Some Nonlinear Perturbation of the Quasilinear hyperbolic Equation*

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Abstract

In this paper we prove existence and uniqueness of global regular solutions of a mixed problem for the perturbed of extensional vibrations of a thin rod equation or torsional vibrations of thin rod.

1 Introduction

Let us consider a Lipschitzian perturbation of vibrations of a thin rod operator (see. Love [1])

$$Ku = \frac{\partial^2 u}{\partial t^2} - \Delta u - M\left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u_{tt},$$

given by $Ku + F(u)$, where F is a Lipschitzian function which satisfies some appropriate conditions fixed in Section 2 and $M \in C^1([0,+\infty))$. We obtain the nonlinear mixed problem:

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} - \Delta u - M\left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u_{tt} + F(u) = f \text{ in } Q, \\ u = 0 \text{ on } \Sigma, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega. \end{gathered} \tag{1}$$

In (1) Ω is a bounded open set of R^n with smooth boundary Γ, Q is the cylinder $\Omega \times (0,T)$, T is a positive real number. The lateral boundary of Q

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is represented by Σ, i.e. $\Sigma = \Gamma \times (0,T)$. By $M(\lambda)$ we denote a real function defined on the positive real numbers with $M(\lambda) \geq \rho > 0$ for all $\lambda \geq 0$, for some $\rho > 0$; ∇u is the gradient of u and Δ is the Laplace operator.

Solving equation (1) is important by the following reasons: equation (1) with $M(\lambda) = 1$ and $F(u) = 0$ arises in the study of the extensional vibrations of thin rods; many authors studied this equation when $M(\lambda) = 1$; among them we mention Pereira [12] and Ferreira and Pereira [13]. Besides, equation (1) with $F(u) = 0$, $M(\lambda) = \lambda_0$, $\lambda_0 = \int_\Omega \phi^2 dx$, where ϕ is the torsion-function, also appears in the study of the torsional vibrations of thin rods (see. Love [1]). Finally, the function $M(\lambda)$ in (1) has its motivation in the mathematical description of the vibrations of an elastic stretched string

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2 \, dx\right) \Delta u = 0, \tag{2}$$

which for $M(\lambda) \geq \rho > 0$, was studied by Pohozaev [16], Nishihara [15] and Lions [11]. When $M(\lambda) \geq 0$ was treated by Arosio and Spagnolo [8], Ebihara, Medeiros and Miranda [4], Yamada [6], Matos [14].When $M(\lambda)$ is a constant C^2, (2) becomes a perturbation of the d'Alembert operator

$$\Box u = \frac{\partial^2 u}{\partial t^2} - C^2 \Delta u$$

(see Lions [11]). In [2], Strauss studied a nonlinear perturbation of this operator of the type:

$$\left| \begin{array}{c} \Box u + F(u) = f \text{ in } Q \\ u = 0 \text{ on } \Sigma \\ u(0) = u_0, \ u'(0) = u_1 \text{ in } \Omega \end{array} \right. \tag{3}$$

where $F : R \to R$ is continuous and $sF(s) \geq 0$ for all $s \in R$.

For others perturbations of Kirchhoff-Carrier operator, see Jörgens [9], Hosoya and Yamada [5], D'Ancona and Spagnolo [7].

In Ebihara and Pereira [3] it was proved that there exists only one classical solution for a quasilinear model, given by following initial-boundary value problem:

$$\left| \begin{array}{c} \frac{\partial^2 u}{\partial t^2} - \Delta u - M\left(\int_\Omega |\nabla u(x,t)|^2 \, dx\right) \Delta u_{tt} = f \text{ in } Q \\ u = 0 \text{ on } \Sigma \\ u(0) = u_0, \ u'(0) = u_1 \text{ in } \Omega \end{array} \right. \tag{4}$$

when the following hypotheses hold:

(H.O) $M(\lambda) \in C^1([0,+\infty))$, and there exist positive constants α, ρ such that

$$M(\lambda) \geq \alpha\sqrt{\lambda} + \rho, \text{ for all } \lambda \in [0,+\infty).$$

(H.1) There exists a nonnegative function $\beta(\lambda)$ satisfying:

$$\left|\frac{d}{d\lambda}M(\lambda)\right|\sqrt{\lambda} \leq \beta(\lambda)M(\lambda), \text{ for all } \lambda \in [0,+\infty).$$

(H.2) The initial conditions are such that:

$$u_0, u_1 \in D(A^l), \ l \geq 2$$

and

$$f, \frac{df}{dt} \in C(0,T; D(A^{\frac{l-1}{2}})), \ l \geq 2$$

where $A = -\Delta$ and by $D(A^s)$ we are denoting the domain of the operator A^s.

Motivated by Hosoya and Yamada [5], Strauss [2] and Ebihara and Pereira [3], we shall investigate in this paper the following perturbation of the vibrations model,

$$\left| \begin{array}{c} Ku + F(u) = f \text{ in } Q \\ u = 0 \text{ on } \Sigma \\ u(0) = u_0, \ u_1(0) = u_1 \text{ in } \Omega \end{array} \right. \tag{5}$$

with F Lipschitzian.

Our main objective is to give a complete, clear and short proof for the existence and uniqueness of global regular solutions to the problem in question. For that, we use the Faedo-Galerkin's method associated with a compactness argument and some technical ideas. In our proof of the existence and uniqueness of global regular solutions to (5) we assume **(H.0)** and substitute **(H.2)** by $f \in C([0,T]; H_0^1(\Omega))$ and $u_0, u_1 \in D(A^{1/2})$ and **(H.1)** becomes unnecessary.

2 Notations and Main Result

For the Hilbert space $L^2(\Omega)$ we denote its inner product and norm, respectively, by $(\ ,\ )$ and $|.|$, defined by:

$$(u,v) = \int_\Omega uv dx \ ; \ \ |u|^2 = \int_\Omega |u|^2 dx.$$

By $H^m(\Omega)$ we represent the Sobolev space on the bounded open set Ω of R^n. $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. In particular, $H^1(\Omega)$ has inner product $((\ ,\ ))$ and norm $\|.\|$ given by

$$((u,v)) = \int_\Omega uv dx + \int_\Omega \nabla u . \nabla v dx$$

and

$$\|v\|^2 = \int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx.$$

In $H_0^1(\Omega)$ we consider the equivalent norm

$$\|v\|^2 = \int_\Omega |\nabla v|^2 dx$$

and the inner product

$$((u,v)) = \int_\Omega \nabla u . \nabla v dx$$

for all $u, v \in H_0^1(\Omega)$. We also observe that if Γ is of class $C^{1,1}$, then the norms of $\|v\|_{H^2}$ and $|\triangle v|$ are equivalent for $v \in H_0^1(\Omega) \cap H_0^2(\Omega)$.

Suppose that the functions $M(\lambda)$ and $F(u)$ satisfy:

(A.1) $M \in C^1([0,+\infty))$ and there exist positive constants α, ρ such that the following inequality is valid:

$$M(\lambda) \geq \alpha\sqrt{\lambda} + \rho, \text{ for all } \lambda \in [0,+\infty).$$

(A.2) $F : R \to R$ is a Lipschitzian function such that

$$sF(s) \geq 0 \text{ for all } s \in R.$$

We have the following result:

THEOREM 1. Suppose $u_0, u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$, $f \in C([0,T]; H_0^1(\Omega))$ and assumptions **(A.1)-(A.2)** hold. Then there exists a unique function $u : [0,T] \to L^2(\Omega)$, satisfying:

$$u \in C^1([0,T]; D(A^\alpha)) \text{ for all } 0 \leq \alpha < 1 \tag{6}$$

$$u' \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \tag{7}$$

$$u'' \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \tag{8}$$

$$Ku + F(u) = f \text{ in } L^\infty(0,T; L^2(\Omega)) \tag{9}$$

$$u(0) = u_0 \tag{10}$$

$$u'(0) = u_1 \tag{11}$$

$$u = 0 \text{ on } \Sigma. \tag{12}$$

As it is well known, the Laplace operator with homogeneous boundary conditions has a set of eigenvectors such that the subspace spanned by them is dense in $L^2(\Omega)$, in $H_0^1(\Omega)$ and in $H_0^1(\Omega) \cap H_0^2(\Omega)$. We may consider these eigenvectors to be orthonormal in $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$ or $H_0^1(\Omega) \cap H_0^2(\Omega)$. Moreover, by the regularity of the boundary Γ, we have $w_j(x) \in H^2(\Omega)$.

Let us denote by $V_m = [w_1, ..., w_m]$ the subspace of $H_0^1(\Omega)$ generated by the fists m eigenvectors w_j $(m \geq 1)$.

Let

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x) \in V_m$$

be a solution of the system:

$$(Ku_m(t), v) + (F(u_m(t), v) = (f(t), v), \ \forall \ v \in V_m, \tag{13}$$

$$u_m(0) = u_{0m} \to u_0 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega) \text{ as } m \to \infty, \tag{14}$$

$$u_m'(0) = u_{1m} \to u_1 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega) \text{ as} m \to \infty, \tag{15}$$

where

$$u_{0m} = \sum_{j=1}^{m} (u_0, w_j) w_j, \ \ u_{1m} = \sum_{j=1}^{m} (u_1, w_j) w_j.$$

Setting $\nu = w_j$ for $j = 1, ..., m$ we obtain a system of ordinary differential equations to which the classical Peano's theorem is applicable. Therefore this system has a local solution $u_m(t)$ on $[0, T_m)$, $0 < T_m \leq T$. The extension to $[0, T]$ is a consequence of the estimate (i) given in the next Section.

To prove our results, we will use the following result due a Simon [17]

Lemma 2.1 *Let $X \subset E \subset Y$ Banach spaces, the imbedding $X \to E$ being compact. Then the following imbedding are compact:*

(i) $L^q(O, T; X) \cap \{\varphi : \frac{\partial \varphi}{\partial t} \in L^1(0, T; Y)\} \to L^q(0, T; E)$ if $1 \leq q \leq \infty$,

(ii) $L^\infty(0, T; X) \cap \{\varphi : \frac{\partial \varphi}{\partial t} \in L^r(0, T; Y)\} \to C([0, t]; E)$ if $1 < r \leq \infty$.

and (see Temam [18])

Lemma 2.2 *Let $X \subset E \subset X^*$ Banach spaces. If $u \in L^2(0, T; X)$ and $u_t \in L^2(0, T; X^*)$ then $u \in C([0, T]; E)$ a.e.*

3 A Priori Estimates

Estimate (i). Setting $v = u''_m(t)$ in (13), we obtain

$$\begin{aligned} |u''_m(t)|^2 + a(u_m(t), u''_m(t)) + M(a(u_m(t)))a(u''_m(t), u''_m(t)) \\ +(F(u_m(t)), u''_m(t)) = (f(t), u''_m(t)). \end{aligned} \tag{16}$$

where

$$a(u,v) = \int_\Omega \nabla u.\nabla v dx \text{ and } a(u) = \int_\Omega |\nabla u|^2 dx.$$

Let us define $\mu(t) = M(a(u_m(t))) \geq \rho > 0$. Dividing both sides of (16) by $\mu(t)$, applying Cauchy-Schwartz's inequality and using the hypothesis **(A.1)**, we get

$$\begin{aligned} \frac{1}{\mu(t)} |u''_m(t)|^2 + |\nabla u''_m(t)|^2 \quad \leq \quad & \frac{\|u_m(t)\| \, |\nabla u''_m(t)|}{\alpha \|u_m(t)\|} + \frac{|f(t)| \, |u''_m(t)|}{\rho} \\ & + \frac{|F(u_m(t))| \, |u''_m(t)|}{\alpha \|u_m(t)\|}. \end{aligned} \tag{17}$$

Since F is Lipschitzian and satisfies $sF(s) \geq 0$ for all $s \in R$ (thus, $F(0) = 0$) we obtain

$$|F(u_m(t))| \leq C \left(\int_\Omega |u_m(t)|^2 dx \right)^{\frac{1}{2}} = C |u_m(t)| . \tag{18}$$

Using the Poincaré's inequality in (18), we have

$$|F(u_m(t))| \leq C |\nabla u_m(t)| , \; C > 0. \tag{19}$$

Applying (19) in (17), yields

$$\begin{aligned} \frac{1}{\mu(t)} |u''_m(t)|^2 + |\nabla u''_m(t)|^2 \quad \leq \quad & \frac{1}{\alpha^2} + \frac{1}{4} |\nabla u''_m(t)|^2 + \frac{C^2}{\rho^2} |\nabla f(t)|^2 \\ & + \frac{1}{4} |\nabla u''_m(t)|^2 + \frac{C^2}{\alpha^2} + \frac{1}{4} |\nabla u''_m(t)|^2 \end{aligned} \tag{20}$$

where C is a positive constant such that $|f(t)| \leq C |\nabla f(t)|$ and $|u''_m(t)| \leq C |\nabla u''_m(t)|$.

Then, from te above estimate and $f \in C([0,T]\,; H_0^1(\Omega))$, we get

$$|\nabla u_m''(t)|^2 \leq 4\left(\frac{1}{\alpha^2} + \frac{C^2}{\alpha^2} + \frac{C^2}{\rho^2} \max_{0\leq t\leq T} |\nabla f(t)|^2\right) < C_1 \tag{21}$$

where $C_1 > 0$ is independent of t and m.

Thus,

$$(u_m'') \text{ is bounded in } L^\infty(0, T_m; H_0^1(\Omega)). \tag{22}$$

By the Fundamental Theorem of Calculus, we have

$$(u_m') \text{ is bounded in } L^\infty(0, T_m; H_0^1(\Omega)) \tag{23}$$

and

$$(u_m) \text{ is bounded in } L^\infty(0, T_m, H_0^1(\Omega)). \tag{24}$$

The above estimates permit us to extend $u_m(t)$ to all interval $[0, T]$.

Estimate (ii). Since we are working with spectral basis, we have $-\Delta u_m'(t) \in V_m$. Then we can take $v = -2\Delta u_m'(t)$ in (13) to obtain

$$\begin{aligned}\frac{d}{dt}\left[|\nabla u_m'(t)|^2 + |\Delta u_m(t)|^2\right] + M(a(u_m(t)))\frac{d}{dt}|\Delta u_m'(t)|^2 \\ +2(\nabla F(u_m(t)), \nabla u_m'(t)) = 2(\nabla f(t), \nabla u_m'(t)).\end{aligned}$$

Setting $\mu(t) = M(a(u_m(t))) \geq \rho > 0$, dividing both sides of the last equality by $\mu(t)$, we obtain

$$\begin{aligned}&\frac{1}{\mu(t)}\frac{d}{dt}\left[|\nabla u_m'(t)|^2 + |\Delta u_m(t)|^2\right] + \frac{d}{dt}|\Delta u_m'(t)|^2 \\ = \;&\frac{-2}{\mu(t)}(\nabla F(u_m(t)), \nabla u_m'(t)) + \frac{2}{\mu(t)}(\nabla f(t), \nabla u_m'(t)).\end{aligned} \tag{25}$$

Defining

$$h_m(t) = \frac{1}{\mu(t)}\left[|\nabla u_m'(t)|^2 + |\Delta u_m(t)|^2\right] + |\Delta u_m'(t)|^2 \tag{26}$$

we have

$$\begin{aligned}h_m'(t) \;=\; &\frac{1}{\mu(t)}\frac{d}{dt}\left[|\nabla u_m'(t)|^2 + |\Delta u_m(t)|^2\right] + \frac{d}{dt}|\Delta u_m'(t)|^2 \\ &\frac{-\mu'(t)}{\mu(t)^2}\left[|\nabla u_m'(t)|^2 + |\Delta u_m(t)|^2\right].\end{aligned} \tag{27}$$

Using (25) in (27), we obtain

$$\begin{aligned} h'_m(t) &\leq \frac{2}{\mu(t)} |\nabla F(u_m(t))| \, |\nabla u'_m(t)| + \frac{2}{\mu(t)} |\nabla f(t)| \, |\nabla u'_m(t)| \\ &+ \frac{|\mu'(t)|}{\mu(t)^2} \left[|\nabla u'_m(t)|^2 + |\Delta u_m(t)|^2 \right]. \end{aligned}$$

We observe that

$$|\nabla F(u_m(t))| = |F'(u_m(t)) \nabla u_m(t)| \leq C \, |\nabla u_m(t)|$$

because F is Lipschitzian.

Consequently,

$$\begin{aligned} h'_m(t) &\leq \frac{2C}{\mu(t)} |\nabla u_m(t)| \, |\nabla u'_m(t)| + \frac{2}{\mu(t)} |\nabla f(t)| \, |\nabla u'_m(t)| \\ &+ \frac{|\mu'(t)|}{\mu(t)^2} \left[|\nabla u'_m(t)|^2 + |\Delta u_m(t)|^2 \right]. \end{aligned} \tag{28}$$

We also observe that

$$\mu'(t) = M'(a(u_m(t))) 2a(u_m(t), u'_m(t))$$

and

$$\begin{aligned} |M'((a(u_m(t)))| &\leq C_2 \text{ if } 0 \leq t \leq T, \ 0 \leq a(u_m(t)) \leq C_1 \\ |a(u_m(t), u'_m(t))| &\leq |\nabla u_m(t)| \, |\nabla u'_m(t)| \, . \end{aligned}$$

Then

$$|\mu'(t)| \leq C_3,$$

where C_3 is a positive constant.

From (28), we get

$$\begin{aligned} h'(t) &\leq \frac{2C}{\mu(t)} |\nabla u_m(t)| \, |\nabla u'_m(t)| + \frac{2}{\mu(t)} |\nabla f(t)| \, |\nabla u'_m(t)| \\ &+ \frac{C_3}{\mu(t)^2} \left[|\nabla u'_m(t)|^2 + |\Delta u_m(t)|^2 \right]. \end{aligned} \tag{29}$$

Using the estimate (i), we get

$$h'(t) \leq \frac{C_4^2}{\mu(t)} + \frac{1}{\mu(t)} \left|\nabla u'_m(t)\right|^2 + \frac{1}{\mu(t)} \left|\nabla f(t)\right|^2 + \frac{1}{\mu(t)} \left|\nabla u'_m(t)\right|^2 + \frac{C_3}{\mu(t)} \frac{1}{\mu(t)} \left[\left|\nabla u'_m(t)\right|^2 + \left|\Delta u_m(t)\right|^2\right]. \quad (30)$$

By the definition of $h_m(t)$,

$$\frac{1}{\mu(t)} \left[\left|\nabla u'_m(t)\right|^2 + \left|\Delta u_m(t)\right|^2\right] \leq h_m(t) \ ,$$

and

$$\frac{1}{\mu(t)^{\frac{1}{2}}} \left|\Delta u_m(t)\right| \leq h_m(t)^{\frac{1}{2}}.$$

From (23), we have

$$\frac{1}{\mu(t)} \left|\nabla u'_m(t)\right|^2 \leq C_{1,}$$

where C_1 is positive constant.

Hence

$$h'_m(t) \leq \frac{C_4^2}{\rho} + 2h_m(t) + \frac{1}{\rho} \max_{0\leq t\leq T} \left|\nabla f(t)\right|^2 + \frac{C_3}{\rho^{\frac{1}{2}}} h_m(t).$$

Thus,

$$\frac{d}{dt} h_m(t) \leq C_5 \left(1 + h_m(t)\right), \ 0 \leq t \leq T, \quad (31)$$

where

$$C_5 = \max\left\{\frac{C_4^2}{\rho} + \frac{1}{\rho} \max_{0\leq t\leq T} \left|\nabla f(t)\right|^2, 2, \frac{C_3}{\rho^{\frac{1}{2}}}\right\} > 0.$$

From (31), we can easily see that, there exists a positive constant C_6 such that for all $t \in [0, T]$

$$h_m(t) \leq C_6 \text{ in } [0, T].$$

This implies that

$$\frac{1}{\mu(t)} \left[\left|\nabla u'_m(t)\right|^2 + \left|\Delta u_m(t)\right|^2\right] + \left|\Delta u'_m(t)\right|^2 < C_6.$$

Therefore,

$$\left[|\nabla u_m'(t)|^2 + |\Delta u_m(t)|^2\right] + \rho\, |\Delta u_m'(t)|^2 < C_7,\ \ 0 \le t \le T \tag{32}$$

where $C_7 > 0$ is a constant independent of t and m.

Whence,

$$(u_m) \text{ is bounded in } L^\infty(0,T;H^2(\Omega)), \tag{33}$$

$$(u_m') \text{ is bounded in } L^\infty(0,T;H^2(\Omega)). \tag{34}$$

Estimate (iii). Setting $v = -\Delta u_m''(t)$ in (13), we obtain

$$\begin{aligned} &|\nabla u_m''(t)|^2 + (\Delta u_m(t), \Delta u_m''(t)) + M(a(u_m(t)))\, |\Delta u_m''(t)|^2 \\ &\quad + (F(u_m(t)), -\Delta u_m''(t)) = (f(t), -\Delta u_m''(t)). \end{aligned}$$

Let $\mu(t) = M(a(u_m(t))) \ge \rho > 0$. Dividing both sides of the last equality by $\mu(t)$, applying Cauchy-Schwartz's inequality and using hypothesis **(A.1)**, we get

$$\begin{aligned} \frac{1}{\mu(t)}\, |\nabla u_m''(t)|^2 + |\Delta u_m''(t)|^2 \ \ \le\ \ & \frac{1}{\rho}\, |\Delta u_m''(t)| + \frac{1}{\rho}\, |f(t)|\, |\Delta u_m''(t)| \\ & + \frac{|F(u_m(t))|\, |\Delta u_m''(t)|}{\alpha\, \|u_m(t)\|}. \end{aligned} \tag{35}$$

Using (19), $|f(t)| \le C\, |\nabla f(t)|$ and recalling that $|\Delta u_m'(t)|$ is bounded by the second estimate in (35), we obtain that

$$\begin{aligned} \frac{1}{\mu(t)}\, |\nabla u_m''(t)|^2 + |\Delta u_m''(t)|^2 \ \ \le\ \ & \frac{C_7^2}{\rho^2} + \frac{1}{4}\, |\Delta u_m''(t)|^2 + \frac{C^2}{\rho^2}\, |\nabla f(t)|^2 \\ & + \frac{1}{4}\, |\Delta u_m''(t)|^2 + \frac{C^2}{\alpha^2} + \frac{1}{4}\, |\Delta u_m''(t)|^2 . \end{aligned} \tag{36}$$

In particular we have

$$|\Delta u_m''(t)|^2 \le 4 \left(\frac{C_7^2}{\rho^2} + \frac{C^2}{\alpha^2} + \frac{C^2}{\rho^2} \max_{0 \le t \le T} |\nabla f(t)|^2 \right) < C_8$$

where $C_8 > 0$ is a constant independent of t and m.

Hence,

$$(u_m'') \text{ is bounded in } L^\infty(0,T;H^2(\Omega)). \tag{37}$$

4 Proof of Theorem

By estimates (22), (23), (24), (33), (34) and (37) there exists a subsequence of $(u_m)_{m\in N}$ which we still denote by $(u_m)_{m\in N}$, and a function u such that:

$$u_m \to u \text{ weak} - \text{star in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{38}$$

$$u_m' \to u' \text{ weak} - \text{star in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{39}$$

$$u_m'' \to u'' \text{ weak} - \text{star in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)). \tag{40}$$

From (38), (39) and (40), via Lemma 1, there exists a subsequence (still represented by $(u_m)_{m\in N}$) such that:

$$u_m \to u \text{ strongly in } C(0,T;D(A^\alpha)). \tag{41}$$

and

$$u_m' \to u' \text{ strongly in } C(0,T;D(A^\alpha)). \tag{42}$$

for all $\alpha \in [\frac{1}{2}, 1)$
As a consequence,

$$M(a(u_m(t))) \to M(a(u(t))) \text{ strongly in } C(0,T;D(A^\alpha)). \tag{43}$$

Now, by (40), we have

$$\Delta u_m'' \to \Delta u'' \text{ weaklyin} L^p(0,T;L^2(\Omega)) \; 1 \le p < \infty, \tag{44}$$

$$\Delta u_m'' \to \Delta u'' \text{ weakly} - \text{startin} L^\infty(0,T;L^2(\Omega)) \; . \tag{45}$$

Thus, by (43) and (44) we get

$$M(a(u_m(t)))\Delta u_m''(t) \to M(a(u(t)))\Delta u''(t) \text{ weakly in } L^p(0,T;L^2(\Omega)), \; 1 \le p < \infty. \tag{46}$$

Since that F is Lipschitzian, the convergence in (41) implies that

$$F(u_m(t)) \to F(u(t)) \text{ strongly in } L^p(0,T;L^2(\Omega)). \tag{47}$$

The established convergences make it easy to pass to the limit in the approximate problem (13) and obtain the existence of a strong solution to problem (1).

To prove the continuity established in the Theorem 1, we observe that (from Lemma 2)

$$u_m \in \ C(0,T; D(A^\alpha)). \tag{48}$$

and

$$u_m' \in \ C(0,T; D(A^\alpha)). \tag{49}$$

Consequently,

$$u_m \in \ C^1(0,T; D(A^\alpha)). \tag{50}$$

In order to check up that $u(0) = u_0$, it is sufficient to use (48). Analogously, by using (49), it can be shown that $u'(0) = u_1$.

remark 4.1 *If M' satisfies **(H.1)** then it is possible to show that $u''' \in L^2(0,T; H^{-1}(\Omega))$. Consequently, by interpolation, we have that $u'' \in C([0,T]; L^2(\Omega))$ and, thus the equation $Ku + F(u) = f$ is satisfied a.e. in t.*

5 Uniqueness

We prove the uniqueness in Theorem 1 as follows.

Let u, v be solutions of (5) in the class of Theorem 1. It follows that $w = u - v$ is a solution of:

$$\begin{aligned} w'' - \Delta w - M(a(u(t)))\Delta w'' - \left[M(a(v(t))) - M(a(u(t)))\right]\Delta v'' \\ +F(u(t)) - F(v(t)) = 0, \end{aligned} \tag{51}$$

$$w(0) = 0, \tag{52}$$

$$w'(0) = 0. \tag{53}$$

We shall prove that $w = 0$ in $[0,T]$. In fact, taking the inner product in $L^2(\Omega)$ of (50) by $2w' \in L^2(0,T; H_0^1(\Omega))$, we get

$$\begin{aligned} \frac{d}{dt}\left[|w'(t)|^2 + |\nabla w(t)|^2 + M(a(u(t)))a(w'(t))\right] = \frac{d}{dt}(M(a(u(t))))a(w'(t)) \\ + \left[M(a(v(t))) - M(a(u(t)))\right] 2a(v''(t), w'(t)) \\ -2(F(u(t)) - F(v(t)), w'(t)) \end{aligned} \tag{54}$$

i.e.,

$$
\begin{aligned}
& \frac{d}{dt}\left[\left|w'(t)\right|^{2}+\left|\nabla w(t)\right|^{2}+M(a(u(t)))a(w'(t))\right] \\
\leq\; & \left|\frac{d}{dt}(M(a(u(t))))\right|\left|a(w'(t))\right| \\
& +\left|M(a(v(t)))-M(a(u(t)))\right| 2\left|a(v''(t),w'(t))\right| \\
& +\left|F(u(t))-F(v(t))\right|\left|w'(t)\right|.
\end{aligned} \tag{55}
$$

Observe also that

$$\left|M(a(v(t)))-M(a(u(t)))\right| = \left|M'(\xi)\right|\left|\left\|v(t)\right\|^{2}-\left\|u(t)\right\|^{2}\right|$$

where

$$\xi=(1-\theta)\left\|u(t)\right\|^{2}+\theta\left\|v(t)\right\|^{2},\ 0\leq\theta\leq 1.$$

Hence,

$$
\begin{aligned}
\left|M'(\xi)\right|\left|\left\|u(t)\right\|^{2}-\left\|v(t)\right\|^{2}\right| &= \left|M'(\xi)\right|\left|(\left\|u(t)\right\|-\left\|v(t)\right\|)(\left\|u(t)\right\|+\left\|v(t)\right\|)\right| \\
&\leq C_{2}\left|\bigtriangledown w(t)\right|(\left\|u(t)\right\|+\left\|v(t)\right\|)
\end{aligned}
$$

because

$$\left|M'(\xi)\right|\leq C_{2} \text{ if } \xi\in(a(u(t)),a(v(t))) \text{ with } t\in[0,T].$$

Then,

$$
\begin{aligned}
& 2\left|M(a(u(t)))-M(a(v(t)))\right|\left|a(v''(t),w'(t))\right| \\
\leq\; & 2C_{2}\left|\nabla w(t)\right|(\left\|u(t)\right\|+\left\|v(t)\right\|)\left|\Delta v''(t)\right|\left|w'(t)\right|.
\end{aligned} \tag{56}
$$

Using (6), (8) and the Poincaré's inequality in (55) we obtain that

$$
\begin{aligned}
& \left|M(a(v(t)))-M(a(u(t)))\right| 2\left|a(v''(t),w'(t))\right| \\
\leq\; & C_{9}\left(\left|\nabla w(t)\right|^{2}+\left|\nabla w'(t)\right|^{2}\right).
\end{aligned} \tag{57}
$$

Also we have that

$$
\begin{aligned}
\left|\frac{d}{dt}M(a(u(t)))\right| &= \left|M'(a(u(t)))\right| 2\left|a(u(t),u'(t))\right| \\
&\leq C_{2}\left|\Delta u(t)\right|\left|u'(t)\right|\leq C_{10};
\end{aligned} \tag{58}
$$

hence

$$\left|\frac{d}{dt}M(a(u(t)))\right| a(w'(t)) \leq C_{10}a(w'(t)). \qquad (59)$$

In the last term of inequality (54), we observe that

$$\begin{aligned} |2(F(u(t)) - F(v(t)), w'(t))| &\leq 2\,|F(u(t)) - F(v(t))|\,|w'(t)| \qquad (60)\\ &\leq 2C\,|w(t)|\,|w'(t)| \\ &\leq C_{11}\left[|\nabla w(t)|^2 + |\nabla w'(t)|^2\right]. \end{aligned}$$

Using (56), (58) and (59) in (54) we obtain

$$\begin{aligned} &\frac{d}{dt}\left[|w'(t)|^2 + |\nabla w(t)|^2 + M(a(u(t)))a(w'(t))\right] \qquad (61)\\ \leq\ & C_{12}\left[|\nabla w(t)|^2 + |\nabla w'(t)|^2\right] \end{aligned}$$

where

$$C_{12} = \max\{C_{9,}\ C_{10},\ C_{11}\} > 0.$$

Integrating (60) from 0 to t and applying the hypothesis **(A.1)** we have

$$\begin{aligned} &|w'(t)|^2 + |\nabla w(t)|^2 + \rho\,|\nabla w'(t)|^2 \qquad (62)\\ \leq\ & C_{12}\int_0^t \left[|\nabla w(s)|^2 + |\nabla w'(s)|^2\right] ds; \end{aligned}$$

hence by Gronwall inequality we conclude

$$w = 0 \text{ in } [0, T].$$

Thus, the proof of Theorem 1 is completed. □

REMARK 2. In the forthcoming work we will try to study the equation (5) when $M(\lambda)$ has zero points, that is, degenerate case and $F(s)$ is a continuous function such that $sF(s) \geq 0$ for all $s \in R$.

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