On the Prasad metric and the topology of the de Sitter universe

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Abstract

We present the Laplace partial differential equation associated to the Prasad metric. We discuss the so called internal and external spaces that correspond to the symmetry groups $SO_{3,2}$ and $SO_{4,1}$, respectively. Using hyperspherical coordinates we show that both radial differential equations can be led to the Riccati ordinary differential equation.

For the Prasad metric with the radius of the universe independent of the parametrization, we associate the solution of the temporal equation with quantum number hypercharge to the internal structure, and for the external structure, we associate the energy eigenvalues.

Finally, when the radius of the universe goes to infinity we reobtain Minkowskian results.

1 Introduction

Translations in a three-dimensional space can be described as rotations associated with the group $SO_{3,1}$. Lorentz postulated a particular transformation, the standard relativistic one[1].

Arcidiacono [2,3,4] proposed a natural extension of these transformations, where translations in a Minkowskian spacetime can be thought as rotations in the symmetry group $SO_{4,1}$, restricted to the de Sitter universe $\mathcal{D}$. We can divide $\mathcal{D}$ in internal and external hyperspaces, associating with each one, the symmetry groups $SO_{4,1}$ and $SO_{3,2}$, respectively[5]. After parametrizing these hyperspaces, we solve the generalized Laplace equation, concluding that the angular part has the spherical harmonics as solutions and the radial equation can be led to the classical field equation, when the radius of $\mathcal{D}$ approaches infinity. By a suitable change of variables this radial equation can be led to a Riccati equation [6].

This paper is organized as follows: in section two we present the Laplace differential equation for the Prasad metric, by the parametrization of the internal and external hyperspaces associated with $\mathcal{D}$, solving the angular differential equation in terms of spherical harmonics and reducing the radial differential field equation to a Riccati equation. In section three we discuss the generalized Laplace differential equation associated with another parametrization of $\mathcal{D}$, proving the same behavior for the angular differential equation as in the first case and obtaining the classical differential field equation when the radius of $\mathcal{D}$ approaches infinity. In an appendix we discuss the topology of $\mathcal{D}$.

2 The Prasad metric

The de Sitter Universe ($\mathcal{D}$) can be described as the sum $\wp_1 + \wp_2$ where $\wp_1$ is the external space and $\wp_2$ is the internal one. To each of these spaces there correspond the symmetry groups $SO_{4,1}$ and $SO_{3,2}$ respectively[7].

2.1 The external space ($\wp_1$)

This hyperspace has constant curvature $1/R^2$ and has a quadratic form associated with it, as follows:

$$R^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2 + (x_5)^2.$$

A possible parametrization[6] for the above quadratic form is

$$\begin{aligned} x_1 &= R\sin\chi\sin\theta\cos\varphi\cosh t \\ x_2 &= R\sin\chi\sin\theta\sin\varphi\cosh t \\ x_3 &= R\sin\chi\cos\theta\cosh t \\ x_4 &= R\sinh t \\ x_5 &= R\cos\chi\cosh t \end{aligned}$$

allowing us to write the line element as

$$ds_+^2 = R^2\cosh^2 t[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)] - R^2 dt^2 \tag{1}$$

The metric tensor is given by

$$g_{ij} = R^2 \begin{pmatrix} \cosh^2 t & 0 & 0 & 0 \\ 0 & \sin^2\chi\cosh^2 t & 0 & 0 \\ 0 & 0 & \sin^2\chi\sin^2\theta\cosh^2 t & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

After substituting these terms in the generalized laplacian[7]

$$\nabla^2\psi = \nabla\cdot\nabla\psi = \frac{1}{g^{1/2}}\frac{\partial}{\partial y^k}\left[g^{1/2}g^{ik}\frac{\partial\psi}{\partial y^i}\right] \tag{2}$$

we obtain the Laplace differential equation

$$(R^2\cosh^2 t\sin^2\chi)\nabla^2\phi = \frac{\sin^2\chi}{\cosh t}\partial_t(\cosh^3 t\partial_t\phi) - \partial_t(\sin^2\chi\partial_\chi\phi) +$$

$$+\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{\sin^2\theta}\partial_{\varphi\varphi}\phi = 0. \tag{3}$$

After writing the scalar field ϕ as

$$\phi(\chi, t, \theta, \varphi) = \Gamma(\chi, t)Y(\theta, \varphi)$$

we can separate the Laplace differential equation in two equations:

$$\frac{1}{Y}\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y) + \frac{1}{Y}\frac{1}{\sin^2\theta}\partial_{\varphi\varphi}Y = -\ell(\ell+1) \tag{4}$$

and

$$\frac{1}{\cosh t}\partial_t(\cosh^3 t\partial_t\Gamma) - \frac{1}{\sin^2\chi}\partial_\chi(\sin^2\chi\partial_\chi\Gamma) - \frac{\ell(\ell+1)}{\sin^2\chi}\Gamma = 0, \tag{5}$$

with $\ell = 0, 1, 2, 3\ldots$

Relatively to eq.(5), we consider the functional product

$$\Gamma(\chi, t) = \Pi(\chi)\Omega(t)$$

and suppose that the separation constant[1] is α^2. Then, eq.(5) can be separated in two equations:

$$\frac{1}{\sin^2\chi}\frac{d}{d\chi}\left(\sin^2\chi\frac{d\Pi}{d\chi}\right) + \frac{\ell(\ell+1)}{\sin^2\chi}\Pi - \alpha^2\Pi = 0 \tag{6}$$

and

$$\frac{1}{\cosh t}\frac{d}{dt}\left(\cosh^3 t\frac{d\Omega}{dt}\right) - \alpha^2\Omega = 0. \tag{7}$$

We emphasize eq.(5), namely, the radial one.[2] We can rewrite eq.(6) as

$$\frac{1}{\sin^2\chi}\left[2\sin\chi\cos\chi\frac{d\Pi}{d\chi} + \sin^2\chi\frac{d^2\Pi}{d\chi^2} + \ell(\ell+1)\Pi - \alpha^2\Pi\sin^2\chi\right] = 0$$

or

$$\frac{d^2\Pi}{d\chi^2} + 2\cot\chi\frac{d\Pi}{d\chi} + \frac{\ell(\ell+1)}{\sin^2\chi}\Pi - \alpha^2\Pi = 0. \tag{8}$$

Introducing the variable $j \equiv \cos\chi$, we obtain[8]

$$\frac{d}{d\chi} = \frac{d}{dj}\frac{dj}{d\chi} = -\frac{d}{dj}\sin\chi \Rightarrow 2\cot\chi\frac{d\Pi}{d\chi} = -2\cos\chi\frac{d\Pi}{dj}$$

and

$$\frac{d^2}{d\chi^2} = \frac{d}{d\chi}\left(-\frac{d}{dj}\sin\chi\right) = -\frac{d}{dj}\frac{dj}{d\chi}\frac{d}{dj}\sin\chi - \frac{d}{dj}\cos\chi = \frac{d^2}{dj^2}(\sin^2\chi) - \frac{d}{dj}\cos\chi.$$

Thus, eq.(8) can be written, with this substitution, as

$$\frac{d^2\Pi}{dj^2}(1-j^2) - 3j\frac{d\Pi}{dj} - \alpha^2\Pi + \frac{\ell(\ell+1)}{1-j^2}\Pi = 0 \tag{9}$$

[1] $\alpha = 0, 1, 2, 3, \ldots$ corresponds to the eingenvalue spectrum of Gegenbauer equation.

[2] The variable χ is defined by $\sin\chi = \dfrac{(x_1)^2 + (x_2)^2 + (x_3)^2}{R^2}$. Although sin$\chi$ is not the radius of $\mathcal{D}$, it is the so-called *adimensional normalized radius.*

Besides, introducing a new variable μ as follows

$$\frac{1}{\Pi}\frac{d\Pi(j)}{dj} \equiv \mu(j),$$

we can rewrite eq.(9) as

$$\mu' = -\mu^2 + \frac{1}{1-j^2}\left[3j\mu + \alpha^2 - \frac{\ell(\ell+1)}{1-j^2}\right] \tag{10}$$

which is a Riccati equation.

2.2 The internal space ($\wp_2$)

This hyperspace has a negative constant curvature $-1/r^2$ (see appendix) and can be characterized by the symmetry group $SO_{3,2}$. To this group is associated a quadratic form

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_6^2 = r^2.$$

Prasad[6] proposed the following parametrization with nonstatic coordinates:

$$\begin{aligned}
x_1 &= r\sinh\chi\sin\theta\cos\varphi\cos t\\
x_2 &= r\sinh\chi\sin\theta\sin\varphi\cos t\\
x_3 &= r\sinh\chi\cos\theta\cos t\\
x_4 &= r\sin t\\
x_6 &= r\cosh\chi\cos t
\end{aligned}$$

Then, we can write the line element as follows:

$$ds_-^2 = r^2\cos^2 t[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)] - r^2dt^2. \tag{11}$$

The metric tensor associated with $\wp_2$ is given by

$$g_{ij} = r^2\begin{pmatrix} \cos^2 t & 0 & 0 & 0\\ 0 & \cos^2 tsh^2\chi & 0 & 0\\ 0 & 0 & \cos^2 tsh^2\chi\sin^2\theta & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

The generalized Laplace differential equation can be written as

$$r^2\cos^2 t\sinh^2\chi\nabla^2\phi = -\frac{\sin^2\chi}{\cos t}\partial_t(\cos^3 t\partial_t\phi) + \partial_\chi(\sinh^2\chi\partial_\chi\phi) +$$

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{\sin^2\theta}\partial_{\varphi\varphi} = 0 \tag{12}$$

Defining the funcional product

$$\phi(\rho,t,\theta,\varphi) = \Lambda(\rho,t)Y(\theta,\varphi) \equiv \Lambda Y,$$

we can separate eq.(12) in radial (temporal) and angular equations:

$$-\frac{1}{\Lambda}\frac{\sinh^2\chi}{\cos t}\partial_t(\cos^3 t\partial_t\Lambda) + \frac{1}{\Lambda}\partial_\chi(\sinh^2\chi\partial_\chi\phi) = \ell(\ell+1) \tag{13}$$

and

$$\frac{1}{Y}\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{Y}\frac{1}{\sin^2\theta}\partial_{\varphi\varphi}\phi = -\ell(\ell+1). \tag{14}$$

It is also worthwhile to mention here that eq.(14) is exactly the angular equation obtained in the precedent case (the internal space $\wp_1$). Once again we separate eq.(13) in radial and temporal equations (we will emphasize the radial equation only).

Taking $\Lambda(\chi,t) = \Xi(\chi)\Psi(t)$ we get

$$\frac{1}{\Psi}\frac{\sin^2\chi}{\cos t}\partial_t(\cos^3 t\partial_t\Psi) + \frac{1}{\Xi}\partial_\chi(\sinh^2\chi\partial_\chi\Xi) = \ell(\ell+1)$$

The above equation can be separated, and using β^2 as the separation constant we obtain[3] the following ordinary differential equations:

$$\frac{1}{\Psi}\frac{1}{\cos t}\frac{d}{dt}\left(\cos^3 t\frac{d\Psi}{dt}\right) = \beta^2 \tag{15}$$

and

$$\frac{1}{\Xi}\frac{1}{\sinh^2\chi}\frac{d}{d\chi}\left(\sinh^2\chi\frac{d\Xi}{d\chi}\right) - \frac{\ell(\ell+1)}{\sinh^2\chi} = -\beta^2. \tag{16}$$

We can write eq.(16) as

$$\frac{d^2\Xi}{d\chi^2} + 2\coth\chi\frac{d\Xi}{d\chi} - \left[\frac{\ell(\ell+1)}{\sinh^2\chi} - \beta^2\right]\Xi = 0. \tag{17}$$

Let $m = \cosh\chi$. With this substitution, eq.(17) can be written as[9]

$$(m^2-1)\frac{d^2\Xi}{dm^2} + 3m\frac{d\Xi}{dm} - \left[\frac{\ell(\ell+1)}{m^2-1} - \beta^2\right]\Xi = 0. \tag{18}$$

[3] $\beta = 0, 1, 2, 3, \ldots$ corresponds to the eigenvalue spectrum of Gegenbauer equation.

Defining

$$\mu(m) = \frac{1}{\Xi}\frac{d\Xi(m)}{dm}$$

we can write eq.(18) as

$$\mu' = -\mu^2 + \frac{1}{1-m^2}\left[3m\mu + \beta^2 - \frac{\ell(\ell+1)}{1-m^2}\right], \tag{19}$$

which is also a Riccati equation.

3 The Prasad metric with radius of the Universe independent of the parametrization

In the former case, with the hyperbolic parametrization of $\mathcal{D}$, the classical radial field equations were not obtained in the limit $r \to \infty$. With a new temporal parametrization we are able to obtain the classical case in this limit.

The group that characterizes the external space $\mathcal{D}_+$ is $SO_{4,1}$, while the internal space $\mathcal{D}_-$ is only affected by the symmetry group $SO_{3,2}$. Using cartesian coordinates, we can express[10] the internal and external line elements,

$$ds_+^2 = d{x_1}^2 + d{x_2}^2 + d{x_3}^2 - d{x_4}^2 + d{x_5}^2,$$

$$ds_-^2 = d{x_1}^2 + d{x_2}^2 + d{x_3}^2 - d{x_4}^2 - d{x_6}^2.$$

From the transformations proposed by Tolman[11]

$$x_5 \pm x_4 = R_+ \exp[\pm t/R_+](1 - r^2/R_+^2)^{1/2}$$

for $\mathcal{D}_+$ and

$$x_4 \pm i x_6 = R_- \exp[\pm it/R_-](1 + r^2/R_-^2)^{1/2}$$

for $\mathcal{D}_-$, we can substitute these expressions in the expressions for the line elements, obtaining

$$ds_\pm^2 = \frac{dr^2}{1 \mp r^2/{R_\pm}^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 - (1 \mp r^2/R_\pm^2)dt^2$$

where the metric tensor can be written as

$$g_{ij}^{\mp} = \begin{pmatrix} (1 \mp r^2/R_{\pm}^2)^{-1} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -(1 \mp r^2/R_{\pm}^2) \end{pmatrix}.$$

For the generalized Laplace differential equation

$$\nabla^2\psi = \nabla \cdot \nabla\psi = \frac{1}{g^{1/2}} \frac{\partial}{\partial y^k} \left[g^{1/2} g^{ik} \frac{\partial \psi}{\partial y^i} \right] \tag{20}$$

with this metric tensor, we obtain

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 (1 \mp \Lambda_{\pm} r^2) \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) +$$

$$\frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \varphi^2} - \frac{1}{1 \mp \Lambda_{\pm} r^2} \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{21}$$

where $\Lambda_{\pm} \equiv 1/R_{\pm}^2$.

Let $\psi(r, t, \theta, \varphi) = Y(\theta, \varphi)\Gamma(r, t)$. Then we can separate eq.(21) as

$$\frac{1}{\Gamma} \frac{\partial}{\partial r} \left[r^2 (1 \mp \Lambda_{\pm} r^2) \frac{\partial \Gamma}{\partial r} \right] - \frac{1}{\Gamma} \frac{r^2}{(1 \mp \Lambda_{\pm} r^2)} \frac{\partial^2 \Gamma}{\partial t^2} = \ell(\ell + 1) \tag{22}$$

for the non-angular equation and

$$\frac{1}{Y} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{Y} \frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial \varphi^2} = -\ell(\ell + 1) \tag{23}$$

for the angular one. If the separation constant in the equations above has integer values, the angular equation has the spherical harmonics as solutions, given by $Y(\theta, \varphi) = P_l^m(\cos\theta) e^{\pm im\varphi}$, where $P_l^m(\cos\theta)$ is the *associated Legendre polynomial*[7,12].

If we put $\Gamma(r, t) = S(r)T(t)$, we obtain from eq.(22)

$$\frac{1}{T} \frac{\partial^2 T}{\partial t^2} = -B.$$

If we are on the internal structure of $(\mathcal{D}_-)$, the temporal equation above has an oscillatory solution

$$T_-(t) = \tau_- \exp(\pm iHt/R_-),$$

associated with the quantum number *hypercharge* (H)[13]. In case we are treating $\mathcal{D}_+$, time will have an exponential character described by the equation associated with the energy eigenvalues (E)

$$T_+(t) = \tau_+ \exp(\pm Et/R_+)$$

For the radial equation we can write

$$\frac{1 \mp \Lambda_\pm r^2}{r^2} \frac{d}{dr} \left[r^2 (1 \mp \Lambda_\pm r^2) \frac{dS}{dr} \right] - \frac{\ell(\ell+1)}{r^2} (1 \mp \Lambda_\pm r^2) S = BS. \tag{24}$$

This equation can be transformed in a classical field equation (for example, the electric or the gravitational fields in $\mathcal{D}$). From the definition $\Lambda_\pm = 1/R_\pm^2$,it is obvious that in the limit where $R \to \infty$, $\Lambda_\pm \to 0$. Using some results of Quantum Mechanics in hyperspherical universes[15] in $\mathcal{D}_+$, with $B = Y^2/R_-^2$, and in $\mathcal{D}_-$,with $B = E^2/R_+^2$, in the limit where $R_\pm \to \infty$ we obtain $B \to 0$. Therefore the radial field equation can be written as

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = 0 \tag{25}$$

in the ground state, where $\ell = 0$.

With the change of variables

$$\frac{1}{R} \frac{dR}{d\rho} = \eta \quad \Rightarrow \quad \frac{dR}{d\rho} = \eta R,$$

we can transform eq.(24) in a Riccati equation:

$$\eta' = -\eta^2 - \eta \left(\frac{2}{r} \mp \frac{2\Lambda_\pm r}{1 \mp \Lambda_\pm r^2} \right) + \frac{\ell(\ell+1)}{r^2(1 \mp \Lambda_\pm r^2)} + \frac{B}{1 \mp \Lambda_\pm r^2}. \tag{26}$$

4 Conclusions

The radial differential field equations have different signs, but both are led to the same Riccati equation. In the first case, with the parametrization, the classical case of a spherically symmetric scalar field cannot be obtained by making the radius of $\mathcal{D}$ approach infinity. This was expected because in cartesian coordinates the radius of $\mathcal{D}$ is independent of the coordinates, while in curvilinear ones the parametrization constrains the radius of $\mathcal{D}$: it is the radial parametric coordinate. Thus the information about the radius is lost, because the Laplace differential equation is homogeneous. In the last treatment, with the parametrization independent of the radius of the Universe, the classical case is obtained. Finally, by a suitable change of variables, all radial differential field equations can be led to a Riccati equation.

We have obtained the following equations:

$$\frac{d^2\Pi}{d\chi^2} + 2\cot\chi\frac{d\Pi}{d\chi} + \frac{\ell(\ell+1)}{\sin^2\chi}\Pi - \alpha^2\Pi = 0, \tag{27}$$

$$\mu' = -\mu^2 + \frac{1}{1-j^2}\left[3j\mu + \alpha^2 - \frac{\ell(\ell+1)}{1-j^2}\right], \tag{28}$$

for the internal Prasad metric, and

$$\frac{d^2\Xi}{d\chi^2} + 2\coth\chi\frac{d\Xi}{d\chi} - \left[\frac{\ell(\ell+1)}{\sinh^2\chi} - \beta^2\right]\Xi = 0, \tag{29}$$

$$\mu' = -\mu^2 + \frac{1}{1-m^2}\left[3m\mu + \beta^2 - \frac{\ell(\ell+1)}{1-m^2}\right], \tag{30}$$

for the external Prasad metric

We note that the two Riccati equations arise from different radial differential equations, but are the same equation, with different names of variables.

We have already obtained (eq.(24))

$$\frac{1\mp\Lambda_\pm r^2}{r^2}\frac{d}{dr}\left[r^2(1\mp\Lambda_\pm r^2)\frac{dS}{dr}\right] - \left[\frac{\ell(\ell+1)}{r^2}(1\mp\Lambda_\pm r^2) + B\right]S = 0, \tag{31}$$

$$\eta' = -\eta^2 - \eta\left(\frac{2}{r} \mp \frac{2\Lambda_\pm r}{1\mp\Lambda_\pm r^2}\right) + \frac{\ell(\ell+1)}{r^2(1\mp\Lambda_\pm r^2)} + \frac{B}{1\mp\Lambda_\pm r^2}, \tag{32}$$

for the Prasad metric with parametrization independent of the radius of $\mathcal{D}$.

Appendix

The topology $S^3 \times \mathcal{R}$ of the de Sitter universe

Definition 1: A *differentiable manifold* of dimension n is a set M and a family of bijective applications $x_a : V_a \subset \mathcal{R}^n \to M$ of open sets V_a of $\mathcal{R}^n$ in M such that:

(1) $\bigcup_a x_a(V_a) = M$.

(2) For all pairs a, b, with $x_a(V_a) \cap x_b(V_b) = W \neq \emptyset$, the sets $x_a^{-1}(W)$ e $x_b^{-1}(W)$ are open in $\mathcal{R}^n$ and the applications $x_b^{-1} \circ x_a$ are differentiable.

The applications x_a for $p \in x_a(U_a)$ are called *parametrizations* of M in p. A family $\{(V_a, x_a)\}$ satisfying (1) and (2) is called a *differentiable structure* on M [14].

Definition 2: Let M_1 and M_2 be differentiable manifolds. An application $\varphi : M_1 \to M_2$ is *differentiable* in $p \in M_1$ if, given a parametrization $y : Z \subset \mathcal{R}^m \to M_2$ in $\varphi(p)$, there exists a parametrization $x : V \subset \mathcal{R}^n \to M_1$ in p such that $\varphi(x(V)) \subset y(Z)$ and the application $y^{-1} \circ \varphi \circ x : V \subset \mathcal{R}^n \to \mathcal{R}^m$ is differentiable in $x^{-1}(p)$.

Definition 3: Let M_1 and M_2 be differentiable manifolds. An application $\psi : M_1 \to M_2$ is a *diffeomorphism* if it is differentiable, bijective, onto and its inverse ψ^{-1} is differentiable.

Definition 4: Let $f : \mathcal{R}^{k+1} \to \mathcal{R}$ be an application of class C^∞ in all its components. A point $p \in \mathcal{R}^{k+1}$ is called *regular point* of f if at least one of the applications $\partial f / \partial x^i$, with $i = 1, 2, \ldots k+1$, is non-null. A real number r is called a *regular value* of f if $f^{-1}(r)$ consists of regular points.

The de Sitter Universe is identified as a manifold with a level hypersurface in $\mathcal{R}^5$. Given two vectors $x, y \in \mathcal{D}$, then the inner product is defined by

$$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 - x_5 y_5,$$

and consequently there is a quadratic form associated with it,

$$Q(x) = (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 - (x_5)^2,$$

whence we have

$$\frac{\partial Q}{\partial x_i} = 2x_i \qquad \text{and} \qquad \frac{\partial Q}{\partial x_5} = -2x_5.$$

Therefore $r = R$ (the non-null radius of the pseudosphere) is a regular value of Q.

We enunciate the following lemma:

Lemma 1: If r is a regular value of $f : \mathcal{R}^{k+1} \to \mathcal{R}$, then $f^{-1}(r)$ is an empty set or a k-dimensional manifold[15].

By lemma 1, the set

$$Q^{-1}(R^2) = \{x \in \mathcal{R}^5 : (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 - (x_5)^2 = R^2\}$$

is a 4-dimensional manifold of $\mathcal{R}^5$. $\mathcal{D} = Q^{-1}(R^2)$ is a one leaf hyperboloid in $\mathcal{R}^5$. Parallel sections to the hyperplan $\{x_1, x_2, x_3, x_4\}$ are *discs* which represent hyperspheres $\mathcal{S}^3$. Thus, we have the theorem[16]:

Theorem 1: The de Sitter spacetime is diffeomorphic to $S^3 \times \mathcal{R}$.

Proof: Consider the set

$$\mathcal{S}^3 \times \mathcal{R} = \{(y_1, y_2, y_3, y_4, t) : y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1, \; -\infty < t < \infty\}$$

(In this definition we consider the radius of $\mathcal{D}$ equal to unit, without loss of generality, which is rigorously true only in the case $t = 0$.) We define a map $F : S^3 \times \mathcal{R} \to \mathcal{R}^5$:

$$F(y_1, y_2, y_3, y_4, t) = [(1 + t^2)^{1/2}] \left(y_1, y_2, y_3, y_4, \frac{t}{(1 + t^2)^{1/2}} \right),$$

whence we obtain the derivatives

$$\frac{\partial F}{\partial y_i} = (1 + t^2)^{1/2}(1, 1, 1, 1, 0) \quad \text{and} \quad \frac{\partial F}{\partial t} = \frac{t}{(1 + t^2)^{1/2}}(y_1, y_2, y_3, y_4, 1).$$

Therefore, F is a smooth function. F is also one-to-one, since if we consider $F(y) = F(y_1, y_2, y_3, y_4, t_1)$ and $F(x) = F(x_1, x_2, x_3, x_4, t_2)$, then for $F(y) = F(x)$, we must have $t_1 = t_2$ and then $x = y$.

We know that

$$Q(F) = (1 + t^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) - t^2 = (1 + t^2) \cdot 1 - t^2 = 1$$

Thus F is a map on $\mathcal{D}$.

Now, we consider $G : \mathcal{D} \to S^3 \times \mathcal{R}$ defined by

$$G(x_1, x_2, x_3, x_4, x_5) = (1 + x_5^2)^{-1/2} \left[x_1, x_2, x_3, x_4, x_5(1 + x_5^2)^{1/2} \right]$$

where G is smooth. Besides, $G[F(y_1, y_2, y_3, y_4, t)] = (y_1, y_2, y_3, y_4, t)$. This proves that $G = F^{-1}$; then F is a diffeomorphism between $\mathcal{D}$ and $S^3 \times \mathcal{R}$.

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