Level-continuity of functions and applications[1]

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Abstract- In this paper, we study the concepts of level-continuity and proper local maximum points of functions defined on a topological space X and, on the one hand, we stablish that, under adequate conditions, f is level-continuous iff f is without proper local maximum points and, on the other, we prove that level-convergence and variational convergence (Γ-convergence) of functions are equivalent when the limit function is level-continuous.

Keywords- Topological spaces, Kuratowski limits, variational convergence.

1. INTRODUCTION

The study of the variational convergence and his applications has been done by many authors, including De Giorgi&Franzoni [1] and Attouch [2] in the setting of the calculus of variations, Greco [3] and Rojas&Román-Flores [4] in convergence of fuzzy sets on locally compact metric spaces and finite dimensional spaces,respectively.

This convergence is based on the Kuratowski limits and one of the most important properties of the Γ-convergence is the preservation of maximum points in Γ-convergents sequences of functions. More precisely: let $\{f_n\}_n$ be a sequence of real functions on X and let x_n be a maximum point of f_n. If $f_n \xrightarrow{\Gamma} f$ and $x_n \to x$, then x is a maximum point of f and $f(x) = \lim_{n\to\infty} f_n(x_n)$.

On the other hand, the level-continuity and level convergence has been used by the author in multivalued characterizations of certain class of maximum points of functions on $\mathbb{R}^n$ ([5]) and compactness of spaces of fuzzy sets on a metric space X ([6]).

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The aim of this paper is, on the one hand, to introduce the concept of level-continuity of functions and to analyze his connections with the existence of proper local maximum points and, on the other, to compare level-convergence (L-convergence) with Γ-convergence.This analysis is carried out in the setting of regular topological spaces, and generalizes the results obtained by the author in [5-6].

This paper is organized as follows. In Section 2 we give the previous results that will be used in the article. In Section 3 we introduce the concept of level-continuity of non-negative real functions defined on X and we study its connections with the existence of proper local maximum points.

Finally, in Section 4 we compare L-convergence with Γ-convergence. Furthermore, some examples are presented.

2. PRELIMINARIES

In the sequel, all topological spaces will be assumed to be *regular* (see [7]), unless specifically stated.

DEFINITION 2.1. *Let $(X, \mathcal{T})$ be a topological space and let $\{A_n\}_{n\in\mathbb{N}}$ a sequence of subsets of X.*

i) *A point $x \in X$ is a limit point of $\{A_n\}_n$ if, for every neighborhood U of x, there is an $n \in \mathbb{N}$ such that for all $m \geq n$, $A_m \cap U \neq \emptyset$.*

ii) *A point $x \in X$ is a cluster point of $\{A_n\}_n$ if, for every neighborhood U of x, and every $n \in \mathbb{N}$, there is an $m \geq n$ such that $A_m \cap U \neq \emptyset$.*

iii) $\liminf\ A_n$ *is the set of all limit points of $\{A_n\}_n$.*

iv) $\limsup\ A_n$ *is the set of all cluster points of $\{A_n\}_n$.*

If $\liminf\ A_n = \limsup\ A_n = A$, then we say A is the limit of the sequence $\{A_n\}_n$, the sequence $\{A_n\}_n$ converges to A (in the Kuratowski sense), and we write $A = \lim\ A_n$ (or $A_n \stackrel{K}{\to} A$).

PROPOSITION 2.2. *If $\{A_n\}_n$ is a sequence of subsets of X, then*

i) $\liminf\ A_n \subseteq \limsup\ A_n$.

ii) $\liminf\ A_n$ *and* $\limsup\ A_n$ *are closed subsets of* X.

iii) $\limsup\ A_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k \geq n} A_k}$

iv) $\liminf\ A_n = \bigcap_{H} \overline{\bigcup_{k \in H} A_k}$, *where* H *denotes an arbitrary cofinal subset of* $\mathbb{N}$ *and the intersection is over all such* H.

For more details see [7-8].

REMARK 2.3. We recall that H is a cofinal subset of $\mathbb{N}$ if $\forall n \in \mathbb{N},\ \exists m \in H$ such that $m > n$.

DEFINITION 2.4. *If* $f : X \to [0, \infty)$ *is a function and* $\alpha \in (0, \infty)$, *then we define the* α*-level and the strict* α*-level of* f *by*

$$\begin{aligned} \{f \ \geq \ \alpha\} &= L_\alpha f = \{x \in X / f(x) \geq \alpha\} \ \textit{and} \\ \{f \ > \ \alpha\} &= \{x \in X / f(x) > \alpha\}, \end{aligned}$$

respectively.

We observe that $\alpha \leq \beta$ implies $L_\alpha f \supseteq L_\beta f$.

DEFINITION 2.5. *Let* $f : X \to [0, \infty)$ *be. Then* $x_0 \in X$ *is said to be a local maximum point of* f *if there is a neighborhood* U *of* x_0 *such that* $f(x) \leq f(x_0)$, *for every* $x \in U$ and $0 < f(x_0) < \sup_{x \in X} f(x)$. DEFINITION 2.6. *Let* $f : X \to [0, \infty)$ *be and* $\sup_{x \in X} f(x) = M$ *(which may be* ∞*). We say that* f *is level-continuous if* $\alpha_p \to \alpha$ *implies* $L_{\alpha_p} f \xrightarrow{K} L_\alpha f,\ \forall \alpha \in (0, M)$.

The following examples shows that continuity and level-continiuty are independent conditions.

EXAMPLE 2.7. Let $X = [0, 1]$ be and $\mathcal{T}$ the usual topology generated by the usual metric on X. Define $f : X \to [0, \infty)$ by

$$f(x) = \begin{cases} 1 - x & if \quad 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & if \quad \frac{1}{2} < x \leq 1. \end{cases}$$

Then is clear that f is continuous.

On the other hand, taking $\alpha_p = \frac{1}{2} + \frac{1}{p}$, $p \geq 2$, we have that

$$L_{\alpha_p} f = \left[0, \frac{1}{2} - \frac{1}{p}\right], \ \forall p.$$

Thus, $\limsup \ L_{\alpha_p} f = \bigcap_{p=1}^{\infty} \overline{\bigcup_{k \geq p} L_{\alpha_k} f} = \bigcap_{p=1}^{\infty} \overline{\bigcup_{k \geq p} \left[0, \frac{1}{2} - \frac{1}{k}\right]} = \left[0, \frac{1}{2}\right]$, whereas $L_{1/2} f = [0, 1]$. Consequently, f is not level-continuous.

Example 2.8. Let $(X, \mathcal{T})$ be as in Example 2.7 and $f : X \to [0, \infty)$ defined by

$$f(x) = \begin{cases} 1 & if \quad x = 1 \\ 0 & if \quad x \neq 1. \end{cases}$$

Then, clearly, f is not continuous.

But, for each $\alpha \in (0, 1)$, we have that $L_\alpha f = \{1\}$. Therefore, f is level-continuous.

Remark 2.9. We observe that $f : X \to [0, \infty)$ it is always left level-continuous, that is, if $\alpha_p \nearrow \alpha$ then $L_{\alpha_p} f \xrightarrow{K} L_\alpha f$. In fact, suppose that $x \in \bigcap_{p=1}^{\infty} \overline{\bigcup_{k \geq p} L_{\alpha_k} f}$. Then

$$x \in \overline{\bigcup_{k \geq p} L_{\alpha_k} f}, \ \forall p. \tag{1}$$

Now, if $f(x) < \alpha$, then there exists p_0 such that $f(x) < \alpha_k$, $\forall k \geq p_0$.

Therefore, $x \notin L_{\alpha_k} f$, $\forall k \geq p_0$.

Because X is regular and $L_{\alpha_{p_0}} f$ is closed, then there exists $U(x)$ such that $U \cap L_{\alpha_{p_0}} f = \emptyset$. But $L_{\alpha_{p_0}} f \supseteq \bigcup_{k \geq p_0} L_{\alpha_k} f$ and, consequently, $U \cap [\bigcup_{k \geq p_0} L_{\alpha_k} f] = \emptyset$, that is, $x \notin \overline{\bigcup_{k \geq p_0} L_{\alpha_k} f}$, in contradiction with (1).

So, must be $f(x) \geq \alpha$ and, consequently, $\limsup \ L_{\alpha_p} f \subseteq L_\alpha f$.

For the reverse inclusion, let $x \in L_\alpha f$ and H a cofinal subset of $\mathbb{N}$.

Then, $x \in L_\alpha f$ implies that $f(x) \geq \alpha \geq \alpha_k$, $\forall k$ and, therefore, $x \in L_{\alpha_k} f$, $\forall k$, which implies $x \in \overline{\bigcup_{k \in H} L_{\alpha_k} f}$.

So, $x \in \bigcap_H \overline{\bigcup_{k \in H} L_{\alpha_k} f}$ where the intersection is over all H cofinal in $\mathbb{N}$.

That is, $x \in \liminf \ L_{\alpha_p} f$. Thus, we can conclude that $\lim \ L_{\alpha_p} f = L_\alpha f$.

3. LEVEL-CONTINUITY AND PROPER LOCAL MAXIMUM POINTS

In this section we shall prove that, under adequate assumptions, level-continuity and non-existence of proper local maximum points are equivalent conditions.

THEOREM 3.1. *Let* $f : X \to [0, \infty)$ *be with* $\sup_{x \in X} f(x) = M$.
If $L_\alpha f$ *is closed* $\forall \alpha$, *then are equivalent:*

i) f *is without proper local maximum points*

ii) $\{f \geq \alpha\} = \overline{\{f > \alpha\}}$, $\forall\ \alpha \in (0, M)$.

iii) f *is level-continuous.*

PROOF. $(i) \to (ii)$. Let $0 < \alpha_0 < M$. Then because $\{f \geq \alpha_0\}$ is closed, it is clear that $\overline{\{f > \alpha_0\}} \subseteq \{f \geq \alpha_0\}$. If we suppose that $\overline{\{f > \alpha_0\}} \neq \overline{\{f > \alpha_0\}}$, then there exists $x_0 \in \{f \geq \alpha_0\} \setminus \overline{\{f > \alpha_0\}}$. Consequently, by regularity of X, there exists $U(x_0)$ such that $U \cap \{f > \alpha_0\} = \emptyset$.

But then, $f(x) \leq \alpha_0 = f(x_0) < M$, $\forall x \in U$. Consequently, x_0 is a proper local maximum point of f, in contradiction with our hypothesis.
$(ii) \to (i)$. Suppose that x_0 is a proper local maximum point of f.

Then, $0 < f(x_0) = \alpha_0 < M$ and there exists a neighborhood $U(x_0)$ of x_0 such that $f(x) \leq f(x_0) = \alpha_0$, $\forall x \in U$.

Therefore, $x_0 \in \{f \geq \alpha_0\}$ and $U \cap \{f > \alpha_0\} = \emptyset$.

Thus, $x_0 \in \{f \geq \alpha_0\} \setminus \overline{\{f > \alpha_0\}}$ and, consequently, $\{f \geq \alpha_0\} \neq \overline{\{f > \alpha_0\}}$.
$(iii) \to (ii)$. Let $\alpha \in (0, M)$ be. We know that $\overline{\{f > \alpha\}} \subseteq \{f \geq \alpha\}$. For the reverse inclusion, let $x_0 \in \{f \geq \alpha\}$ and choose $\alpha_p \searrow \alpha$ (*strictly*).
Thus, by level-continuity of f, must be $L_{\alpha_p} f \xrightarrow{K} L_\alpha f$, that is, $L_\alpha f = \bigcap_{p=1}^{\infty} \overline{\bigcup_{k \geq p} L_{\alpha_k} f}$.

Now, let $U(x_0)$ be an arbitrary neighborhood of x_0.

If $U \cap \{f > \alpha\} = \emptyset$, then $U \cap \{f \geq \alpha_k\} = \emptyset$, $\forall k$, and this implies that $U \cap [\bigcup_{k \geq p} L_{\alpha_k} f] = \emptyset$, $\forall p$. But then, $x_0 \notin \overline{\bigcup_{k \geq p} L_{\alpha_k} f}$, $\forall p$.

Consequently, $x_0 \notin \bigcap_{p=1}^{\infty} \overline{\bigcup_{k \geq p} L_{\alpha_k} f} = L_\alpha f$ which is a contradiction.

Therefore $U \cap \{f > \alpha\} \neq \emptyset$ and $x_0 \in \overline{\{f > \alpha\}}$.

$(ii) \to (iii)$. Suppose that f is not level-continuous in $\alpha_0 \in (0, M)$.

Then there exists a sequence $\{\alpha_p\}$ such that $\alpha_p \to \alpha_0$ and

$$L_{\alpha_p} f \not\to L_{\alpha_0} f. \tag{2}$$

Without loss of generality, due to Remark 2.9, we can suppose $\alpha_p \searrow \alpha_0$ (*strictly*). Thus, $L_{\alpha_k} f \subseteq L_{\alpha_0} f$, $\forall k$, and because $L_{\alpha_0} f$ is closed, we have $\overline{\bigcup_{k \geq p} L_{\alpha_k} f} \subseteq L_{\alpha_0} f$ for all p, that is,

$$\bigcap_{p=1}^{\infty} \overline{\bigcup_{k \geq p} L_{\alpha_k} f} = \limsup\ L_{\alpha_p} f \subseteq L_{\alpha_0} f\ . \tag{3}$$

On the other hand, if $x \in \{f > \alpha_0\}$ then there is p_0 such that $f(x) > \alpha_k$, for all $k \geq p_0$.

Therefore, $x \in L_{\alpha_k} f$, $\forall k \geq p_0$, and this implies that $x \in \overline{\bigcup_{k \in H} L_{\alpha_k} f}$ for every cofinal subset H of $\mathbb{N}$.

Consequently, $x \in \bigcap_{H} \overline{\bigcup_{k \in H} L_{\alpha_k} f} = \liminf\ L_{\alpha_p} f$.

Because $\liminf\ L_{\alpha_p} f$ is closed and $\{f > \alpha_0\} \subseteq \liminf\ L_{\alpha_p} f$, we can conclude that

$$\overline{\{f > \alpha_0\}} = L_{\alpha_0} f \subseteq \liminf\ L_{\alpha_p} f. \tag{4}$$

Thus, by (3) and (4), we have that $L_{\alpha_0} f = \lim\ L_{\alpha_p} f$ which contradicts (2), and the proof of our theorem is complete.■

Remark 3.2. Due to Theorem 3.1, we can conclude that if f is level-continuous then any local maximum of f is a global maximum.

4. LEVEL-CONVERGENCE AND Γ-CONVERGENCE

Let $\mathcal{F}(X) = \{f : X \to [0,\infty)/L_\alpha f \ \textit{closed}, \ \forall \alpha\}$.

DEFINITION 4.1. *(level-convergence). Let* $f_n, f \in \mathcal{F}(X)$. *We say that* f_n *level-converges to* f *(for short :* $f_n \xrightarrow{L} f$*) iff* $\lim L_\alpha f_n = L_\alpha f, \ \forall \alpha$.

DEFINITION 4.2. *(Γ-convergence). Let* $f_n, f \in \mathcal{F}(X)$. *We say that* f_n *Γ-converges to* f *(for short :* $f_n \xrightarrow{\Gamma} f$*) iff* $\lim End(f_n) = End(f)$, *where*

$$End(f) = \{(x,\alpha) \in X \times [0,\infty)/f(x) \geq \alpha\}.$$

THEOREM 4.3. *Let* $f_n, f \in \mathcal{F}(X)$, f *level-continuous. Then, the following conditions are equivalents:*

(i) $f_n \xrightarrow{L} f$

(ii) $f_n \xrightarrow{\Gamma} f$.

PROOF. $(i) \to (ii)$. In order to prove that $f_n \xrightarrow{\Gamma} f$ it is sufficient to prove that

$$\limsup End(f_n) \subseteq End(f) \subseteq \liminf \ End(f_n).$$

Let $(x,\alpha) \in \limsup \ End(f_n)$. Then

$$(x,\alpha) \in \bigcap_{p\geq 1} \overline{\bigcup_{k\geq p} End(f_k)} \ . \tag{5}$$

We want to prove that $(x,\alpha) \in End(f)$, that is, $f(x) \geq \alpha$.

If we suppose that $f(x) < \alpha$, then there is $\epsilon > 0$ such that $f(x) < \alpha - \epsilon < \alpha$.

So, due to $f_n \xrightarrow{L} f$, we obtain that $x \notin L_{\alpha-\epsilon} f = \bigcap_{p\geq 1} \overline{\bigcup_{k\geq p} L_{\alpha-\epsilon} f_k}$.

This implies that $\exists p_0$ such that $x \notin \overline{\bigcup_{k\geq p_0} L_{\alpha-\epsilon} f_k}$ and, therefore, there exists $U(x)$ such that

$$U \cap [\bigcup_{k\geq p_0} L_{\alpha-\epsilon} f_k] = \emptyset. \tag{6}$$

Now, we assure that $[U \times (\alpha - \epsilon)] \cap [\bigcup_{k\geq p_0} End(f_k)] = \emptyset$.

In fact,

$$(y,\beta) \in U\times(\alpha-\epsilon,\infty)\cap[\bigcup_{k\geq p_0} End(f_k)] \Rightarrow \begin{cases} \beta > \alpha-\epsilon \quad and \\ \exists k_0 \geq p_0 \ such\ that\ (y,\beta) \in End(f_{k_0}). \end{cases}$$

Therefore, $f_{k_o}(y) \geq \beta > \alpha - \epsilon$.

But, due to (6), $y \in U(x)$ implies that $y \notin \bigcup_{k\geq p_0} L_{\alpha-\epsilon} f_k$.

That is, $f_k(y) < \alpha - \epsilon$, $\forall k \geq p_0$, which is absurd.

Thus, $U(x) \times (\alpha - \epsilon, \infty)$ is an open set in the product topology nonintersecting to $\bigcup_{k\geq p_0} End(f_k)$.

Because $(x,\alpha) \in U(x) \times (\alpha-\epsilon,\infty)$, we obtain that $(x,\alpha) \notin \overline{\bigcup_{k\geq p_0} End(f_k)}$.

Therefore, $(x,\alpha) \notin \bigcap_{p\geq 1} \overline{\bigcup_{k\geq p} End(f_k)}$, in contradiction with (5).

So, must be $f(x) \geq \alpha$ and, consequently, $(x,\alpha) \in End(f)$.

On the other hand, let $(x,\alpha) \in End(f)$. Then $f(x) \geq \alpha$ and, due to $f_n \xrightarrow{L} f$, we obtain that

$$x \in \liminf\ L_\alpha f_n = \bigcap_H \overline{\bigcup_{k\in H} L_\alpha f_k}. \tag{7}$$

If we suppose that $(x,\alpha) \notin \overline{\liminf\ End(f_n)}$, then there exists H_0 cofinal such that $(x,\alpha) \notin \overline{\bigcup_{k\in H_0} End(f_k)}$.

Therefore, there must exist to exist $V(x,\alpha)$ such that

$$V(x,\alpha) \cap [\bigcup_{k\in H_0} End(f_k)] = \emptyset. \tag{8}$$

Without loss of generality, we can suppose that V is a basic open set of the product topology, that is, V is an open set of form $U \times (\theta,\eta)$ where U is an open in X and (θ,η) is an open interval in $\mathbb{R}^+$contains α. We note that if $y \in U$, then $V = U \times (\theta,\eta)$ contains the segment $\{y\} \times (\theta,\eta)$.

Now, we assure that the projection $p_X(V(x,\alpha))$ is an open set in X , nonintersecting $\bigcup_{k\in H_0} L_\alpha f_k$ (we recall that p_X is an open mapping).

In fact, if we suppose that $p_X(V(x,\alpha)) \cap [\bigcup_{k\in H_0} L_\alpha f_k] \neq \emptyset$, then there exists $y \in p_X(V(x,\alpha))$ such that $f_{k_0}(y) \geq \alpha$, for some $k_0 \in H_0$.

Therefore, $y \in U$ and there is $\beta \leq \alpha$ such that $(y,\beta) \in V(x,\alpha) = U \times (\theta,\eta)$.

But then, $(y,\beta) \in V(x,\alpha) \cap End(f_{k_0}) \subseteq V(x,\alpha) \cap [\bigcup_{k\in H_0} End(f_k)]$, in contradiction with (8).

Because $p_X(V(x,\alpha)) \cap [\bigcup_{k\in H_0} L_\alpha f_k] = \emptyset$ and $x \in p_X(V(x,\alpha))$, we conclude that $x \notin \overline{\bigcup_{k\in H_0} L_\alpha f_k}$ which, due to (7), is absurd.

Summarizing, we must have $(x,\alpha) \in \liminf\ End(f_n)$.

Therefore, $\lim\ End(f_n) = End(f)$, which implies that $f_n \xrightarrow{\Gamma} f$, completing the first part of our proof.

$(ii) \to (i)$. Let $\alpha \in [0,\infty)$ and suppose that $f_n \xrightarrow{\Gamma} f$.

We want to prove that $f_n \xrightarrow{L} f$ and, for this, it is sufficient to prove that

$$\limsup\ L_\alpha f_n \subseteq L_\alpha f \subseteq \liminf\ L_\alpha f_n,\ \forall\alpha.$$

Let

$$x \in \limsup\ L_\alpha f_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k\geq n} L_{\alpha_k} f}. \tag{9}$$

If $f(x) < \alpha$, then $(x,\alpha) \notin \bigcap_{n=1}^{\infty} \overline{\bigcup_{k\geq n} End(f_k)}$.

Therefore, $\exists n_0$ such that $(x,\alpha) \notin \overline{\bigcup_{k\geq n_0} End(f_k)}$.

Consequently, $\exists V(x,\alpha)$ such that

$$V \cap [\bigcup_{k\geq n_0} End(f_k)] = \emptyset. \tag{10}$$

Also, without loss of generality, we can suppose that V is an openv set of form $V = U \times (\theta,\eta)$.

But then, the projection $U = p_X(V(x,\alpha))$ is a neighborhood of x which non-intersecting $\bigcup_{k\geq n_0} L_\alpha f_k$.

In fact, if $y \in U \cap \bigcup_{k\geq n_0} L_\alpha f_k$ then $\exists\beta \leq \alpha$ such that $(y,\beta) \in V$, and $\exists k_0 \geq n_0$ such that $f_{k_0}(y) \geq \alpha \geq \beta$, that is, $(y,\beta) \in End(f_{k_0})$.

Thus, $(y,\beta) \in V \cap [\bigcup_{k\geq n_0} End(f_k)]$ which contradicts (10).

So, $U \cap [\bigcup_{k \geq n_0} L_\alpha f_k] = \emptyset$ but, because $x \in U$, this implies that $x \notin \overline{\bigcup_{k \geq n_0} L_\alpha f_k}$, in contradiction with (9).

Hence $f(x) \geq \alpha$ and, consequently, $x \in L_\alpha f$.

Therefore, $\limsup \ L_\alpha f_n \subseteq L_\alpha f$.

On the other hand, let $x \in L_\alpha f$ and suppose that $f(x) > \alpha$.

Then there is $\epsilon > 0$ such that $f(x) > \alpha + \epsilon$.

So, due to $f_n \xrightarrow{\Gamma} f$, we have that

$$(x, \alpha + \epsilon) \in End(f) = \liminf \ End(f_n) = \bigcap_H \overline{\bigcup_{k \in H} End(f_k)}. \tag{11}$$

Now, if we suppose that $x \notin \liminf \ L_\alpha f_n$, then $\exists H_0$ cofinal such that $x \notin \overline{\bigcup_{k \in H_0} L_\alpha f_k}$ and, therefore, $\exists U(x)$ such that

$$U \cap [\bigcup_{k \in H_0} L_\alpha f_k] = \emptyset. \tag{12}$$

We assure that $[U \times (\alpha, \infty)] \cap \bigcup_{k \in H_0} End(f_k) = \emptyset$.

In fact, if $(y, \beta) \in [U \times (\alpha, \infty)] \cap \bigcup_{k \in H_0} End(f_k)$ then $f_{k_0}(y) \geq \beta > \alpha$ for some $k_0 \in H_0$, and this implies that $y \in U \cap L_\alpha f_{k_0} \subseteq U \cap [\bigcup_{k \in H_0} L_\alpha f_k]$, in contradiction with (12). Thus, because $(x, \alpha + \epsilon) \in U \times (\alpha, \infty)$, we obtain that $(x, \alpha + \epsilon) \notin \overline{\bigcup_{k \in H_0} End(f_k)}$ and, therefore, $(x, \alpha + \epsilon) \notin \liminf \ End(f_n) = End(f)$ which, due to (11), is absurd. So, necessarily, we must have $x \in \liminf \ L_\alpha f_n$ and, consequently, $\{f > \alpha\}$ is contained in $\liminf \ L_\alpha f_n$.

Finally, because $\liminf \ L_\alpha f_n$ is closed and f is level-continuous, by Theorem 3.1, we obtain

$$\overline{\{f > \alpha\}} = \{f \geq \alpha\} = L_\alpha f \subseteq \liminf \ L_\alpha f_n.$$

Consequently, $f_n \xrightarrow{L} f$ and the proof is complete.■

The following example shows that, in Th.4.3 above, the level-continuity condition on f can not be avoided.

EXAMPLE 4.4. Let $X = [0, 2]$ endowed with the usual topology and define (for all $n \geq 2$):

$$f_n(x) = \begin{cases} \frac{x}{n} + 1 - \frac{1}{n} & if \quad 0 \leq x \leq 1 \\ \frac{n}{n-1}(x-1) + 1 & if \quad 1 < x \leq 2 - 1/n \\ 2 & if \quad 2 - 1/n < x \leq 2. \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} & if \quad 0 \leq x \leq 1 \\ 1 & if \quad 1 < x \leq 2. \end{cases}$$

Firstly, we observe that f is not level-continuous. In fact, taking $\alpha_p = 1 + \frac{1}{p}$ we have that $\alpha_p \to 1$ and $L_{\alpha_p} f = [1 + \frac{1}{p}, 2]$, $\forall p$. Therefore $\lim L_{\alpha_p} f = [1, 2]$ whereas $L_1 f = [0, 2]$.

On the other hand, it is easy to see that $L_\alpha f_n, L_\alpha f$ are closed sets $\forall n, \alpha$; f_n is level-continuous for each n and $f_n \xrightarrow{\Gamma} f$, but $\{f_n\}$ does not converge levelwise to f. In fact, for $\alpha = 1$ we have $L_1 f = [0, 2]$ whereas $L_1 f_n = [1, 2]$, $\forall n$, consequently, $\lim L_1 f_n = [1, 2]$.

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