Harmonic non $\pm$-holomorphic tori into complex Grassmann manifolds

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Abstract

In this note we construct via an adaptation of Uhlenbeck'separation of variables method (see, for example [22] and also [17] or [19]) families of explicit examples of harmonic and non $\pm$-holomorphic tori in complex Grassmann manifolds.

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§1 Introduction

The modern study of hamonic surfaces in Riemannian homogeneous spaces started with Calabi [8], Chern [9] and Eells [13] and now, after Uhlenbeck [23] is very well understood in the case of harmonic maps from S^2 to a homogeneous symmetric space. It was a very important step to complexity the problem, and this was done by Eells-Wood [14], Din-Zakarewski [11] and Glaser-Stora [15].

Much less attention was given to the case where the target is a non homogeneous non-symmetric spaces like flag manifolds. Black's book [2] discusses this case, relating this study with the understanding of f-structures on flag manifolds which is intimaly connected with the Eells-Wood's Theorem, this study therefore gives a natural relationship between Theory of twistors and harmonic maps into flags. The main interest in this case relies heavily with its connection with symmetric spaces like Grassmannians, as well their similarities with the variational approach to problems in low dimensional topology. For related material see, for example [1], [12] or [20].

In this paper we give some explicit examples of harmonic and non $\pm$-holomorphic tori on $G_r(\mathbb{C}^n)$ for arbitrary values of $r, 1 \leq r \leq n-1$. There are several theories of such tori in projective spaces as its is described by Burstall in [6] or Jensen-Liao in [16]. The case $G_2(\mathbb{C}^n)$ was discussed by Udagawa [21].

Let $(L_1, L_2, \ldots, L_n)$ an arbitrary element of $F(n)$. We define:

$$\begin{aligned} k(i_1, i_2, \ldots, i_r) : F(n) &\longrightarrow G_r(\mathbb{C}^n) \\ (L_1, L_2, \ldots, L_n) &\longmapsto L_{i_1} \wedge L_{i_2} \wedge \ldots \wedge L_{i_r} \end{aligned}$$

Now consider $u = (\pi_1, \pi_2, \ldots, \pi_n) : \mathbb{R}^2 \to (F(n), ds^2_{\Lambda=(\lambda_{ij})})$ given by: $\pi_i(x, y) = \exp(B_1 x + B_2 y).E_i.\exp(-B_1 x - B_2 y)$ with $B_1, B_2 \in u(n)$ and $[B_1, B_2] = 0$. The main result in this note is:

Theorem. There are infinitely many B_1, B_2 like above, with $[B_1, \sum \lambda_{ij} E_i B_1 E_j] + [B_2, \sum \lambda_{ij} E_i B_2 E_j] = 0$ such that $k(i_1, i_2, \ldots, i_r) \circ u : T^2 \to G_r(\mathbb{C}^n)$ is harmonic and non $\pm$-holomorphic.

Using the above theorem, we can produce explicit families of harmonic and non $\pm$-holomorphic tori in $G_r(\mathbb{C}^n)$ for arbitrary values of r. In particular, these examples generalize the tori in $\mathbb{R}P^n$ obtained by Uhlenbeck in [22]. Here we complexify the problem: we just see $\mathbb{R}P^n$ being totally geodesically embeded in $\mathbb{C}P^n$ as the set of real points.

These examples may be related to a Moser question as described by Uhlenbeck in [22].

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§2 On the complex geometry of Grassmannians and flag manifolds

The complex Grassmann manifold, denoted by $G_k(\mathbb{C}^n)$, can be seen as the homogeneous symmetric space obtained as a quotient of $U(n)$ via its natural action on

$G_k(\mathbb{C}^n)$. We can easily see that the isotropy subgroup of such action can be considered as $U(k) \times U(n-k)$.

If we endow $U(n)$ with its canonical Killing form metric (given by the trace) then the normal metric on $G_k(\mathbb{C}^n)$ $\left(\text{wich is diffeomorphic to } \dfrac{U(n)}{U(k) \times U(n-k)}\right)$, is the well known Fubini-Studi metric. Throughout this paper, $G_k(\mathbb{C}^n)$ is equipped with this Kähler metric (or a constant multiple of it).

We will now describe some basic properties enjoyed by the maximal flag manifold $F(n) = \dfrac{U(n)}{T}$, where $T = \underbrace{U(1) \times \cdots \times U(1)}_{n \text{ times}}$, is any maximal torus in $U(n)$. It is entirely similar to deal with the generalized flag manifold $F(N; r_1, \ldots, r_k)$ $\left(\text{which is diffeomorphic to } \dfrac{U(N)}{U(r_1) \times \cdots \times U(r_k)}(r_1 + \cdots + r_k = N)\right)$ and $F(k)$.

We consider now invariant almost complex structures $J : p \to p; J^2 = -I$. Borel and Hirzebruch [5] showed that there are $2^{\binom{n}{2}}$ such invariant structures.

Example 2.1. We consider $n = 3$ and $J : p \to p$ defined in the following way

$$J\left[\begin{pmatrix} 0 & a_{12} & a_{13} \\ -\overline{a}_{12} & 0 & a_{23} \\ -\overline{a}_{13} & -\overline{a}_{23} & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & \varepsilon_1\sqrt{-1}a_{12} & \varepsilon_2\sqrt{-1}a_{13} \\ \varepsilon_1\sqrt{-1}\overline{a}_{12} & 0 & \varepsilon_3\sqrt{-1}a_{23} \\ \varepsilon_2\sqrt{-1}\overline{a}_{13} & \varepsilon_3\sqrt{-1}\overline{a}_{23} & 0 \end{pmatrix}$$

where $\varepsilon_i = \pm 1, i = 1, 2$ and 3. There are $2^{\binom{3}{2}} = 2^3 = 8$ distinct invariant almost complex structures.

Such a choice clearly defines a tournament τ_J with n players $\{1, 2, \ldots, n\}$. More precisely, we define: $J[(a_{ij})] = (a'_{ij}), 1 \leq i \neq j \leq n$ where

$$i \to j (i < j) \Leftrightarrow a'_{ij} = \sqrt{-1}a_{ij}$$

or

$$i \leftarrow j (i < j) \Leftrightarrow a'_{ij} = -\sqrt{-1}a_{ij}.$$

. Hence there is a $1 - 1$ correspondence between J and τ_J.

Fixing the usual Hermitian inner product on $\mathbb{C}^n$ then $\mathbb{C}^n = \bigoplus_{i=1}^{n} E_i$. At the Lie algebra level we have:

$$\begin{aligned} u(n)^{\mathbb{C}} &\cong \mathbb{C}_n \cong Hom(\mathbb{C}^n, C^n) \cong (\mathbb{C}^n)^* \otimes C^n \cong \\ &\cong (\overline{E}_1 \oplus \cdots \oplus \overline{E}_n) \otimes (E_1 \oplus \cdots \oplus E_n) \cong \underbrace{(u(1) \oplus \cdots \oplus u(1))}_{n \text{ times}} \oplus \end{aligned}$$

$\Big(\oplus_{i<j}(\overline{E}_i E_j \oplus \overline{E}_j E_i)\Big)$, where E_i is equal to the subspace of $\mathbb{C}^n$ generated by e_i.

Each real vector space D_{ij} has two invariant almost complex structures. According to [7], we see that any $U(n)$-invariant almost complex structure J on $F(n)$ is characterized by choosing one of these two structures on each D_{ij}. Let $u(n)^{\mathbb{C}} \cong p^{(1,0)} \oplus p^{(0,1)}$. Then $p^{(1,0)} = \bigoplus_{i \to j} \overline{E}_i E_j$. See [7] for more details.

In [4], Borel has studied the left invariant metrics on $F(n)$: let A and B in p and consider the following collection of inner products:

$$\langle A, B\rangle_{ds^2_{\Lambda=(\lambda_{ij})}} := \sum tr(\lambda_{ij} E_i A E_j B^*),$$

where E_i is the matrix with 1 in the (i,i)-position and zero elsewhere, $\lambda_{ij} = \lambda_{ji} > 0$ and $\lambda_{ii} = 0, 1 \leq i, j \leq n$.

§3 Harmonic and ±-holomorphic equations on $F(n)$ Equi-harmonic maps

From now on, M will always denote a closed and oriented Riemann surface. Let $\tilde{\phi} : M \to U(n)$ be the lift map of $\phi : M \to F(n)$, i.e. $\phi = \pi \circ \tilde{\phi}$ where $\pi : U(n) \to F(n)$ is the natural projection. Let $e_1, \ldots, e_n$ the canonical basis on $\mathbb{C}^n$.

We denote by π_j the matrix of the orthogonal projection onto the subspace of $\mathbb{C}^n$ generated by e_j which is denoted by E_j. Then $\pi_j : M \to gl(n,\mathbb{C}) \cong M(n \times n, \mathbb{C}) = \mathbb{C}_n$ satisfies that $A'_{ji}(e_1, \ldots, e_n) = (e_1, \ldots, e_n) A^{ij}_z$ where $A^{ij}_z = \pi_i \dfrac{\partial \pi_j}{\partial z}$.

For $V \in \Gamma(\phi^* T(F(n)))$ we set $q = \phi^*\beta(V)$ where $\phi^*\beta : \phi^* T(F(n)) \to M \times u(n)$ is the pull-back of the Maurer-Cartan form on $U(n)$. Define a variation of ϕ by:

$$\phi_t(x) := \pi(\exp(-tq)\tilde{\phi})$$

Denote its associate objects by $\pi_j(t), A^{ij}_z(t)$, etc. We have:

Lemma 3.1. 1) $\delta \pi_j = \dfrac{\partial}{\partial t}\Big|_{t=0} \pi_j(t) = [\pi_j, q]$

2) $\dfrac{\partial}{\partial z}[\pi_j, q] = \left[\dfrac{\partial \pi_j}{\partial z}, q\right] + \left[\pi_j, \dfrac{\partial q}{\partial z}\right]$

3) $\delta(A^{ij}_z) = \dfrac{\partial}{\partial t}\Big|_{t=0} A^{ij}_z(t) = [A^{ij}_z, q] - \pi_i \dfrac{\partial q}{\partial z} \pi_j$

Proof. See [17] or [18].

The Killing inner product on $gl(n,\mathbb{C})$ is defined by:

$$\langle A, B\rangle := tr(A.B^*), \forall A, B \in gl(n,\mathbb{C}) \cong \mathbb{C}_n$$

It is easy to see that $\langle A, B\rangle = \overline{\langle B, A\rangle}$ and $\langle A, [B, C]\rangle = \langle [B^*, A], C\rangle$. In particular we have:

$$\langle A, B\rangle + \langle B, A\rangle = 2Re\langle A, B\rangle.$$

Definition 3.2. Let $\phi = (\pi_i) : M^2 \to (F(n), ds^2_{\Lambda=(\lambda_{ij})})$. We define the energy of ϕ as:

$$E(\phi) := \int_M \sum \lambda_{ij} |A^{ij}_z|^2 v_g.$$

Let (ϕ_t) the variation of ϕ above defined. Then

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} E(\phi_t) = \int_M \sum \lambda_{ij} \frac{\partial}{\partial t}\Big|_{t=0} |A^{ij}_z(t)|^2 v_g = \\
&2Re \int_M \sum \lambda_{ij} \left\langle A^{ij}_z, \frac{\partial}{\partial t}\Big|_{t=0} A^{ij}_z(t) \right\rangle v_g = \\
&= 2Re \int_M \sum \lambda_{ij} \left\langle A^{ij}_z, [A^{ij}_z, q] - \pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle v_g
\end{aligned}$$

so we obtain: $\frac{1}{2}\delta E(\delta\phi) = \frac{1}{2}\frac{d}{dt}\Big|_{t=0} E(\phi_t) = I + II$ where

$$\begin{aligned}
I &= Re \int_M \sum \lambda_{ij} \langle A^{ij}_z, [A^{ij}_z, q]\rangle v_g \\
II &= -Re \int_M \sum \lambda_{ij} \left\langle A^{ij}_z, \pi_i \frac{\partial q}{\partial z} \pi_j \right\rangle v_g
\end{aligned}$$

But we can prove that $I = 0$ (See [17] or [18] for details). On the other hand, if we use Stokes'theorem we have:

$$\begin{aligned}
II &= -Re \int_M \sum \lambda_{ij} \left\langle A^{ij}_z, \frac{\partial q}{\partial z} \right\rangle v_g = \\
&= Re \int_M \sum \lambda_{ij} \left\langle \frac{\partial A^{ij}_z}{\partial \overline{z}}, q \right\rangle - Re \int_M \sum \lambda_{ij} \frac{\partial}{\partial \overline{z}} \langle A_z, q\rangle V_g \\
&= Re \int_M \left\langle \frac{\partial A^{\Lambda}_z}{\partial \overline{z}}, q \right\rangle v_g, \quad \text{where} \quad A^{\Lambda}_z := \sum \lambda_{ij} A^{ij}_z
\end{aligned}$$

and $\dfrac{\partial A_z^\lambda}{\partial \overline{z}} : M \to u(n)$.

Proposition 3.3. $\phi = (\pi_i)_{i=1}^n : (M, g) \to (F(n), ds^2_{\Lambda=(\lambda_{ij})})$ is harmonic if, and only if $\dfrac{\partial A_x^\Lambda}{\partial x} + \dfrac{\partial A_y^\Lambda}{\partial y} = 0$ where $A_x^\Lambda := \sum \lambda_{ij}\pi_i \dfrac{\partial \pi_j}{\partial x}$, $A_y^\Lambda := \sum \lambda_{ij}\pi_i \dfrac{\partial \pi_j}{\partial y}$.

Proof. In fact

$$
\begin{aligned}
4\frac{\partial A_z^\Lambda}{\partial \overline{z}} &= \sum \lambda_{ij}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)(A_x^{ij} - \sqrt{-1}A_y^{ij}) = \\
&= \frac{\partial}{\partial x}(A_x^\Lambda) + \frac{\partial}{\partial y})(A_y^\Lambda) + (*) \\
\text{where} \qquad & \frac{1}{\sqrt{-1}}(*) = \sum \lambda_{ij}\left(\frac{\partial A_x^{ij}}{\partial y} - \frac{\partial A^{ij}}{\partial x}\right) = \\
&= \sum \lambda_{ij}\left[\frac{\partial}{\partial y}\left(\pi_i \frac{\partial \pi_j}{\partial x}\right) - \frac{\partial}{\partial x}\left(\pi_i \frac{\partial \pi_j}{\partial y}\right)\right] = 0
\end{aligned}
$$

because $\lambda_{ij} = \lambda_{ji}$.

Definition 3.4. Let E_∂ and $E_{\overline{\partial}}$ denote the ∂-and $\overline{\partial}$-energy respectively, defined by:

$$
E_\partial(\phi) = \sum_{i\to j}\int_M \lambda_{ij}|A_z^{ij}|^2_{v_g} \quad \text{and} \quad E_{\overline{\partial}}(\phi) = \sum_{i\to j}\int_M \lambda_{ij}|A_{\overline{z}}^{ij}|^2_{v_g}
$$

Therefore $\phi = (\pi_1, \ldots, \pi_n) : (M^2, J_1) \to (F(n), J)$ is holomorphic with respect to the almost complex structure determined by τ_J if, and only if $E_{\overline{\partial}}(\phi) = \sum_{i\to j}\int_M \lambda_{ij}|A_{\overline{z}}^{ij}|^2 v_g = 0$ i.e. $A_{\overline{z}}^{ij}.A_z^{ji} = 0, \forall i \to j$.

Definition 3.5. A map $\phi : M^2 \to G/H$ with a non-empty set of G-invariant metrics is said to be equi-harmonic if it is harmonic with respect to each G-invariant metric on G/H.

We now consider the family of Borel-type metrics in $F(n)$ (i.e. metrics $U(n)$-invariants) and $\psi = (\pi_i)_i : M^2 \to F(n) = U(n)/T$ an equi-harmonic map. We define:

$$
\begin{aligned}
k_{(i_1,i_2,\ldots,i_r)} : F(n) &\longrightarrow G_r(\mathbb{C}^n) \quad \text{by :} \\
(L_1, L_2, \ldots, L_n) &\longmapsto L_{i_1} \wedge L_{i_2} \wedge \ldots \wedge L_{i_r}
\end{aligned}
$$

where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$.

Theorem 3.6 ([2]) The map $k_{(i_1,i_2,\ldots,i_r)} \circ \psi : (M^2, g) \to (G_r(\mathbb{C}^n)$, Killing form metric) is harmonic.

Proof: See [2] for details.

§4 Examples of tori in $G_r(\mathbb{C}^n)$

We will produce families of harmonic and non $\pm$-holomorphic tori in $G_r(\mathbb{C}^n)$ for arbitrary values of r. For $r = 1$ we will see that all these examples have according to Chern-Wolfson [10] Kähler angle 0.

Our method was based on Uhlenbeck's one as it is given in [22].

Suppose $\phi = (\pi_1, \ldots, \pi_n) : \mathbb{R}^2 \to F(n)$ is defined by: $\phi = \pi \circ \widetilde{\phi}$ where $\widetilde{\phi}(x, y) = \exp(B_1 x + B_2 y), B_1, B_2 \in u(n)$ and $[B_1, B_2] = 0$. Then:

$$\widetilde{\phi}(x, y) = \exp(B_2 y) \exp(B_1 x)$$
$$\frac{\partial \widetilde{\phi}}{\partial x} = \widetilde{\phi} B_1$$
$$\frac{\partial \widetilde{\phi}^*}{\partial x} = \left(\frac{\partial \widetilde{\phi}}{\partial x}\right)^* = -B_1 \widetilde{\phi}^*$$

Therefore $\dfrac{\partial \pi_i}{\partial x} = \dfrac{\partial}{\partial x}(\widetilde{\phi}.E_i.\widetilde{\phi}^*) = \widetilde{\phi}[B_1, E_i]\widetilde{\phi}^*$. So $A_x^{ji} = \pi_j \dfrac{\partial \pi_i}{\partial x} = \widetilde{\phi} E_j [B_1, E_i] \widetilde{\phi}^* = \widetilde{\phi} E_j B_1 E_i \widetilde{\phi}^*$. Similarly: $A_y^{ji} = \widetilde{\phi} E_j B_2 E_i \widetilde{\phi}^*$. Hence

$$A_z^{ij} = \widetilde{\phi} E_j \chi E_i \widetilde{\phi}^* \quad \text{where} \quad \chi = \frac{1}{2}(B_1 - \sqrt{-1} B_2)$$

Now we can investigate the harmonicity of ϕ. We have:

$$\frac{\partial A_x^{ij}}{\partial x} = \frac{\partial}{\partial x}(\widetilde{\phi} E_i B_1 E_j \widetilde{\phi}^*) = \widetilde{\phi}[B_1, E_i B_1 E_j]\widetilde{\phi}^*$$

and

$$\frac{\partial A_y^{ij}}{\partial x} = \widetilde{\phi}[B_2, E_i B_2 E_j]\widetilde{\phi}^*$$

Therefore according to Proposition 3.3 we have:

Proposition 4.1. Suppose that $\phi : I\!R^2 \to F(n)$ is doubly periodic. Then ϕ is harmonic with respect to $ds^2_{\Lambda=(\lambda_{ij})}$ if, and only if

$$[B_1, \sum \lambda_{ij} E_i B_1 E_j] + [B_2, \sum \lambda_{ij} E_i B_2 E_j] = 0$$

We can now prove the following result:

Theorem 4.2. Let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in Q - \{0\}$ and

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_j^1 = \begin{pmatrix} \alpha_j X & 0 \\ 0 & \beta_j X \end{pmatrix}$$
$$B_j^2 = \begin{pmatrix} \beta_j X & 0 \\ 0 & \alpha_j X \end{pmatrix}, j = 1, \ldots, k \leq \frac{n}{4}.$$

We consider

$$B_1 = \sqrt{-1} \begin{pmatrix} B_1^1 & & & & & \\ & \ddots & & & & \\ & & B_k^1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

$$B_2 = \sqrt{-1} \begin{pmatrix} B_1^2 & & & & & \\ & \ddots & & & & \\ & & B_k^2 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Then:
(i) $\phi(x, y) = \pi(\exp(B_1 x + B_2 y))$ is doubly periodic.
(ii) $\phi : T^2 \to F(n)$ is equi-harmonic.

Proof: (i) For $l \in \{1, 2, \ldots\}$

$$B_2^l = (\sqrt{-1})^l \begin{pmatrix} \alpha_1^l X^l & & & & & & & \\ & \beta_1^l X^l & & & & & & \\ & & \ddots & & & & & \\ & & & \alpha_k^l X^l & & & & \\ & & & & \beta_k^l X^l & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

where $X^l = \begin{cases} X & \text{if } l \text{ is odd} \\ I_2 & \text{if } l \text{ is even} \end{cases}$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

So

$$\exp(B_1 x) = I + B_1 x + \frac{B_1^2 x^2}{2!} + \cdots =$$

$$= \begin{pmatrix} \cos\alpha_1 x I_2 & & & & & & & \\ & \cos\beta_1 x I_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\alpha_k x I_2 & & & & \\ & & & & \cos\beta_k x I_2 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

$$+\sqrt{-1} \begin{pmatrix} \sin\alpha_1 X & & & & & & & \\ & \sin\beta_1 x X & & & & & & \\ & & \ddots & & & & & \\ & & & \sin\alpha_k x X & & & & \\ & & & & \sin\beta_k x X & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

Combining with $[B_1, B_2] = 0$, there exists a $\nu \in Q - \{0\}$ such that

$$\phi(x+2\pi n\nu, y+2\pi n\nu) = \phi(x,y).$$

Hence we have

$$\phi : T^2 = \frac{I\!R^2}{2\pi\nu(Z\!\!Z \oplus Z\!\!Z)} \to F(n)$$

(ii) For any left-invariant metric ds^2_Λ on $F(n)$ we have

$$\sum \lambda_{ij} E_i B_1 E_j = \sqrt{-1} \begin{pmatrix} \alpha_1\lambda_{12}X & & & & & & & \\ & \beta_1\lambda_{34}X & & & & & & \\ & & \ddots & & & & & \\ & & & \alpha_k\lambda_{4k-3,4k-2}X & & & & \\ & & & & \beta_k\lambda_{4k-1,4k}X & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

so

$$B_1(\sum \lambda_{ij} E_i B_1 E_j) = - \begin{pmatrix} \alpha_1^2\alpha_{12}X^2 & & & & & & & \\ & \beta_1^2\lambda_{34}X^2 & & & & & & \\ & & \ddots & & & & & \\ & & & \alpha_k^2\lambda_{4k-3,4k-2}X^2 & & & & \\ & & & & \beta_k^2\lambda_{4k-1,4k}X^2 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

$= (\sum \lambda_{ij} E_i B_1 E_j).B_1)$. So $[B_1, \sum \lambda_{ij} E_i B_1 E_j] = 0$. Similarly we have $[B_2, \sum \lambda_{ij} E_i B_2 E_j] = 0$. Therefore according to Proposition 4.1 we see that ϕ is equi-harmonic.

Theorem 4.3. Let $\alpha_1, \ldots, \alpha_k \in Q - \{0\}, 2k \leq n, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$B_1 = B_2 = \begin{pmatrix} \alpha_1 X & & & & & & \\ & \alpha_2 X & & & & & \\ & & \ddots & & & & \\ & & & \alpha_k X & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

Then $\phi : T^2 \to F(n)$
$(x, y) \mapsto \pi(\exp(B_1(x+y)))$ is an equi-harmonic map.

Proof: We proceed like in the prove of Theorem 4.2.

Theorem 4.4. Let $\alpha_1 = \alpha_2 = \cdots = \alpha_k \in \mathbb{R} - \{0\}, 2k \leq n, X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
and

$$B_1 = B_2 = \begin{pmatrix} \alpha_1 X & & & & & & \\ & \alpha_1 X & & & & & \\ & & \ddots & & & & \\ & & & \alpha_1 X & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

Then $\phi : T^2 \to F(n)$ defined by:
$(x, y) \mapsto \pi(\exp(B_1(x+y)))$ is an equi-harmonic map.

Proof: Just use proposition 4.1.

We are now in the position of stating and prove the main result of this note:

Theorem 4.5. Let $\phi = (\pi_1, \ldots, \pi_{i_1}, \ldots, \pi_{i_r}, \ldots, \pi_n) : T^2 \to F(n)$ families of equi-harmonic tori as in theorems 4.2, 4.3 or 4.4 and $i_\alpha \leq n \quad \forall\ \alpha \in [1, r] \cap \mathbb{N}$. Then $u = k_{(i_1, i_2, \ldots, i_r)} \circ \phi = \pi_{i_1} \wedge \ldots \wedge \cdots \wedge \pi_{i_r} : T^2 \to G_r(\mathbb{C}^n)$ is a full harmonic map but not $\pm$-holomorphic.

Proof: Since ϕ is equi-harmonic we can use Theorem 3.6 and we obtain that u is harmonic.

On the other hand:

$$\pi_{i_1-1}\frac{\partial u}{\partial \overline{z}} = \pi_{i_1-1}\frac{\pi_{i_1}}{\partial \overline{z}} + \pi_{i_1-1}\frac{\partial \pi_{i_2}}{\partial \overline{z}} + \cdots + \pi_{i_1-1}\frac{\partial \pi_{i_r}}{\partial z} = A_{\overline{z}}^{i_1-1,i_1}$$

which is non-zero in any example so $\frac{\partial u}{\partial \overline{z}} \neq 0$. Similarly $\frac{\partial u}{\partial z} \neq 0$. So u is not $\pm$-holomorphic.

We will now consider $r = 1$. Therefore in this case we have $\pi_i : T^2 \to \mathbb{C}P^{n-1}, 1 \leq i \leq 4k$.

Let dA the area form on T^2 defined by $\pi_i^*(ds^2_{\mathbb{C}P^{n-1}})$ and the orientation of M and let $S_{\pi_i} = \{x \in T^2, dA(x) = 0\}$.

The Kähler angle of π_i is a function on $T^2 - S_{\pi_i}$, which takes values in $[0, \pi]$: it is defined by $\pi_i^*\Omega = \cos\theta dA$ where Ω is the Kähler form on $\mathbb{C}P^{n-1}$.

We can prove that $\theta = 0$ if, and only if π_i is $+$-holomorphic and $\theta = \pi$ if, and only if π_i is $-$-holomorphic. Therefore the Kähler angle gives a measurament of the distance that our function from a $\pm$-holomorphic one.

Definition 4.6. We say that $\pi_i : T^2 \to \mathbb{C}P^{n-1}$ is totally real or weakly Lagrangian or super-conformal if $\theta = \pi/2$.

Proposition 4.7. Every $\psi = \pi_i : T^2 \to \mathbb{C}P^{n-1}$ full harmonic but not $\pm$-holomorphic map obtained applying theorem 4.5 is totally real.

Proof: Is is known that $\psi^*\Omega = (|A_z^\psi| - |A_{\overline{z}}^\psi|^2).dA$ then $\cos\theta = |A_z^\psi| - |A_{\overline{z}}^\psi|^2$. But:

$$|A_z^{ij}|^2 = \langle A_z^{ij}, A_z^{ij}\rangle = \langle \widetilde{\psi} E_i X E_j \widetilde{\psi}^*, , \widetilde{\psi} E_i X E_j \widetilde{\psi}^*\rangle =$$
$$tr(\widetilde{\psi} E_i X E_j \widetilde{\psi}^* \widetilde{\psi} E_j X^* E_i \widetilde{\psi}^* E_i \widetilde{\psi}^*) = \langle E_i X E_j, E_i X E_j\rangle$$

But $E_i \chi E_j = \frac{1}{2} E_i (B_1 - \sqrt{-1} B_2) E_j = \frac{1}{2}(B_1^{ij} - \sqrt{-1} B_2^{ij})$.

Without loss of generality we can assume $\psi = \pi_1$. Therefore:

$|A_z^\psi|^2 = \sum_{j=2}^{n} \langle E_1 \chi E_j, E_1 \chi E_j\rangle = \frac{1}{2}\sum_j (|B_1^{1j}|^2 + |B_2^{1j}|^2)$. Similarly $|A_{\overline{z}}^\psi|^2 = \frac{1}{2}\sum_j (|B_1^{1j}|^2 + |B_2^{1j}|^2)$.

Therefore $\cos\theta = |A_z^\psi|^2 - |A_{\overline{z}}^\psi|^2 = 0$ then $\theta = \pi/2$ i.e. ψ is weakly Lagrangian.

Remark 4.8. • It is a nice question to classify every tori obtained via this adpted Uhlenbeck's separation of variables argument.

- It will be interesting to understand the stability properties of these tori in $G_r(\mathbb{C}^n)$.
- We notice that in particular the harmonic, totally real tori in $\mathbb{C}P^n$ found here could only exist if $n \geq 3$. This fact was expected since they are not $\pm$-holomorphic neither satisfy Toda's equations [3].

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