A FAMILY OF PARTITIONS WITH ATTACHED PARTS AND "N COPIES OF N"

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Abstract

In this paper we find two distinct combinatorial interpretations for a family of summations with several free parameters. In one case we used partition with attached parts and in the other partitions with "N copies of N". There are interesting special cases including another description for an infinite family of partitions identities in parts that are congruent to $j \pmod{k}$.

1 Introduction

In a series of two papers Slater, [5] and [6], gave more than one hundred identities. To prove those identities it was fundamental the use of the q-anologue of Gauss summation formula.

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty} \tag{1.1}$$

where we use the standard notation

$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1})$$

and

$$(A;q)_\infty = \prod_{n=1}^{\infty}(1-Aq^n), \ |q|<1.$$

To prove our main theorem we use, also, this formula to get two distinct descriptions for partitions in parts that are congruent to j (mod k).

Among the identities given by Slater are the two famous Rogers-Ramanujan identities that were generalized by Gordon [3] in 1961.

In a recent paper by Andrews and Santos [2] in 1997 a new family of partition identities was given including two of the Stater's identitives as special cases. In this family there are descripitions of partitions with attached odd parts.

In theorems 3.1 and 3.2 we have, also, results in which attached parts occurs.

Some results in the theory of partitions have been obtained by using multisets. Among them we have, due to MacMahon [4], that

$$\sum_{n=0}^{\infty} \pi(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n},$$

where $\pi(n)$ denote the number of plane partition of n which is equal to the number of partitions of n with parts in the multiset

$$M_0 = \{1_1, 2_1, 2_2, 3_1, 3_2, 3_3, 4_1, 4_2, 4_3, 4_4, ...\}.$$

In Section 2 we give a combinatorial interpretation for a sum with four free parameters. In Section 3 we have a combinatorial interpretation, using attached parts for a specialization of the sum given in Section 2. In Section 4 we present our main result. In Section 5 we have some interesting special cases.

2 Partitions with parts in multisets

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We define for $l \geq 1, 1 \leq j < k$ and $i \leq 2(l-1)$ the following set:

$$A_{i,j,k,l} = \left\{(ak + rj + 2l - i - 1)_{rj+2l-i-1} \in M_0 | a, r \geq 0\right\}.$$

Theorem 2.1 Let $f(n)$ denote the number of partitions of n in distinct parts belonging to $A_{i,j,k,l}$ such that when $\alpha = (ak + rj + 2l - i - 1)_{rj+2l-i-1}$ and $\beta = (bk + sj + 2l - i - 1)_{sj+2l-i-1}$ are consecutive parts, $\alpha > \beta, ak \geq (b+s)k + 2l$.

Then,

$$\sum_{n=0}^{\infty} f(n) q^n = \sum_{n=0}^{\infty} \frac{q^{ln^2+(l-i-1)n}}{(q^j;q^k)_n (q^k;q^k)_n} \tag{2.1}$$

Proof. We define $f(m,n)$ to be the number of partitions of the type enumerated by $f(n)$ with the added restriction that the number of parts is exactly m. Then the following equation is verified by $f(m,n)$:

$$f(m,n) = f(m,n-km) + f(m-1,n-2lm+i+1) + \\ f(m,n-km+k-j) - f(m,n-2km+k-j). \tag{2.2}$$

To prove this we split the partitions enumerated by $f(m,n)$ into three classes: (a) those in which there is no part of the form R_R, (b) those in which $(2l-i-1)_{2l-i-1}$ is a part and (c) those in which R_R is a part for $R > 2l-i-1$.

If in those from class (a) we substract k from each part without changing the subscript we are left with a partition of $n-km$ in exactly m parts and these are the ones enumerated by $f(m,n-km)$. From those in class (b) we drop the element $(2l-i-l)_{2l-i-1}$ and subtract $2l$ from each of the remaining parts keeping the subscript. In doing so we are left with a partition of $n-2lm+i+1$ in exactly $m-1$ parts and these are the ones enumerated by $f(m-1,n-2lm+i+1)$. Finally from those in class (c) if we replace the element R_R by $(R-j)_{(R-j)}$ and substract k from each of the remaining parts without changing the subscript, we get a partition of $n-km+k-j$ which is enumerated by $f(m,n-km+k-j)$. We have to observe that by this transformation we get only those partitions of $n-km+k-j$ into m parts which contain a part of the form "R_R" with $R \geq 2l-i-1$. For this reason the partitions of class (c) can be put in an one to one correspondence with those that are enumerated by $f(m,n-km+k-j) - f(m,n-2km+k-j)$.
The transformations just described are possible by the following reasons:

(i) the elements in class (a) for not having part of the form "R_R" all of its parts are, then, of the form $(ak+rj+2l-i-1)_{rj+2l-i-1}$ with $a \geq 1$.

(ii) the parts of a partition in class (b) that are distinct from $(2l-i-1)_{2l-i-1}$ are greater than or equal to $4l-i-1$ because of the restriction given in the theorem *i.e.*, for $(ak+rj+2l-i-1)_{rj+2l-i-1} + (2l-i-1)_{(2l-i-1)}$ we have $ak \geq 2l$.

(iii) due to the transformation done in class (a) the partitions of $n-km+k-j$ with m parts without having part of the form "R_R" are enumerated by
$f(m,n-2km+k-j)$.

Now we define

$$F(z;q)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n)z^mq^n$$

and using (2.2) we get:

$$\begin{aligned}
&F(z;q)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n-km)z^mq^n+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m-1,n-2lm+i+1)z^mq^n\\
&+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n-km+k-j)z^mq^n-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n-2km+k-j)z^mq^n\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n-km)(zq^k)^mq^{n-km}
\end{aligned}$$

$$\begin{aligned}
&+zq^{2l-i-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m-1,n-2lm+i+1)(zq^{2l})^{m-1}q^{n-2lm+i+1}\\
&+q^{j-k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n-km+k-j)(zq^k)^mq^{n-km+k-j}\\
&-q^{j-k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n-2km+k-j)(zq^{2k})^mq^{n-2km+k-j}\\
&=F(zq^k;q)+zq^{2l-i-1}F(zq^{2l};q)+q^{j-k}F(zq^k;q)-q^{j-k}F(zq^{2k};q). \qquad (2.3)
\end{aligned}$$

Assuming that

$$F(z;q)=\sum_{n=0}^{\infty}\gamma(q,n)z^n$$

and using (2.3) we may compare coefficients of z^n obtaining:

$$\begin{aligned}
&\gamma(q,n)=q^{kn}\gamma(q,n)+q^{2ln-i-1}\gamma(q,n-1)+q^{kn-k+j}\gamma(q,n)-q^{2kn-k+j}\gamma(q,n)\\
&i.e\\
&(1-q^{kn}-q^{kn-k+j}+q^{2kn-k+j})\gamma(q,n)=q^{2ln-i-1}\gamma(q,n-1).
\end{aligned}$$

Therefore,

$$\gamma(q,n)=\frac{q^{2ln-i-1}}{(1-q^{kn})(1-q^{kn-k+j})}\gamma(q,n-1) \tag{2.4}$$

and observing that $\gamma(q,0)=1$ we may iterate (2.4) to get

$$\gamma(q,n)=\frac{q^{ln^2+(l-i-1)n}}{(q^j;q^k)_n(q^k;q^k)_n}.$$

From this

$$F(z;q)=\sum_{n=0}^{\infty}\frac{q^{ln^2+(l-i-1)n}z^n}{(q^j;q^k)_n(q^k;q^k)_n}$$

and we can finish the proof by observing that

$$\sum_{n=0}^{\infty}f(n)q^n=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(m,n)q^n=F(1;q)=\sum_{n=0}^{\infty}\frac{q^{ln^2+(l-i-1)n}}{(q^j;q^k)_n(q^k;q^k)_n}$$

□

3 Partitions with attached parts

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If we take $l=k$ on the right side of (2.1) we have

$$\sum_{n=0}^{\infty}\frac{q^{kn^2+(k-i-1)n}}{(q^j;q^k)_n(q^k;q^k)_n} \tag{3.1}$$

In our next theorem we give a different combinatorial interpretation for(3.1). In order to do this we define for $1\le j<k$ and $2k-i-1\ge 1$ the following sets:

$$B_{i,j,k}=\{rk+j,(2+s)k-(i+1)|r,s\ge 0\}$$

Theorem 3.1 Let $h(n)$ denote the number of partitions of n with parts in $B_{i,j,k}$ which satisfy:
(a) if "$rk-(i+1)$" and "$sk-(i+1)$" are parts then $|r-s|\ge 2$,
(b) $tk+j$ is a part (repetitions allowed)only if "$(t+1)k-(i+1)$" or "$(t+2)k-(i+1)$" occurs as a part.
Then for $1\le j<k$ and $2k-i-1\ge 1$ we have $f(n)=h(n)$, *i.e.*:

$$\sum_{n=0}^{\infty} f(n)q^n = \sum_{n=0}^{\infty} h(n)q^n = \sum_{n=0}^{\infty} \frac{q^{kn^2+(k-i-1)n}}{(q^j;q^k)_n(q^k;q^k)_n} \tag{3.2}$$

where $i \neq 2k-j-1$ and $i \neq k-j-1$.

Proof. First we define $h(m,n)$ as the number of partitions of the type enumerated by $h(n)$ with the added condition that the number of parts in each partition is exactly m.

Our goal is to prove that.

$$U_{i,j,k}(z) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{z^n q^{kn^2+(k-i-1)n}}{(zq^j;q^k)_n(q^k;q^k)_n} := V_{i,j,k}(z) \tag{3.3}$$

We have the following functional equation:

$$\begin{aligned} V_{i,j,k}(z) - V_{i,j,k}(zq^k) &= \sum_{n=0}^{\infty} \frac{z^n q^{kn^2+(k-i-1)n}}{(zq^{k+j};q^k)_{n-1}(q^k;q^k)_n}\left(\frac{1}{1-zq^j} - \frac{q^{kn}}{1-zq^{kn+j}}\right) \\ &= \frac{1}{(1-zq^j)(1-zq^{k+j})}\sum_{n=1}^{\infty} \frac{z^n q^{kn^2+(k-i-1)n}}{(zq^{2k+j};q^k)_{n-1}(q^k;q^k)_{n-1}} \\ &= \frac{zq^{2k-i-1}}{(1-zq^j)(1-zq^{k+j})} V_{i,j,k}(zq^{2k}) \end{aligned}$$

that is,

$$V_{i,j,k}(z) = V_{i,j,k}(zq^k) + \frac{zq^{2k-i-1}}{(1-zq^i)(1-zq^{k+j})} V_{i,j,k}(zq^{2k}). \tag{3.4}$$

We observe that (3.4), together with $V_{i,j,k}(0) = 1$, uniquely determine $V_{i,j,k}(z)$ as a double power series in z and q.

On the other side, due to the definition of $B_{i,j,k}$ and the condition (b) of the theorem, $\cup_{i,j,k}(z) - \cup_{i,j,k}(zq^k)$ enumerates all those partitions of the type enumerated by $\cup_{i,j,k}(z)$ that contain any number of j's attached to an appearence of "$2k-(i+1)$" or any $(k+j)'s$ attached to an appearence of "$2k-(i+1)$" and not to an appearence of "$3k-(i+1)$". This, together with condition (a), tell us that the partitions in $\cup_{i,j,k}(z) - \cup_{i,j,k}(zq^k)$ are generated by

$$\frac{zq^{2k-i-1}}{(1-zq^j)(1-zq^{j+k})} \cup_{i,j,k} (zq^{2k}).$$

Considering that $\cup_{i,j,k}(0) = 1$ we may conclude that (3.3) is true, and to finish the proof we just observe that

$$\sum_{n=0}^{\infty} h(n)q^n = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h(m,n)q^n = \cup_{i,j,k}(1) = \sum_{n=0}^{\infty} \frac{q^{kn^2+(k-i-1)n}}{(q^k;q^k)_n(q^j;q^k)_n}.$$

$\square$

In our next theorem we have a combinatorial interpretation for the sum on the right side of (3.2) for the cases not considered in the theorem (3.1), *i.e*, for $i = 2k - j - 1$ and $i = k - j - 1$. To do this we define

$$\mathbf{B}_{i,j,k} = \{[(2+s)k - (i+1)]_1, [rk+j]_2 | r, s \geq 0\}.$$

Theorem 3.2 Let $\mathbf{h}(n)$ denote the number of partitions of n with parts in $\mathbf{B}_{i,j,k}$ which satisfy: (a) if $[rk - (i+1)]_1$ and $[sk - (i+1)]_1$ are parts then $|r - s| \geq 2$. (b) $[tk+j]_2$ is a part (repetitions allowed) only if $[(tk+1)k - (i+1)]_1$ or $[(t+2)k - (i+1)]_1$ occurs as a part.

Then, for $1 \leq j < k$ and $i = 2k - j - 1$ or $i = k - j - 1$, $f(n) = \mathbf{h}(n)$.

The proof of this theorem follows by the same arguments used in the proof of Theorem (3.1) just adding the subscript 1 to the elements $sk - (i+1)$ and the subscript 2 to $tk + j$. For $i = 2k - j - 1$ we get

$$\sum_{n=0}^{\infty} \mathbf{h}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{kn^2+(j-k)n}}{(q^j;q^k)_n(q^k;q^k)_n} \tag{3.5}$$

and for $i = k - j - 1$

$$\sum_{n=0}^{\infty} \mathbf{h}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{kn^2+jn}}{(q^j;q^k)_n(q^k;q^k)_n} \tag{3.6}$$

4 The main theorem

If in (1.1) we make the following substitution $q \to q^k, a, b \to \infty$ and, after this, take $c = q^j$ we have:

$$\sum_{n=0}^{\infty} \frac{q^{kn^2+(j-k)n}}{(q^j;q^k)_n(q^k;q^k)_n} = \frac{1}{(q^j;q^k)_\infty} \tag{4.1}$$

Note that the left side of (4.1) is the same as the right side of (3.5) This observation, together with Theorems 2.1 and 3.2, can be used to prove our next theorem.

In order to do that we define

$$C_{j,k} = \{(ak + rj)_{rj} \in M_0 | a \geq 0, r \geq 1\}.$$

We have to observe that this set $C_{j,k}$ is, in fact, the set $A_{i,j,k,l}$ for $i = 2k - j - 1$ and $l = k$.

Defining $g(n)$ as the number of partitions of n with parts congruent to j (mod k) we have:

Theorem 4.1 Let $c(n)$ be the number of partitions of n in distinct parts belonging to $C_{j,k}$ such that when $\alpha = (ak + rj)_{rj}$ and $\beta = (bk + sj)_{sj}$ are consecutive parts, $\alpha > \beta, a > b + s$. Then for $1 \leq j < k$ we have

$$c(n) = \mathbf{h}(n) = g(n)$$

Proof. The fact that $\mathbf{h}(n) = g(n)$ follows from (4.1) and (3.5) and the equality $c(n) = \mathbf{h}(n)$ from Theorems 2.1 and 3.2

$\square$

The table below has the partitions of 20 for $j = 2$ and $k = 3$ when $i = 2k - j - 1$.

$\mathbf{h}(20)$	$c(20)$	$g(20)$
20_1	20_{20}	20
$14_1 + 2_2 + 2_2 + 2_1$	20_{14}	14+2+2+2
$11_1 + 5_2 + 2_2 + 2_1$	20_8	11+5+2+2
$8_2 + 8_1 + 2_2 + 2_1$	20_2	8+8+2+2
$8_1 + 5_2 + 5_2 + 2_1$	$18_{12} + 2_2$	8+5+5+2
$8_1 + 2_2 + \ldots + 2_2 + 2_1$	$18_6 + 2_2$	$8 + 2 + 2 + \ldots + 2$
$5_2 + 5_2 + 2_2 + \ldots + 2_2 + 2_1$	$16_4 + 4_4$	5+5+5+5
$5_2 + 5_2 + 5_2 + 5_1$	$15_6 + 5_2$	$5 + 5 + 2 + \ldots + 2$
$2_2 + \ldots + 2_2 + 2_1$	$14_2 + 6_6$	$2 + 2 + \ldots + 2$

5 Particular Cases

If we take $i = 0; j = 1; k = 2$ and $l = 1$ in Theorem 2.1 we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} f(n) q^n.$$

Considering that, by identity 79 of Slater [6], this sum is equal to:

$$\prod_{n=0}^{\infty} \frac{(1-q^{20n-8})(1-q^{20n-12})(1-q^{20n})(1+q^{2n-1})}{(1-q^{2n})}$$

we have the following theorem:

Theorem 5.1 For $i = 0; j = 1; k = 2$ and $l = 1$ the number of partitions of n in parts that are distinct odd or even $\not\equiv 0, \pm 8 \pmod{20}$ is equal to $f(n)$.
We illustrate this for $n = 7$

7_7	7
7_5	6+1
7_3	5+2
7_1	4+3
$6_4 + 1_1$	4+2+1
$6_2 + 1_1$	3+2+2
$5_1 + 2_2$	2+2+2+1

We mention now an interesting case that we get by taking $i = 2; j = 1; k = l = 2$ in Theorem 4.1. In this case we have the identity

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-n}}{(q;q)_{2n}} = \frac{1}{(q;q^2)_{\infty}}$$

and from our Theorem 4.1 we get, by using Euler's theorem, the following result:

Theorem 5.2 The number of partitions of n in distinct parts is equal to $c(n)$ for $i = 2; j = 1$ and $k = l = 2$.

We illustrate this for $n = 8$

8_8	8
8_6	7+1
8_4	6+2
8_2	5+3
7_3+1_1	5+2+1
7_1+1_1	4+3+1

Now, by taking $i=j=1$ and $k=l=2$ in (2.1), we get:

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}$$

which, by identity (39) of Slater [6], is equal to:

$$\prod_{n=1}^{\infty}\frac{(1+q^{8n-3})(1+q^{8n-5})(1-q^{8n})}{(1-q^{2n})}$$

For the case we are considering the set of partitions enumerated by $h(n)$ in our Theorem 3.1 is exactly the same as $C_{2,2}(n)$ given in Theorem 1 of Andrews and Santos [2] that is :

"the number of partitions of n wherein: 2 appears as a part at most 1 time, (b) the total number of appearences of $2j$ *and* $2j+2$ *together is at most* 1 *and* $(c) 2j+1$ *is allowed to appear (and may be repeated if it appears) only if the total number of appearences of* $2j$ *and* $2j+2$ *together is precisely* 1*"*
and from these results we have proved the following theorem:

Theorem 5.3 The number of partitions of n into parts that are either even but $\not\equiv 0, \pm 6, 8 \pmod{16}$ or odd and $\equiv \pm 3 \pmod 8$ is equal to the number of partitions of n into parts that are even but $\not\equiv 0 \pmod 8$ or distinct, odd and $\equiv \pm 3 \pmod 8$ and this number is, also, equal to $h(n)$ and $f(n)$.

The table below has the partitions of 10 in the order presented in the Theorem 5.3

5+5	10	10	10_{10}
5+3+2	6+4	8+2	10_8
4+4+2	6+2+2	6+2+1+1	10_6
4+3+3	5+3+2	4+3+3	10_4
4+2+2+2	4+4+2	3+3+2+1+1	10_2
3+3+2+2	4+2+2+2	3+2+1+...+1	8_4+2_2
2+...+2	2+...+2	2+1+...+1	8_2+2_2

We mention that there are more sums in the literature that can be interpreted by our results by particular values of the parameters i, j, k and l. For instance the case $i = -1; j = 1;\ k = 2$ and $l = 1$ give us a combinatorial interpretation for identity (99) of Slater [6].

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