Invariant measures for some spatial birth and death processes

Nancy L. Garcia
Universidade Estadual de Campinas

Thomas G. Kurtz
University of Wisconsin - Madison

Abstract

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1 Graphical construction

Consider a spatial birth and death process with a birth rate given by

$$\lambda(x,\eta) = c_1 - c_2 \mathbf{1}\{\min d(x,y_i) \le t_0\} \tag{1.1}$$

where $0 \le c_2 \le c_1$ and $\eta = \{y_1, y_2, \ldots\}$ can be identified with the point process on $\mathbb{R}^d$ given by $\eta = \sum_i \delta_{y_i}$ and a constant death rate equals to 1.

The above process has generator given by

$$Af(\eta) = \int (f(\eta+\delta_x) - f(\eta))\lambda(x,\eta)dx + \int (f(\eta-\delta_x) - f(\eta))\eta(dx) \tag{1.2}$$

for "suitable" functions f.

1.1 Marked Poisson processes

In order to get a graphical construction for the process with generator (1.2), we begin with a c_1-homogeneous Poisson point process on $\mathbb{R}^d \times [0,\infty)$. Denote $N = \{(\xi_1, T_1), (\xi_2, T_2), \ldots\}$. For each point (ξ_i, T_i), associate two independent marks $S_i \sim \exp(1)$ and $Z_i \sim b(1, c_2/c_1)$.

Interpretation We can see the marked point process $\mathbf{C} = (\{(\xi_i, T_i, S_i, Z_i), i = 1, 2, \ldots\}$ as the graphical representation of a birth and death process with constant birth rate c_1 and constant death rate 1 (call this process α) and Z_i will be used as the indicator of "allowed" births.

From now on, a marked point (ξ_i, T_i, S_i, Z_i) will be identified with a marked *cylinder* $((\xi_i + B(0, t_0)) \times [T_i, T_i + S_i), Z_i)$ with *basis* ξ_i, *birth time* T_i, *lifetime* S_i and *flag* Z_i. Calling $C = (\xi, t, s, z)$, we use the notation

$$\text{Basis}\,(C) = \xi, \quad \text{Birth}\,(C) = t, \quad \text{Life}\,(C) = [t, t+s], \quad \text{Flag}\,(C) = z. \tag{1.3}$$

Define incompatibility between cylinders C and C' by

$$C' \not\sim C \quad \text{if and only if} \quad d(\text{Basis}\,(C), \text{Basis}\,(C')) \le t_0 \;\text{ and }\; \text{Life}\,(C) \cap \text{Life}\,(C') \neq \emptyset, \tag{1.4}$$

otherwise $C' \sim C$ (compatible).

1.2 Finite-volume construction

The construction of the spatial birth and death process in a finite box Λ with an initial configuration $\eta_0 = \{\varphi_1, \varphi_2, \ldots\}$ using the Poisson processes is straightforward. We use only the finite set $\{(\xi_i, T_i, S_i, Z_i) : \xi_i \in \Lambda\}$. Let $\mathbf{C}^\Lambda = \{C \in \mathbf{C} : \text{Basis}\,(C) \in \Lambda\}$. To each point φ_j present in the initial configuration η_0 we independently associate an exponential time $\tilde{S}_j$ and a cylinder $(\varphi_j, 0, \tilde{S}_j, 0)$. The collection of initial cylinders is called $\mathbf{C}_0^\Lambda$. We realize the dynamics η_t as a (deterministic) function of $\mathbf{C}^\Lambda$ and $\mathbf{C}_0^\Lambda$.

We construct inductively $\mathbf{K}^\Lambda_{[0,t]}$, the set of *kept* cylinders at time t. The complementary set corresponds to *erased* cylinders. First include all cylinders of $\mathbf{C}_0^\Lambda$ in $\mathbf{K}^\Lambda_{[0,t]}$. Then, move forward in time and consider the first T_i: The corresponding cylinder C_i is erased if: Flag $(C_i) = 1$ and it is incompatible with any of the cylinders already in $\mathbf{K}^\Lambda_{[0,t]}$, otherwise it is kept. This procedure is successively performed mark by mark until all cylinders born before t are considered. Define $\eta_t^\Lambda \in \mathcal{X}^\Lambda$ as

$$\eta_t^\Lambda(\gamma) \quad = \quad \{\text{Basis}\,(C) : C \in \mathbf{K}^\Lambda_{[0,t]},\, \text{Life}\,(C) \ni t\}, \tag{1.5}$$

that is, η_t^Λ is the point process that contains all basis of a kept cylinder that is alive at time t. We leave to the reader to show that η_t^Λ has generator A^Λ defined as in (1.2) restricting the sums to the configurations contained in Λ. It is easy to find an invariant measure μ^Λ for this process (through the equation $\int A\mu(\eta) = 0$). Some regeneration argument should show that η_t^Λ converges in distribution to μ^Λ for any initial configuration η. This in particular implies that μ^Λ is the unique invariant measure for η_t^Λ.

Using the same Poisson marks for η_t^Λ and α_t (the process with constant birth rate c_1 and constant death rate 1), we have

$$\eta_t^\Lambda(A) \;\leq\; \alpha_t(A), \tag{1.6}$$

for all $A \subset \Lambda$ because in the process α_t all cylinders are kept. This implies

$$\mu^\Lambda\{\eta : \eta(A) = 0\} \geq \mathbb{P}\{\alpha : \alpha(A) = 0\}. \tag{1.7}$$

1.3 Backwards oriented percolation

If we try to perform an analogous construction in infinite volume we are confronted with the problem that there is not a first mark. To overcome this we follow the original approach of Harris (1972) (see also Durrett (1997)) and introduce the notion of percolation. The goal is to partition the set of cylinders in finite subsets to which the previous mark-by-mark construction can be applied.

Consider the total order $\prec$ in the set of cylinders induced by the birth times. That is $C \prec C'$ if and only if $\mathrm{Birth}\,(C) \leq \mathrm{Birth}\,(C')$.

For an arbitrary space-time point (x,t) define the set

$$\mathbf{A}_1^{x,t} \;=\; \{C \in \mathbf{C}\,;\, d(x, \mathrm{Basis}\,(C)) \leq t_0,\ \mathrm{Life}\,(C) \ni t\} \tag{1.8}$$

the set of cylinders containing the point (x,t).

For any cylinder C define the set of *ancestors* of C as the set

$$\mathbf{A}_1^C = \{C' \in \mathbf{C}\,;\, C' \prec C\,;\, C' \not\sim C\} \tag{1.9}$$

Notice that the definition of ancestor does not depend on the lifetime of C. Recursively for $n \geq 1$, the nth generation of ancestors are defined as

$$\mathbf{A}_n^{x,t} = \{C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^{x,t}\}. \tag{1.10}$$

and for a given cylinder C,

$$\mathbf{A}_n^C = \{C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^C\}. \tag{1.11}$$

We say that there is *backward oriented percolation* in $\mathbf{C}$ if there exists a space-time point (x,t) such that $\mathbf{A}_n^{x,t} \neq \emptyset$ for all n, that is, there exists a point with infinitely many generations of ancestors. Let the *clan* of the space-time point (x,t) be the union of its ancestors:

$$\mathbf{A}^{x,t} = \bigcup_{n\geq 1} \mathbf{A}_n^{x,t} \tag{1.12}$$

and $\mathbf{C}[0,t] = \{C \in \mathbf{C} : \text{Birth}\,(C) \in [0,t]\}$.

In the next theorem we give a sufficient condition for the existence of the infinite-volume process in any finite time interval in terms of backwards percolation.

Theorem 1.13 *If with probability one* $\mathbf{A}^{x,t} \cap \mathbf{C}[0,t]$ *is finite for any* $x, \in \mathbb{R}^d$ *and* $t \geq 0$*, then for any box* $\Lambda \subset \mathbb{R}^d$*, the process with generator* A^Λ *is well defined and has at least one invariant measure* μ^Λ*.*

Proof. We construct the process for $\Lambda = \mathbb{R}^d$. The construction for other Λ is analogous. The initial distribution is denoted $\eta_0 = \{\varphi_1, \varphi_2, \ldots\}$. For each $\varphi_j \in \eta_0$ let S_j be an independent exponentially distributed random time of mean 1. The time S_j represents the lifetime of the cylinder with basis φ_j, birth time 0 and flag 0. We call $\mathbf{C}(0)$ the set of cylinders $\{(\varphi_j, 0, S_j, 0); \varphi_j \in \eta_0\}$. Since the cylinders in $\mathbf{C}(0)$ have no ancestors in $\mathbf{C}[0,t]$, under the hypothesis of the theorem, every cylinder in $\mathbf{C}(0) \cup \mathbf{C}[0,t]$ has a finite number of ancestors in $\mathbf{C}[0,t]$. We partition this set as follows. As in the finite-volume case, we construct a set of *kept* cylinders, denoted by $\mathbf{K}$, and a set of *deleted* cylinders denoted by $\mathbf{D}$. Let

$$\mathbf{K}_0^{(1)}[0,t] = \mathbf{C}(0) \cup \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C = \emptyset\} \tag{1.14}$$

be the set of cylinders with no ancestors and

$$\mathbf{K}_0^{(2)}[0,t] = \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C \in \mathbf{K}_0[0,t] \quad \text{and} \quad \text{Flag}\,(C) = 0\} \tag{1.15}$$

and

$$\mathbf{D}_0[0,t] = \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C \in \mathbf{K}_0[0,t] \quad \text{and} \quad \text{Flag}\,(C) = 1\} \tag{1.16}$$

be the set of cylinders with ancestors in $\mathbf{K}_0^{(1)}[0,t]$, the ones with Flag equals to 0 are kept, otherwise they are erased. Let

$$\begin{aligned}\mathbf{K}_0[0,t] &= \mathbf{K}_0^{(1)}[0,t] \cup \mathbf{K}_0^{(2)}[0,t] && (1.17)\\ \mathbf{U}_0[0,t] &= \mathbf{K}_0[0,t]. && (1.18)\end{aligned}$$

Inductively, for $k \geq 1$ let

$$\begin{aligned}\mathbf{K}_k^{(1)}[0,t] &= \mathbf{K}_{k-1}[0,t] \cup \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] = \emptyset\Big\} && (1.19)\\ \mathbf{K}_k^{(2)}[0,t] &= \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] \neq \emptyset \quad \text{and} \quad \text{Flag}\,(C) = 0\Big\} && (1.20)\\ \mathbf{D}_k[0,t] &= \mathbf{D}_{k-1}[0,t] \cup \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] \neq \emptyset \quad \text{and} \quad \text{Flag}\,(C) = 1\Big\} && (1.21)\\ \mathbf{K}_k[0,t] &= \mathbf{K}_k^{(1)}[0,t] \cup \mathbf{K}_k^{(2)}[0,t] && (1.22)\end{aligned}$$

where

$$\mathbf{U}_k[0,t] = \Big\{C \in \mathbf{C}[0,t] \setminus \Big[\cup_{i=0}^{k-1}\mathbf{U}_i[0,t]\Big] : \mathbf{A}_1^C \cap \mathbf{U}_{k-1}[0,t] \neq \emptyset\Big\} \tag{1.23}$$

is the set of cylinders being classified at step k. Defining

$$\mathbf{K}[0,t] = \cup_{k\geq 0}\mathbf{K}_k[0,t], \qquad \mathbf{D}[0,t] = \cup_{k\geq 0}\mathbf{D}_k[0,t], \tag{1.24}$$

we have

$$\mathbf{C}[0,t] = \mathbf{K}[0,t] \,\dot\cup\, \mathbf{D}[0,t]. \tag{1.25}$$

The process is now defined as in (1.5) by

$$\eta_t = \{\text{Basis}\,(C) : C \in \mathbf{K}[0,t],\ \text{Life}\,(C) \ni t\}\}. \tag{1.26}$$

The reader can check that if we apply the above construction to the set of cylinders in $C^\Lambda[0,t]$ we obtain $K^\Lambda[0,t]$ as defined in Section 1.2.

Display (1.26) says that the presence in η_t of a point ξ can be established by classifying the ancestors of (ξ,t) in kept and erased cylinders. Since there is no backwards oriented percolation, this classification can be accomplished in a finite number of steps. This idea is used in Ferrari and Garcia (1998) to prove ergodicity of a one-dimensional loss network and in Fernández, Ferrari and Garcia (1998) to get graphical representation of Peierls contours.

It is possible to show that η_t has generator A given by (1.2). $\square$

We show in the next theorem that the process can be constructed for times in the whole real line. Since the construction is time-translation invariant, the distribution of η_t will be invariant.

Theorem 1.27 *If with probability one there is no backwards oriented percolation in* $\mathbf{C}$*, then the process with generator* A *can be constructed in* $(-\infty, \infty)$ *in such a way that the marginal distribution of* η_t *is invariant.*

Proof. The proof follows exactly the same steps as Theorem ?? of Fernández, Ferrari and Garcia (1998).

The lack of percolation allows us to construct a set $\mathbf{K} \subset \mathbf{C}$ as $\mathbf{K}[0,t]$ was constructed from $\mathbf{C}(0) \cup \mathbf{C}[0,t]$ in the proof of the previous theorem. Note that $\mathbf{K}$ is both space and time-translation invariant by construction. Analogously to the previous theorem we define η_t as the section of $\mathbf{K}$ at time t:

$$\eta_t = \{\text{Basis}\,(C) : C \in \mathbf{K},\ \text{Life}\,(C) \ni t\}\}. \tag{1.28}$$

By construction, the distribution of η_t does not depend on t, hence its distribution is an invariant measure for the process. $\square$

Definition 1.29 *The distribution of* η_t *is called* μ*.*

Obs.: Mimicking the proof of Theorem 4.2 one concludes that the spatial birth and death process in a finite box Λ can be constructed for all $t \in \mathbb{R}$. In this case one constructs the set $\mathbf{K}^\Lambda$ using the same specification used to construct $\mathbf{K}$ but using only cylinders in $\mathbf{C}^\Lambda$.

2 Percolation

2.1 The key theorem

The following theorem shows that all that is needed is the absence of backwards and non-oriented percolation. We need a continuous-time construction of the backwards percolation clan. To do this in the infinite-volume case, we need to introduce a notion of non-oriented percolation in a time interval. Fix a time interval (s,t) and for any space-time point (x,t') define

$$\mathbf{G}_0^{x,t'}[s,t] = \{C' \in \mathbf{C}[s,t] : d(x, \text{Basis}\,(C)) \le t_0, \text{Life}\,(C') \ni t'\} \tag{2.1}$$

and

$$\mathbf{G}_n^{x,t'}[s,t] = \{C \in \mathbf{C}[s,t] : d(\text{Basis}\,(C), \text{Basis}\,(C')) \le t_0, \text{ for some } C' \in \mathbf{G}_{n-1}^{x,t'}\}. \tag{2.2}$$

Notice that in the definition of $\mathbf{G}_n$ there is no exigency that the birth time of C' be previous to the birth time of C or that the lifetimes intersect. Let

$$\mathbf{G}^{x,t'}[s,t] = \cup_{k\ge 0} \mathbf{G}_k^{x,t'}. \tag{2.3}$$

We say that there is no (non-oriented) percolation in $[s,t]$ if for any space-time point (x,t'), $\mathbf{G}^{x,t'}[s,t]$ contains a finite number of cylinders.

We will show later that the condition $c_1 \le m^d(B(0,t_0))$ is sufficient for the existence of an h such that the probability that there is no non-oriented percolation in $[0,h]$ is one.

Let the time-length and the space-width of the family of cylinders $\mathbf{A}^{x,t}$ be respectively

$$\begin{aligned} \text{TL}\,(\mathbf{A}^{x,t}) &= t - \sup\{s : \text{Life}\,(C) \ni s, \text{ for some } C \in \mathbf{A}^{x,t}\}, & (2.4)\\ \text{SW}\,(\mathbf{A}^{x,t}) &= |\cup_{C\in\mathbf{A}^{x,t}} \text{Basis}\,(C)|. & (2.5)\end{aligned}$$

We say that two sets of cylinders $\mathbf{A}$ and $\mathbf{A}'$ are incompatible if there is a cylinder in $\mathbf{A}$ incompatible with a cylinder in $\mathbf{A}'$:

$$\mathbf{A} \not\sim \mathbf{A}' \text{ if and only if } C \not\sim C' \text{ for some } C \in \mathbf{A} \text{ and } C' \in \mathbf{A}'. \tag{2.6}$$

Theorem 2.7 *Assume that there is no backwards oriented percolation with probability one. Then,*

1. *Uniqueness. The measure μ is the unique invariant measure for the process η_t.*

2. *Time convergence. For any compact set A,*

$$\lim_{t\to\infty} \sup_A |\mathbb{E}\eta_t^\eta(A) - \mathbb{E}\eta(A)| = 0. \tag{2.8}$$

Furthermore,

$$\begin{aligned}
&\sup_A |\mathbb{E}\eta(A) - \mathbb{E}\eta_t^\eta(A)| \\
&\quad \le \ \mathbb{P}(\cup_{x\in A}\{\mathbf{A}^{x,t} \not\sim \mathbf{C}(0) \ \textit{or} \ \mathrm{TL}\,(\mathbf{A}^{x,t}) > t\}) && (2.9)\\
&\quad \le \ \Big(\mathbb{P}(\cup_{x\in A}\mathrm{TL}\,(\mathbf{A}^{x,0}) > bt) + e^{-(1-b)t}\,\mathbb{E}(\mathrm{SW}\,(\mathbf{A}^{x,0}))\Big) && (2.10)
\end{aligned}$$

for any $b \in (0,1)$.

Existence of μ has been proven in Theorem 4.2. In the rest of the section we prove uniqueness, space and time convergence.

2.2 Time convergence and uniqueness

We use the same Poisson marks to construct simultaneously the stationary process η_t and a process starting at time zero with an arbitrary initial configuration η. The second process is called η_t^η, where $\eta_0^\eta = \eta$. The process η_t^η ignores the cylinders in $\mathbf{C}$ with birth times less than 0 and considers $\mathbf{C}(0) = \{(\varphi_j, 0, S_j, 0) : \varphi_j \in \eta\}$, the set of cylinders with basis given by the initial configuration η and birth time zero —the times S_j are exponentially distributed with mean 1 and independent of everything.

It is enough to prove that

$$\sup_A \mathbb{P}(|\eta_t(A) - \eta_t^\eta(A)| > 0) \to 0 \tag{2.11}$$

as $t \to \infty$.

Since we are using $\mathbf{C}$ to construct η_t and $\mathbf{C}[0,t] \cup \mathbf{C}(0)$ to construct η_t^η, it follows

$$|\eta_t^\eta(A) - \eta_t(A)| \ \le \sum_{x\in A} \mathbf{1}\Big\{\Big(\mathbf{A}^{x,t} \not\sim \mathbf{C}(0) \text{ or } \mathrm{TL}\,(\mathbf{A}^{x,t}) > t\Big)\Big\} \tag{2.12}$$

Note that $\mathbf{A}^{x,t} \neq \emptyset$ for finitely many $x \in A$. The proof of the above results is done similarly as in Fernández, Ferrari and Garcia (1998). The difference is in estimates for the moments of $\mathrm{TL}\,(\mathbf{A}^{x,t}$ and $\mathrm{SW}\,(\mathbf{A}^{x,t})$, which is done through the a dominating branching process (Section 3).

The arguments prove that the process converges, uniformly in the initial configuration, to the invariant measure μ. An immediate consequence is that μ is the unique invariant measure.

3 Branching processes. Time length and space width

In this section we show that the condition $c_1 < (m^d(B(0,t_0)))^{-1}$ implies hypothesis of Theorems 1.13 and 2.7.We also show that under those conditions there is an exponential upper bound for the time length and space width of $\mathbf{A}^{x,t}$. The tool is a domination of the backwards percolation process with a branching process.

3.1 Branching processes

Note that the collection of hypercubes

$$\mathcal{C} = \{(\text{Basis}\,(C) + B(0,t_0)) \times Life(C); C \in \mathbf{C}\} \tag{3.1}$$

is a boolean model (Hall, 1985) and for any $x \in \mathbb{R}^d$ and $t \geq 0$ we have

$$\begin{aligned} \mathbb{P}((x,t) \quad \text{not covered}) &= \mathbb{P}((x,t) \notin ((\text{Basis}\,(C) + B(0,t_0)) \times Life(C) \quad \text{for any } C \in \mathbf{C}) \\ &= e^{-m^d(B(0,t_0))c_1}. \end{aligned} \tag{3.2}$$

Therefore, the number of hypercubes that cover (x,t) is Poisson distributed with mean $m^d(B(0,t_0))c_1$.

Define a Galton-Watson branching process $B_n \in \mathbb{N}$ as follows. Let Y_i^n be i.i.d. non negative integer valued random variables with Poisson distribution with mean $m^d(B(0,t_0))c_1$. Define $B_0 = 1$ and

$$B_{n+1} = \sum_{i=1}^{B_n} Y_i^n \tag{3.3}$$

(with the convention $\sum_{i=1}^{0} Y_i^n = 0$). It is possible to couple the BO-cluster $\mathbf{A}^{x,t}$ and $(B_n)_{n\geq 0}$ in such a way that the number of ancestors in the nth generation of (x,t) is less than or equal to B_n. The total number of ancestors of (x,t) is bounded by

$$\|\mathbf{A}^{x,t}\| \leq \sum_{n\geq 0} B_n. \tag{3.4}$$

Therefore, there is no backward oriented percolation if the process is subcritical, that is,

$$c_1 < (2t_0)^{-d} \tag{3.5}$$

Defining the time length and space width of this clan as in (2.4) and (2.5), we get

$$\mathrm{SW}\,(A^{x,0}) \leq m^d(B(0,t_0))B \tag{3.6}$$

$$\mathrm{TL}\,(A^{x,0}) \leq \sum_{i=1}^{B} \tilde{S}_i \tag{3.7}$$

where

$$B = \sum_{n\geq 0} B_n \tag{3.8}$$

and $\tilde{S}_i, i \geq 1$ are i.i.d. exponentially distributed random variables with mean 1.

Since

$$\mathbb{E}[B] = \frac{1}{1 - m^d(B(0,t_0))c_1} \tag{3.9}$$

we have

$$\mathbb{E}[\mathrm{SW}\ (A^{x,0})] \leq \frac{1}{(2t_0)^{-d} - c_1} \tag{3.10}$$

$$\mathbb{E}[\mathrm{TL}\ (A^{x,0})] \leq \frac{1}{1 - m^d(B(0,t_0))c_1}. \tag{3.11}$$

Moreover, the moment generating function of $\mathrm{TL}\,(A^{x,0})$ is given by

$$\mathbb{E}[a^{\mathrm{TL}\ (A^{x,0})}] = F_B[(1 - \log a)^{-1}] \tag{3.12}$$

where by (13.3) of Harris (1963) $F(b)$, the generating function of Z, must satisfy the equation

$$F(b) = bf(F(b)). \tag{3.13}$$

The largest solution of this is

$$\bar{b} = \bar{a}/f(\bar{a}) \tag{3.14}$$

where $\bar{a}$ is the solution of

$$f'(a) = \frac{f(a)}{a}. \tag{3.15}$$

In this case, it is easy to see that

$$f'(a) = m^d(B(0,t_0))c_1 f(a) \tag{3.16}$$

and $\bar{a}$ is given by

$$\bar{a} = \frac{1}{m^d(B(0,t_0))c_1} \tag{3.17}$$

Therefore,

$$\bar{b} = \frac{1}{m^d(B(0,t_0))c_1 e^{1-m^d(B(0,t_0))c_1}}. \tag{3.18}$$

and consequently,

$$\mathbb{P}[\mathrm{TL}\,(A^{x,0}) > bt] \leq F_B(\bar{b})e^{-bt}. \tag{3.19}$$

4 Another birth rate

Consider a spatial birth and death process with a birth rate

$$\lambda(x,\eta) = c_1 + c_2\mathbf{1}\{\min d(x,y_i) \leq t_0\} \tag{4.1}$$

where $\eta = \{y_1, y_2, \ldots\}$ can be identified with $\eta = \sum_i \delta_{y_i}$ the point process on $\mathbb{R}^d$ and a constant death rate equals to 1.

The graphical construction in this case is very similar, except that we begin with a superposition of two independent homogeneous Poisson point process on $\mathbb{R}^d \times [0,\infty)$ with rates c_1 and c_2. For simplicity we call green the process with rate c_1 and red the process with rate c_2. Denote $N^j = \{(\xi_1^j, T_1^j), (\xi_2^j, T_2^j), \ldots\}$, $j \in \{G, R\}$. For each point (ξ_i^G, T_i^G), associate a independent mark $S_i \sim \exp(1)$ and for each point (ξ_i^G, T_i^G) associate two independent marks $S_i \sim \exp(1)$ and $Z_i \sim b(1, c_2/(c_1 + c2))$.

As before, a marked point (ξ_i, T_i, S_i, Z_i) will be identified with a marked *cylinder* $((\xi_i + [-t_0, t_0]^d) \times [T_i, T_i + S_i), Z_i)$ with *basis* ξ_i, *birth time* T_i, *lifetime* S_i and *flag* Z_i, we will say that all green cylinders have flag equal to 0. Incompatibility between cylinders will be considered only for red cylinders and it is defined as (1.4).

Interpretation: The construction, in this case, will follow similar steps as the previous case, except that all green cylinders will be kept and we have to decide which red cylinders are going to be erased.

In order to construct the process in infinite volume, in particular $\mathbb{R}^d$, we define the set of ancestors of cylinders and points considering both types of cylinders and considering only red cylinders denoting these with a subscript R. In the next theorem we prove that no backward percolation of red cylinders is a sufficient condition for the graphical construction of the birth and death process with rate given by (4.1) and existence of an invariant measure.

Theorem 4.2 *If with probability one $\mathbf{A}_R^{x,t} \cap \mathbf{C}[0,t]$ is finite for any $x, \in \mathbb{R}^d$ and $t \geq 0$, then for any box $\Lambda \subset \mathbb{R}^d$, the process with generator A^Λ is well defined and has at least one invariant measure μ^Λ.*

Proof. We construct the process for $\Lambda = \mathbb{R}^d$. The construction for other Λ is analogous. The initial distribution is denoted $\eta_0 = \{\varphi_1, \varphi_2, \ldots\}$. For each $\varphi_j \in \eta_0$ let S_j be an independent exponentially distributed random time of mean 1. The time S_j represents the lifetime of the cylinder with basis φ_j, birth time 0 and flag 0. We call $\mathbf{C}(0)$ the set of cylinders $\{(\varphi_j, 0, S_j, 0); \varphi_j \in \eta_0\}$. Call these cylinders green. We partition $\mathbf{C}(0) \cup \mathbf{C}[0,t]$ as follows. As before, we construct a set of *kept* cylinders, denoted by $\mathbf{K}$, and a set of *deleted* cylinders denoted by $\mathbf{D}$. Put all green cylinders in K. Let

$$\mathbf{K}_0^{(1)}[0,t] = \mathbf{C}(0) \cup \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C \text{ contains a green cylinder}\} \tag{4.3}$$

be the set of cylinders with green ancestors and

$$\mathbf{K}_0^{(2)}[0,t] = \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C = \emptyset \quad \text{Flag}\,(C) = 0\} \tag{4.4}$$

$$\mathbf{D}_0[0,t] = \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C = \emptyset \quad \text{Flag}\,(C) = 1\} \tag{4.5}$$

be the set of cylinders with no ancestors, the ones with Flag equals to 0 are kept, otherwise they are erased. Let

$$\begin{aligned} \mathbf{K}_0[0,t] &= K_0^{(1)}[0,t] \cup K_0^{(2)}[0,t] && (4.6) \\ \mathbf{U}_0[0,t] &= K_0[0,t]. && (4.7) \end{aligned}$$

Inductively, for $k \geq 1$ let

$$\begin{aligned} \mathbf{K}_k^{(1)}[0,t] &= \mathbf{K}_{k-1}[0,t] \cup \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] \neq \emptyset\Big\} && (4.8) \\ \mathbf{K}_k^{(2)}[0,t] &= \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] = \emptyset \quad \text{and} \quad \text{Flag}\,(C) = 0\Big\} && (4.9) \\ \mathbf{D}_k[0,t] &= \mathbf{D}_{k-1}[0,t] \cup \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] = \emptyset \quad \text{and} \quad \text{Flag}\,(C) = 1\Big\} && (4.10) \\ \mathbf{K}_k[0,t] &= K_k^{(1)}[0,t] \cup K_k^{(2)}[0,t] && (4.11) \end{aligned}$$

where

$$\mathbf{U}_k[0,t] = \left\{C \in \mathbf{C}[0,t] \setminus \left[\cup_{i=0}^{k-1}\mathbf{U}_i[0,t]\right] : \mathbf{A}_1^C \cap \mathbf{U}_{k-1}[0,t] \neq \emptyset\right\} \tag{4.12}$$

is the set of cylinders being classified at step k. Defining

$$\mathbf{K}[0,t] = \cup_{k\geq 0}\mathbf{K}_k[0,t], \qquad \mathbf{D}[0,t] = \cup_{k\geq 0}\mathbf{D}_k[0,t], \tag{4.13}$$

we have

$$\mathbf{C}[0,t] = \mathbf{K}[0,t] \,\dot{\cup}\, \mathbf{D}[0,t]. \tag{4.14}$$

The process is now defined as in (1.26) by

$$\eta_t = \{\text{Basis}\,(C) : C \in \mathbf{K}[0,t],\ \text{Life}\,(C) \ni t\}\}. \tag{4.15}$$

5 Representation through stochastic equations

Consider a birth and death process η_t with birth rate given by $\lambda(x,\eta)$ and constant death rate can be represented as the solution of a system of stochastic equations, see Garcia (1995). Consider N to be a Poisson random measure on $\mathbb{R}^d \times [0,\infty)^3$ with mean measure $m^d \times m \times e^{-r} \times m$ and N_0 be a Poisson random measure, independent of N on $\mathbb{R}^d \times [0,\infty)$ with mean measure $m^d \times e^{-r}$ (to represent the initial configuration). The birth and death process η_t satisfies the stochastic equation:

$$\eta_t(A) = \int_{A\times[0,\infty)^2\times[0,t]} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)\,\mathbf{1}_{[t-s,\infty)}(r)N(dx,du,dr,ds) + \int_{A\times[t,\infty)} N_0(dx,dr) \tag{5.1}$$

Results

R1. The process η_t is stationary, that is

$$\mathbb{E}[\eta_t(A)] = c\, m^d(A) \tag{5.2}$$

for some constant c.

R2. If $\lambda(x,\cdot)$ is non-decreasing, in the sense $\lambda(x,n_1) \leq \lambda(x,n_2)$ for $n_1 \leq n_2$, then the process η_t is attractive, that is

$$\eta_0^1 \leq \eta_0^2 \quad \text{implies} \quad \eta_t^1 \leq \eta_t^2 \tag{5.3}$$

for all $t \geq 0$.

R3. It is true that

$$\mathbb{E}[\lambda(x,\eta_t] = c \tag{5.4}$$

for the same constant c as (5.2).

Proof. We have

$$\begin{aligned}
\mathbb{E}[\eta_t(A)](1-e^{-t}) &= \int_0^t \int_A \mathbb{E}[\lambda(x,\eta_s)]e^{-(t-s)}\,dx\,ds \\
km^d(A)(1-e^{-t}) &= \int_0^t \int_A \mathbb{E}[\lambda(x,\eta_s)]e^{-(t-s)}\,dx\,ds
\end{aligned}$$

and (5.4) follows. □

R4. Let $\eta_t^\emptyset$ be the birth and death process with empty initial configuration, then for any process with arbitrary initial configuration η_0 and for A compact we have

$$\begin{aligned}
\mathbb{E}[\eta_t(A) - \eta_t^\emptyset(A)] &= \int_{A\times[0,\infty)^2\times[0,t]} (\mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u) - \mathbf{1}_{[0,\lambda(x,\eta_s^\emptyset)]}(u))\mathbf{1}_{[t-s,\infty)}(r)N(dx,du,dr,ds) \\
&\quad +\mathbb{E}[\int_{A\times[t,\infty)} N_0(dx,dr)] \\
&= \int_0^t \int_A \mathbb{E}[\lambda(x,\eta_{s-}) - \lambda(x,\eta_s^\emptyset)]e^{-(t-s)}\,dx\,ds + e^{-t}\nu(A)
\end{aligned} \tag{5.5}$$

R5. These processes can be defined starting at time $-T$. In fact let N to be a Poisson random measure on $\mathbb{R}^d \times [0,\infty)^2 \times (-\infty,\infty)$ with mean measure $m^d \times m \times e^{-r} \times m$ and N_0 be a Poisson random measure, independent of N on $\mathbb{R}^d \times [0,\infty)$ with mean measure $m^d \times e^{-r}$ (to represent the initial configuration). Define a birth and death process η_t^T satisfying the stochastic equation:

$$\eta_t^T(A) = \int_{A\times[0,\infty)^2\times[-T,t-T]} \mathbf{1}_{[0,\lambda(x,\eta_{s-}^T)]}(u)\mathbf{1}_{[t-T-s,\infty)}(r)N(dx,du,dr,ds) + \int_{A\times[t,\infty)} N_0(dx,dr) \tag{5.6}$$

then

$$N_T^T \overset{\mathcal{D}}{=} N_T. \tag{5.7}$$

R6. If $\lambda(x,\cdot)$ is non-decreasing for each x and

$$\bar{\lambda} = \sup_{x,n} \lambda(x,n) < \infty. \tag{5.8}$$

then we have ergodicity in certain cases.

Proof. In fact, we can construct two processes $\overline{N}$ and $\underline{N}$ such that

$$\overline{N} = \lim_{T\to\infty} \overline{N}^T \tag{5.9}$$

$$\underline{N} = \lim_{T\to\infty} \underline{N}^T \tag{5.10}$$

where $\overline{N}^T$ and $\underline{N}^T$ satisfy (R5) with initial configurations $\underline{N}_0 \equiv \delta_\emptyset$ and $\overline{N}_0 \sim \text{Poisson}(\mathbb{R}^d \times [0,\infty), \bar{\lambda} m^d \times e^{-r})$. Since $\overline{N}^T$ is stochastically bounded by the birth and death process with constant birth rate $\bar{\lambda}$ and unit death rate, the limit in (5.9) exists. The monotonicity of λ guarantees the existence of the limit (5.10).

Therefore, for initial configurations all full (homogeneous birth and death process) and all empty we have convergence in distribution. The question now is: are the two limits equal?

Construct a sequence of processes $\overline{N}^k$ and $\underline{N}^k$ as

$$\overline{N}^0_t(A) = \int_{A\times[0,\infty)^2\times(-\infty,t]} \mathbf{1}_{[0,\bar{\lambda}]}(u)\mathbf{1}_{[-s,\infty)}(r) N(dx,du,dr,ds) \tag{5.11}$$

$$\overline{N}^k_t(A) = \int_{A\times[0,\infty)^2\times(-\infty,t]} \mathbf{1}_{[0,\lambda(x,\overline{N}^{k-1}_s -)]}(u)\mathbf{1}_{[-s,\infty)}(r) N(dx,du,dr,ds) \tag{5.12}$$

and

$$\underline{N}^0_t(A) = 0$$

$$\underline{N}^k_t(A) = \int_{A\times[0,\infty)^2\times(-\infty,t]} \mathbf{1}_{[0,\lambda(x,\underline{N}^{k-1}_s -)]}(u)\mathbf{1}_{[-s,\infty)}(r) N(dx,du,dr,ds). \tag{5.13}$$

Therefore, it is easy to see that

$$\lim_{k\to\infty} \overline{N}^k_0 = \lim_{T\to\infty} \overline{N}^T_T \tag{5.14}$$

$$\lim_{k\to\infty} \underline{N}^k_0 = \lim_{T\to\infty} \underline{N}^T_T \tag{5.15}$$

and ergodicity follows if (5.14) and (5.15) coincide.

But for any A compact we have by the stationarity of the processes, see (5.2) $\overline{N}$ and $\underline{N}$

$$c^{k+1}\, m^d(A) = \mathbb{E}[\overline{N}^{k+1}_0(A) - \underline{N}^{k+1}_0(A)] = \int_{-\infty}^0 \int_A \mathbb{E}[\lambda(x,\overline{N}^k_s) - \lambda(x,\underline{N}^k_s)]\, e^s\, dx\, ds. \tag{5.16}$$

Case 1: If $\lambda(x,n) = c_1 + c_2 \mathbf{1}\{d(x,n) \le t_0\}$ then (5.16) is bounded by

$$\int_{-\infty}^{0} \int_A c_2 \mathbb{E}[\overline{N}_s^k(B(x,t_0)) - \underline{N}_s^k(B(x,t_0))]\, e^s\, dx\, ds \tag{5.17}$$

$$\le \int_{-\infty}^{0} \int_A c_2\, m^d(B(x,t_0))\, c^k\, m^d(A). \tag{5.18}$$

and

$$c^{k+1} \le c_2\, m^d(B(x,t_0))\, c^k \tag{5.19}$$

which converges if $c_2\, m^d(B(x,t_0)) < 1$.

Case 2: Take any $\lambda(x,n) = \lambda(d(x,n))$, with $\lambda(\cdot)$ non-increasing real function. Then, for any $N^2 \ge N^1$ we have

$$\lambda(d(x,N^2)) - \lambda(d(x,N^1)) \le \int (\lambda(x-y) - \lambda(\infty))(N^2 - N^1)(dy) \tag{5.20}$$

then, by (R1) we have

$$\mathbb{E}[\lambda(d(x,N^2)) - \lambda(d(x,N^1))] \le c \int (\lambda(x-y) - \lambda(\infty))\, dy. \tag{5.21}$$

Substituting (5.21) into (5.16) we get

$$c^{k+1}\, m^d(A) \le \int_{-\infty}^{0} \int_A (\lambda(x-y) - \lambda(\infty))\, e^s\, dx\, ds = c^k \int (\lambda(x-y) - \lambda(\infty))\, dy\ m^d(A) \tag{5.22}$$

which converges if $\int (\lambda(x-y) - \lambda(\infty))\, dy < 1$.

R7. If $\lambda(x,\cdot)$ is non-increasing for each x and

$$\bar{\lambda} = \sup_{x,n} \lambda(x,n) < \infty. \tag{5.23}$$

then we have ergodicity in certain cases.

Proof. Under these hypothesis construct a sequence of processes $\overline{N}^k$ as (5.11) and (5.12). This sequence has the property:

$$\overline{N}^0 \ge \overline{N}^2 \ge \ldots \ge \overline{N}^{2\ell} \ge \ldots \ge \overline{N}^{2\ell+1} \ge \ldots \ge \overline{N}^3 \ge \overline{N}^1. \tag{5.24}$$

Ergodicity will follow from

$$\lim_{\ell\to\infty}] \overline{N}_0^{2\ell} = \lim_{\ell\to\infty}] \overline{N}_0^{2\ell+1}. \tag{5.25}$$

Taking any $\lambda(x,n) = \lambda(d(x,n))$, with $\lambda(\cdot)$ non-decreasing real function. From (5.2) we have

$$c^\ell \, m^d(A) = \mathbb{E}[\overline{N}_0^{2\ell}(A) - \overline{N}_0^{2\ell+1}(A)] \leq c^{\ell-1} \, m^d(A) \int (\lambda(\infty) - \lambda(y)) \, dy \tag{5.26}$$

which converges if $\int (\lambda(\infty) - \lambda(y)) \, dy < 1$.

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