0 < \sup_{\Omega_0} z \leq \sup_{\partial\Omega_0} z^+$, a contradiction. □

Let us recall the homotopical version of the Leray-Schauder Fixed Point Theorem.

**LEMMA 5**. Let $E$ be a Banach space and $H : E \times [0,1] \to E$ be a compact continuous mapping such that $H(u, 0) = 0$ for every $u \in E$. If there exists a constant $K$ such that $\|u\|_E < K$ for every pair $(u, \sigma) \in E \times [0,1]$ satisfying $u = H(u, \sigma)$, then the mapping $H(., 1) : E \to E$ has a fixed point.

We adapt for our needs some homotopical techniques widely used to solve elliptic quasilinear boundary value problems for partial differential equations. The a priori estimates required in the above lemma are usually obtained from classical Schauder estimates, see [4]. The situation here is different, we menage to solve an nonliner two-point boudary value problem, so the a priori estimates we need are derived from the behaviour of the real functions involved in the problem.

**LEMMA 6**. Assume (19) on $f$ and let $\alpha$, $a$ and $b$ be functions with the following properties:

$\alpha \in C^0(I\!R)$ with $\alpha(0) = 0$ is an increasing extention of the function $\alpha$ satisfying (11)–(15)

$$a \in C^1[0,T] \text{ and } a > 0 \text{ in } [0,T]$$

$$b \in C^0[0,T],\ b > 0 \text{ in } [0,T] \text{ and injective}$$

$$ab - a' \int_0^r b(s)ds \geq 0 \text{ in } [0,T].$$

There exists a solution $u \in C^1[0,T]$ of the problem

$$\begin{cases} -(a(r)\alpha(u'(r)))' + b(r)f(u(r)) = 0 \text{ in } (0,T) \\ u(0) = u_0, u(T) = u_T. \end{cases} \tag{34}$$

Moreover, if $u_0 = 0$ and $u_T > 0$, then $u \geq 0$ and $u' \geq 0$ in $[0,T]$. According either (12) or (13) is fulfilled we get either $u \in C^2(0,T]$ or $C^2[0,T]$, and either $u'' \geq 0$ in $(0,T]$ or $u'' \geq 0$ in $[0,T]$.

Suplementing the above hypotheses we get deeper informations. If (20) holds and $a'(r) < 0$ for $r \in [0,T]$, then $u'(0) > 0$. Hence $u' > 0$ in $[0,T]$, $u > 0$ in $(0,T]$ and $u'' > 0$ either in $(0,T]$ or in $[0,T]$ according either (12) or (13) is fulfilled, respectively. Futhermore, the solution $u$ is unique.

**PROOF OF LEMMA 6**. Extend $f$ continuously and nondecreasing to the whole $I\!R$. The problem is equivalent to finding a fixed point of the mapping

$$J(u)(r) = u_0 + \int_0^r \alpha^{-1}\Big(\frac{1}{a(s)}\Big(K + \int_0^s b(y)f(u(y))dy\Big)\Big)ds$$

where the constant $K$ satisfies

$$\int_0^T \alpha^{-1}\Big(\frac{1}{a(s)}\Big(K + \int_0^s b(y)f(u(y))dy\Big)\Big)ds = u_T - u_0.$$

Such a constant $K$ do exists because the first member defines an unbounded monotone continuous function of $K$. The mapping $J$ acts from the space $C^0[0,T]$, endowed with the usual Banach norm $\|.\|_{L^\infty(0,T)}$, into itself. We insert a parameter $\sigma \in [0,1]$ into the operator $J$ in order to generate a homotopy. So we define

$$J_\sigma(u)(r) = u_0 + \int_0^r \alpha^{-1}\Big(\frac{1}{a(s)}\Big(K_\sigma + \sigma\int_0^s b(y)f(u(y))dy\Big)\Big)ds$$

with $K_\sigma$ satisfying

$$\int_0^T \alpha^{-1}\Big(\frac{1}{a(s)}\Big(K_\sigma + \sigma\int_0^s b(y)f(u(y))dy\Big)\Big)ds = \sigma(u_T - u_0).$$

We define the mapping $H$ by setting $H(v,\sigma) = J_\sigma(v) = u$. Equivalently, the inserction of the parameter $\sigma$ into the problem (34) produces the class of problems

$$\begin{cases} -(a(r)\alpha(u'(r)))' + \sigma b(r)f(u(r)) = 0 \text{ in } (0,T) \\ u(0) = \sigma u_0, u(T) = \sigma u_T, \end{cases}$$

where $H$ associates to each pair $(u,\sigma)$ the solution $v$ of

$$\begin{cases} -(a(r)\alpha(v'(r)))' + \sigma b(r) f(u(r)) = 0 \text{ in } (0,T) \\ v(0) = \sigma u_0, v(T) = \sigma u_T. \end{cases}$$

Note that $H(u,0) = 0$ for every $u \in C^0[0,T]$. Indeed, this is equivalent to show that

$$\begin{cases} -(a(r)\alpha(u'(r)))' = 0 \text{ in } (0,T) \\ u(0) = 0, u(T) = 0. \end{cases}$$

possesses $u \equiv 0$ as unique solution. Clearly, if $u$ is a solution, then

$$\int_0^T a(r)\alpha(u')\zeta(r)dr = 0$$

for every $\zeta \in C_c^\infty(0,T)$. And hence $a(r)\alpha(u') \equiv 0$ in $[0,T]$. Since $a(r) > 0$ in $[0,T]$, we get $\alpha(u') \equiv 0$ in $[0,T]$. The homogeneous boundary conditions and applying the increasing function $\alpha^{-1}$ imply $u \equiv 0$ in $[0,T]$.

The continuity and compactness of $H$ follows as usual, the main ingredient is the Arzelá-Ascoli Theorem. It remains to show that there exists a constant $M$ such that $\|u\|_{L^\infty(0,T)} < M$ for every pair $(u,\sigma) \in C^0[0,T] \times [0,1]$ satisfying $J_\sigma(u)(r) = u(r)$ for every $r \in [0,T]$. So let $(u,\sigma)$ be a such pair and let us analyse the equivalent boundary value problem

$$\begin{cases} -(a(r)\alpha(u'(r)))' + \sigma b(r) f(u(r)) = 0 \text{ in } (0,T) \\ u(0) = \sigma u_0, u(T) = \sigma u_T. \end{cases}$$

If $|u|$ attains its maximum on the boundary of the interval $[0,T]$, then $\|u\|_{L^\infty(0,T)} \leq \sigma \ \max\{| \ u_0 \ |, | \ u_T \ |\} \leq \max\{| \ u_0 \ |, | \ u_T \ |\}$. This is the unique possibility, the maximum is attained only on the boundary of the interval. Indeed, if the maximum of $| \ u \ |$ is attained, say, at a point $r_0 \in (0,T)$ and $| \ u(r_0) \ | > 0$, then a contradiction is rised, because the equation provides $u(r_0)(a(r)\alpha(u'(r)))'\Big|_{r=r_0} > 0$. If $u(r_0) > 0$, then $(a(r)\alpha(u'(r)))' > 0$ in a neighbourhood $(r_0 - \epsilon, r_0 + \epsilon)$ for $\epsilon > 0$ small. Since $a(r) > 0$ and $u'(r_0) = 0$, we either have $\alpha(u'(r)) < 0$ in $(r_0 - \epsilon, r_0)$ or $\alpha(u'(r)) > 0$ in $(r_0, r_0 + \epsilon)$. Apllying the increasing function $\alpha^{-1}$ we conclude that either $u'(r) < 0$ in $(r_0 - \epsilon, r_0)$ or $u'(r) > 0$ in $(r_0, r_0 + \epsilon)$, a contradiction. On the other hand if $u(r_0) < 0$, we get a similar contradiction.

Thus we have proved that $J$ has a fixed point $u \in C^0[0,T]$ by applying Lemma 3. A bootstrap reasoning shows that $u \in C^1[0,T]$, because $J(u)(r) = u(r)$. From now on we assume $u_0 = 0$ and $u_T > 0$.

Let us prove that $u'(0) > 0$. If on the contrary we assume $u'(0) = 0$, then for $T_0 = \sup\{\tau \in [0,T] : u'(\tau) = 0\}$ we have $u > 0$ in $(T_0, T]$, $u'(T_0) = 0$ and $u$ is bijective from $[T_0, T]$ to $[0, u_T]$. We have the following calculations valid in the interval $(0, T)$:

$$(a\alpha(u'))' = bf(u)$$

$$a(\alpha(u'))' + a'\alpha(u') = bf(u)$$

$$(\alpha(u'))'u' + (a'/a)\alpha(u')u' = (b/a)f(u)u'$$

$$(\alpha(u')u')' + (a'/a)\alpha(u')u' = (b/a)f(u)u' + \alpha(u')u''$$

$$(\alpha(u')u')' + (a'/a)\alpha(u')u' = (b/a)(F(u))' + (\int_0^{u'} \alpha(s)ds)'$$

Since

$$\alpha(u') \le (\int_0^r b(s)ds/a)f(u)$$

we get

$$(\alpha(u')u')' - (\int_0^{u'} \alpha(s)ds)' \le [b/a - (a'/a)(\int_0^r b(s)ds/a)](F(u))'$$

$$\chi(u') \le cF(u)$$

$$\int_{T_0}^T [\chi^{-1}(F(u(r)))]^{-1} u'(r)dr = \int_0^{u_T} [\chi^{-1}(F(s))]^{-1} ds < \infty,$$

a contradiction.

Our solution satisfies the integral equation

$$u(r) = \int_0^r \alpha^{-1}\Big(\frac{1}{a(s)}\Big(K + \int_0^s b(y)f(u(y))dy\Big)\Big)ds$$

then, as remarked above, it is clear that $u \in C^1[0,T]$ and $K = a(0)\alpha(u'(0))$. Since

$$u'(r) = \alpha^{-1}\Big(\frac{1}{a(r)}\Big(a(0)\alpha(u'(0)) + \int_0^r b(s)f(u(s))dy\Big)\Big) \tag{35}$$

we see that either $u \in C^2(0,T]$ or $C^2[0,T]$ according the $C^1$ regularity of $\alpha^{-1}$ at 0, see (12)–(13). Let us see that $u > 0$ in $[0,T]$. Actually, if by contradiction $u^- \not\equiv 0$, there exist $r_1 < r_2$ in $[0, r_2]$ such that $u(r_1) = u(r_2) = 0$, so then multiplying the equation in (34) by $u^-$ and integrating, produces

$$0 \le \int_{r_1}^{r_2} a(s)\alpha(-(u^-)'(s))[-(u^-)'(s)]ds = -\int_{r_1}^{r_2} b(s)f(-u^-(s))(-u^-(s))ds \le 0,$$

then

$$\alpha(-(u^-)'(r))[-(u^-)'(r)] = 0,$$

for $r \in [r_1, r_2]$. Since $\alpha'(t) > 0$ for $t > 0$, we conclude that $u^- \equiv 0$ in $[r_1, r_2]$. From relation (35) we see that if $u \ge 0$ in $[0,T]$, then $u' > 0$ in $[0,T]$. Deriving the equation in (35) we get

$$a'(r)\alpha(u'(r)) + a(r)\alpha'(u'(r))u''(r) = b(r)f(u(r))$$

on $(0,T]$, so that $u'' > 0$ on $(0,T]$, provided (12) holds. Under condition (13) we get $u'' > 0$ on $[0,T]$. Finally, we prove by contradiction the uniqueness of $u$. If $u$ and $v$ satisfy (34), then $tu'(r) + (1-t)v'(r) \ne 0$ for $t \in [0,1]$ and $r \in [0, T_0]$, hence

$$0 \le \int_0^T a \int_0^1 (\alpha'(tu' + (1-t)v'))dt \mid u' - v' \mid^2 ds =$$

$$= \int_0^T a(\alpha(u') - \alpha(v'))(u' - v')ds = -\int_0^T b(f(u) - f(v))(u - v)ds \le 0,$$

then $u \equiv v$ in $[0,T]$. $\square$

**PROOF OF THEOREM 1**. Let us begin with a remark in advance. In what follows we need a function $w$ such that

$$\begin{cases} (\alpha(w'))' + k_1\alpha(w') = k_2 f(w) \text{ in } (R/2, R) \\ w(R) = 0, w(R/2) = m > 0 \text{ and } w'(R) < 0. \end{cases}$$

The function $w$ is the translated reflection $w(r) = u(R-r)$ of the solution $u$ obtained in the previous lemma, with $u_0 = 0$, $u_T = m$, $a(r) = e^{-k_1 r}$ and

$b(r) = k_2 e^{-k_1 r}$. It is also a solution of (34) because of the oddness of $\alpha$. Note that $w'(R) = -u'(0)$ and, since $w'$ is decreasing, we have $\mid w'(r) \mid\geq u'(0)$ in $[R/2, R]$.

Let us start the proof of our result. If $u \geq \varepsilon$ in $\Omega$, we are finished. In case $u \leq \varepsilon$ for some point of $\Omega$ we proceed as follows. If $u(x_0) = 0$ for some $x_0 \in \Omega$, we construct a ball such that $B = B_R(z) \subset\subset \Omega$ and $u(y) = 0$ for some $y \in \partial B$, note that $\nabla u(y) = 0$. Picking a point $\overline{x} \in \Omega$, $\overline{x} \neq x_0$ and a ball $B_r(\overline{x})$, we may increase the radius $r$ untill finding a point $y \in \Omega$ such that $u(y) = 0$, possibly we may have $y = x_0$. Along the segment $ty + (1-t)\overline{x}$, $t \in [0, 1]$, we shrink $B_r(\overline{x})$ into $B_R(z)$ mantaining $y$ fixed. We may take $R$ small enough in order to put $B_R(z) \subset\subset \Omega$, since $B_r(\overline{x})$ may not be inside $\Omega$. Also, we may diminish $R$ so that $0 < u(x) < \varepsilon$ in $B_R(z)$. Take now the annulus $Z = \{x \in \Omega : R/2 < |x - z| < R\}$, $m = \inf\{u(x) \in I\!R :\mid x - z \mid= R/2\}(> 0)$, $k_1 \geq 2(\gamma N - 1 + C(x) \cdot (x - z))/R \geq (N - 1)/r$ for $r =\mid x - z \mid$ and $x \in \overline{Z}$, $k_2 \geq \|d\|_{L^\infty(Z)} \geq d(x)$ for $x \in \overline{Z}$. The above mentioned function $w$ satisfy $Lw \leq 0$ in $Z$, $\nabla w(y) \neq 0$ and $w \leq u$ on $\partial Z$. Arguing by comparison with Lemmas 1, 2 and 3 in each case, $w \leq u$ in $\overline{Z}$. Hence,

$$\lim_{t\to 0^+} \frac{u(y + t(z - y)) - u(y)}{t} \geq \lim_{t\to 0^+} \frac{w(R - tR) - w(R)}{t} = -Rw'(0) > 0$$

thus $\nabla u(y) \neq 0$ constitutes a contradiction. It rests to explain the inequality $Lw \leq 0$ in $Z$:

$$\mathrm{div} A(x, w, \nabla w) = \sum_{j=1}^{N} D_{\eta_j}(A_j(x, w, \nabla w)) =$$

$$= \sum_{i,j=1}^{N} D_{x_i} A_j(x, w, \nabla w)\delta_{ij} + \sum_{i,j=1}^{N} D_\mu A_j(x, w, \nabla w)\ D_{x_j} w\ \delta_{ij} + \sum_{i,j=1}^{N} D_{\eta_i} A_j(x, w, \nabla w) D^2_{x_j x_i} w \geq$$

$$\geq \sum_{i,j=1}^{N} D_{\eta_i} A_j(x, w, \nabla w) D^2_{x_j x_i} w =$$

$$= \Big((w''(r)r - w'(r))/r^3\Big) \sum_{i,j=1}^{N} D_{\eta_i} A_j(x, w, \nabla w)(x_i - z_i)(x_j - z_j) + (w'(r)/r) \sum_{i,j=1}^{N} D_{\eta_i} A_j(x, w, \nabla w)\delta_{ij} \geq$$

$$\geq \Big((w''(r)r - w'(r))/r^3\Big)\rho(\mid w'(r) \mid)r^2 + \gamma N(w'(r)/r)\rho(\mid w'(r) \mid) \geq$$

$$\geq (\rho(|\ w'(r)\ |)w'(r))' + ((\gamma N - 1)/r)\rho(|\ w'(r)\ |)w'(r).$$

Therefore

$$\mathrm{div} A(x, w, \nabla w) - \rho(|\ \nabla w\ |)\nabla w \cdot C(x) \geq$$

$$\geq (\rho(|\ w'(r)\ |)w'(r))' + ((\gamma N - 1 + C(x) \cdot (x - z))/r)\rho(|\ w'(r)\ |)w'(r) \geq$$

$$\geq (\rho(|\ w'(r)\ |)w'(r))' - k_1\rho(|\ w'(r)\ |)w'(r) =$$

$$= k_2 f(w(r)) \geq$$

$$\geq d(x)f(w(r)). \ \square$$

**PROOF OF THEOREM 2**. It is similar to the above repeating the steps with $D_{\eta_i}A_j$ replaced by $a_{ij}$, because $D_{x_j x_i}w$ is the same. The conclusion follows from Lemma 4. $\square$

**PROOF OF THEOREM 3**. Under assumption (21) on $f$, then $f(t) = 0$ for $0 \leq t \leq t_0$ where $t_0 \leq \varepsilon$ and $t_0$ is the largest number such that $f$ vanishes. By the calculations of Lemma 6, (34) has a unique solution $u \in C^2(0, T] \cap C^1[0, T]$ such that $u' > 0$ in $[0, T]$ and $u, u'' > 0$ in $(0, T]$. Indeed, let us see that $u'(0) > 0$. Since $e^{-k_1 r}\alpha(u'(r)) = \alpha(u'(0))$ for $0 \leq r \leq \overline{T}$ and some $\overline{T} \leq T$. There exists a constant $\Gamma > 0$ such that $0 \leq u(r) \leq \Gamma$ for $0 \leq r \leq T$. If $\Gamma \leq t_0$, then $f(u) = 0$ in $[0, T]$ and we make $T = \overline{T}$, so there exists $T^* \in [0, T]$ such that $u'(T^*) > 0$, then $u'(0) > 0$. If $\Gamma > t_0$, then $f(u) = 0$ in $[0, \overline{T}]$, where $u(\overline{T}) = t_0$ and $\overline{T} < T$. Since $u'(\overline{T}) > 0$, we have $u'(0) > 0$. $\square$

**PROOF OF THEOREM 4**. Let $Z$ be the annulus, as in the proof of Theorem 1, corresponding to the ball $B \subset \Omega$ of the hypothesis, such that $x_0 \in \partial B$. As before $Lw \leq 0 \leq Lu$ in $Z$ and $w \leq u$ on $\partial Z$. Note that $\nabla u(x_0) = 0$ and repeat the same procedure. $\square$

**PROOF OF THEOREM 5**. In the case $N = 1$, the function $M(\tau) = \int_0^\tau [\chi^{-1}(F(s))]^{-1} ds$ defined in $[0, \delta]$ is invertible, then $v(r) = M^{-1}(r)$ is a $C^1$ solution of

$$\begin{cases} -(\alpha(v'(r)))' + f(v(r)) = 0 \\ v(0) = v'(0) = 0 \end{cases} \tag{36}$$

defined in $[0, I]$, where $I = \displaystyle\int_0^\delta [\chi^{-1}(F(s))]^{-1} ds$. Furthermore, $v, v' > 0$ in $(0, I]$. Take $\theta \in (0, I)$ and set the translate reflection $w_\theta(t) = v(-t+\theta)$ for $t \in [\theta - I, \theta]$ and $w_\theta(t) = 0$ for $t \geq \theta$. Thus $w_\theta$ satisfies the equation in (36) in $[0, \infty)$, is a $C^1$ function and $C^2$ except possibly in $t = \theta$. Moreover $w_\theta$ is strictly increasing for $0 < t < \theta$. Given $x_0 \in \partial\Omega$ for all $0 < \theta < I$ such that $\theta \leq R$ the function $u(x) = w_\theta(\mid x - x_0 \mid)$ is a solution of the equation in (36) in $I\!R \setminus \{x_0\}$ which vanishes iff $\mid x - x_0 \mid \geq \theta$.

Whenever $N \geq 2$, if $x_0 \in \partial(I\!R^N \setminus \overline{\Omega}) \subset \partial\Omega$ and $R > 0$, there exists $x_1 \notin \Omega$ such that $0 < \rho = \mathrm{dist}(x_1, \Omega) < R/2$. The problem is solved if we construct a function $u \geq 0$ satisfying $Lu = 0$ in $I\!R^N \setminus B_\rho(x_1)$ and such that $u(x) = 0$ iff $\mid x - x_1 \mid \geq b$ for some $b \in (\rho, R/2)$. By a similar reasoning of Lemma 6, using Degree Theory, the two-point boundary value problem

$$\begin{cases} -(r^{N-1}\alpha(z'(r)))' + r^{N-1} f(z(r)) = 0 \text{ in } (\rho, \sigma) \\ z(\rho) = w_\sigma(\rho), z(\sigma) = 0 \end{cases} \tag{37}$$

where $\sigma = \min(\rho + I/2, R/2)$, has a unique solution $z \in C^2[\rho, \sigma) \cap C^1[\rho, \sigma]$ and $z \geq 0$ in $[\rho, \sigma]$. Let $\overline{z}(r) = w_\sigma(r)$. Since $-(r^{N-1}\alpha(\overline{z}'(r)))' + r^{N-1} f(\overline{z}(r)) \geq 0$ for $\rho < r < \sigma$ and $\overline{z}(\rho) = z(\rho), \overline{z}(\sigma) = z(\sigma)$ and by comparison $z(r) \leq \overline{z}(r)$ for $\rho < r < \sigma$. Since $\overline{z}'(\sigma) = 0$ we have $z'(\sigma_-) = 0$ and $z$ can be extended by 0 for $r \geq \sigma$ to a solution in $(\rho, \infty)$. Now we set $u(x) = z(\mid x - x_1 \mid)$ for $x \in \Omega$ and $b = \sup\{r \in [\rho, +\infty) : z(r) > 0\} (\leq \sigma)$, establishing the result. $\square$

**PROOF OF THEOREM 6**. There exists a neighbourhood $V_{x_0}$ of $x_0$ such that $\langle Hu(x)w, w \rangle < 0$ for every $w \in I\!R^N - 0$ and $x \in \Omega$, where $Hu$ is the Hessian matrix of $u$. Furthermore, if $u(x) \neq u(x_0)$, then there exists a point $z \in \Omega$ and a neighbourhood $V_z$ of $z$ such that $\nabla u(x) \neq 0$, $\Delta u(x) < 0$ and $\langle Hu(x)w, w \rangle < 0$ for every $w \in I\!R^N - 0$ and $x \in V_z$. Our claim is true, because if for every $x \in \Omega$ we have $\nabla u = 0$ or $\Delta u(x) \geq 0$ or $\langle Hu(x)w, w \rangle \geq 0$ for every $w \in I\!R^N$. These three cases imply $\Delta u(x) \geq 0$ in $\Omega$. the maximum principle for subharmonic functions asserts that $u(x) = u(x_0)$, a contradiction.

An easy computation furnishes

$$Tu = \sum_{j=1}^{N} D_{x_j}(f_j(\mid \nabla u \mid) D_{x_j} u) =$$

$$= \sum_{j=1}^{N} \mid \nabla u \mid^{-1} f_j'(\mid \nabla u \mid) \Big( \sum_{i=1}^{N} D_{x_i} u D_{x_i x_j} u \Big) D_{x_j} u + \sum_{j=1}^{N} f_j(\mid \nabla u \mid) D_{x_j x_j} u =$$

$$= \mid \nabla u \mid^{-1} \langle Hu \nabla u, Au \nabla u \rangle + \sum_{j=1}^{N} f_j(\mid \nabla u \mid) D_{x_j x_j} u =$$

$$= \mid \nabla u \mid^{-1} \langle AuHu \nabla u, \nabla u \rangle + \sum_{j=1}^{N} f_1'(\mid \nabla u \mid),$$

where $Au$ is the diagonal matrix $\mathrm{diag}(f_1'(\mid \nabla u \mid), ..., f_N'(\mid \nabla u \mid))$. Suppose on the contray that $u(x) \not\equiv u(x_0)$. There is a point $z \in \Omega$ such that $f_j(\mid \nabla u(z) \mid) D_{x_j x_j} u(z) < 0$, for $j = 1, ..., N$. Assume first $Au(x)$ is positive semidefinite for every $x \in \Omega$. Since $Hu(z)$ is negative definite, $Au(z)Hu(z)$ is negative semidefinite. Hence $\langle Au(z)Hu(z)\nabla u(z), \nabla u(z) \rangle < 0$, thus $Lu(z) < 0$, a contradiction. Otherwise, if $Au(x)$ is negative semidefinite for every $x \in \Omega$, then $\mid \nabla u \mid^{-1} \langle Au(z)Hu(z)\nabla u(z), \nabla u(z) \rangle \leq -\|Au(z)\| \Delta u(z) \mid \nabla u(z) \mid^2$, because $\|Hu\| \leq -$ sum of eigenvalues $= -\mathrm{trace}(Hu(z)) = -\Delta u(z)$. Hence

$$Lu(z) < [\min f_i(\mid \nabla u(z) \mid) - \max(-f_j'(\mid \nabla u(z) \mid)) \mid \nabla u(z) \mid] \Delta u(z) =$$

$$= [\min f_i(\mid \nabla u(z) \mid) + \min(f_j'(\mid \nabla u(z) \mid)) \mid \nabla u(z) \mid] \Delta u(z) \leq$$

$$= \min(f_j(\mid \nabla u(z) \mid) \mid \nabla u(z) \mid)' \Delta u(z) \leq 0,$$

again a contradiction. $\square$

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