On the Convergence Rate of Spectral Approximations for the Equations of Nonhomogeneous Incompressible Fluids,*†

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Abstract

We study the convergences rate of solutions of spectral semi-Galerkin approximations for the equations for the motion of a nonhomogeneous incompressible fluids in a bounded domain. The model allows density dependent viscosity.

Key words: Convergence rate, semi-Galerkin method.
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1. Introduction

In this paper, we focus on mathematical aspects of a model of the motion of a viscous, nonhomogeneous incompressible fluids by assuming that the kinematic viscosity is density dependent.

This model inclued as particular case the classical Navier-Stokes equations. This case has been much studied (see for instance, Ladyzhenskaya [17], Lions [20], Temam [33] and the referencees there in).

When the kinematic viscosity is a positive constante was studied by Kazhikov [15], (see also [3],[21]), there he proved the existence of weak solution of "Hopf-Type". These results were generalized by Simon [30], [31],[32] and Kim [16] to case that the density is a non-negative constant. The mencionated authors used the semi-Galerkin method.

Stronger local and global solutions were obtained by Ladyzhenskaya and Solonnikov [18] by linearization, fixed point theorem and potential theory and, by Okamoto [23] using evolution operators techniques and also fixed point theorem.

The more constructive spectral semi-Galerkin method was used by Salvi [29] to obtain local in time strong solutions and to study conditions for regularity at $t = 0$ and, by Boldrini and Rojas-Medar [4], [5] to obtain global strong solutions. Others works were done by [10], [24], [25] and [12].

The case when the kinematic viscosity depend up density was studied by Antonzev, Kazhikov and Monakhov [3] p.119 under the Hölder condition.

Analogous results has been obtained by An Ton [2], he proved the existence of a weak solution of Hopf-Type, by using linearization and the method of sucessive approximations; but he assumed restrict hypotheses on the dependence of the kinematic viscosity on the density (see [2], p.101). An Ton, also write "That the Galerkin method or of its variants seems to give rise to difficulties". This apparent difficult was avoided by Fernández-Caras and Guillén [11] (see also Lions [22]).

The strong solutions were showed by Lemoine [19] by using analogous arguments those given by Ladyzhenskaya and Solonnikov [18]. In fact, these results can be proved by spectral semi-Galerkin approximations as in [29] and [5].

In this work, we study the convergences rates in several norms for the spectral semi-Galerkin approximations. Although this is not a too interesting case from the practical point view, we hope that the techniques that we developed here could be adapted in the important case where the full discretizations are used. This question is presently under investigation.

Before we describe our results, let us briefly comment related results.

Rautmann[26] gave a systematic development of error estimates for spectral Galerkin approximations of the classical Navier-Stokes equations(see also[13],[27]). Boldrini and Rojas-Medar gave analogous error estimates for the model of nonhomogeneous viscous incompressible fluids when the viscosity is an one positive constant[6],[7](see also [28])

The paper is organized as follows: in Section 2 we state some preliminaries results that will be useful in the rest of the paper; we describe the approximation method and state the results of existence and uniqueness of Lemoine[19] as also some estimates apriori that form the theoretical basis for the problem. In Section 3 we derive a L^2-error estimate for the velocity and a L^p-error estimate for the density. Finally, in Section 4 we derive H^1-error estimates for velocity .

Finally, we would like to say that, as it usual in this context, to simplify the notation in the expressions we will denote by C, M generics finites positives constants depending only on Ω and the other fixed parameters of the problem (like the initial data) that may have different values in different expressions. In a few points, to emphasize the fact that the constants are different we use $C_1, C_2, \ldots, M_1, M_2, \ldots$ and so on.

2. Preliminaries

The equations for the motion of a nonhomogeneous incompressible fluid are as follows. Being $\Omega \subset I\!R^n, n = 2$ or 3, a $C^{1,1}$-regular bounded domain, $T > 0$, these equations are

$$
\begin{aligned}
\rho\frac{\partial u}{\partial t} + \rho(u.\nabla)u - \nabla(\mu(\rho)(\nabla u +^t \nabla u)) + \nabla p &= \rho f \quad \text{in } \Omega,\\
\text{divu} &= 0 \quad \text{in } \Omega,\\
\frac{\partial \rho}{\partial t} + u.\nabla\rho &= 0 \quad \text{in } \Omega,\\
u &= 0 \quad \text{on } \partial\Omega\times]0,T[,\\
\rho(x,0) &= \rho_0(x) \quad \text{in } \Omega,\\
u(x,0) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
\tag{2.1}
$$

where $[0,T[$ is the interval of time being considered; Ω is the container where the fluid is in; $u(x,t) \in I\!R^n$ denotes the velocity of the fluid at point $x \in \Omega$ and time

$t \in [0,T[; \rho(x,t) \in I\!R$ and $p(x,t) \in I\!R$ denote respectively, the density and the hydrostatic pressure of the fluid; $u_0(x)$ and $\rho_0(x)$ are the initial velocity and density respectively; $f(x,t)$ is the density by unit of mass of the external force acting on the fluid and $\mu(.)$ is the viscosity of fluid ; the fluid adheres to the wall $\partial\Omega$ of the container which is at rest. The expressions ∇, Δ and div denote the gradient, Laplacian and divergence operators, respectively (we also denote $\dfrac{\partial u}{\partial t}$ by u_t); the i^{th} component of $u.\nabla u$ is given by $[(u.\nabla)u]_i = \displaystyle\sum_j u_j \frac{\partial u_i}{\partial x_j}$ and $u.\nabla\rho = \sum_j u_j \frac{\partial \rho}{\partial x_j}$. The first equation in (1.1) corresponds to the balance of linear momentum; the third equation to the balance of mass and the second one states that fluid is incompressible. The unknowns in the problem are u, ρ and p.

Let $\Omega \subset I\!R^n,\ n = 2$ or 3, be a bounded domain with smooth boundary Γ (class $C^{1,1}$ is enough).

We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D),\ ||\partial^\alpha f||_{L^q(D)} < +\infty,\ |\alpha| \leq m\}$$

$m = 0, 1, 2, \ldots, 1 \leq q \leq +\infty,\ D = \Omega$ or $\Omega \times]0,T[,\ 0 < T < +\infty$, with the usual norm. When $q = 2$, we denote by $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(D)$ in $H^m(D)$. If B is a Banach space, we denote by $L^q(0,T;B)$ the Banach space of the B-valued functions defined in the interval $[0,T]$ that are L^q-integrables in the sense of Bochner. We shall consider the following spaces of divergence free functions

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &= \{v \in (C_0^\infty(\Omega))^n \ /\ \text{div } v = 0 \quad \text{in} \quad \Omega\}, \\ H &= \quad \text{closure of} \quad C_{0,\sigma}^\infty(\Omega) \quad \text{in} \quad (L^2(\Omega))^n \\ V &= \quad \text{closure of} \quad C_{0,\sigma}^\infty(\Omega) \quad \text{in} \quad (H^1(\Omega))^n. \end{aligned}$$

Throughout the paper, P denotes the orthogonal projection from $(L^2(\Omega))^n$ into H and $A = -P\Delta$ with $D(A) = V \cap H^2(\Omega)$ is the usual Stokes operator.

We will denote by $\varphi^n(x)$ and λ_n the eigenfunction and eigenvalue of A. It is well know that $\{\varphi^n\}_{n=1}^\infty$ form an orthogonal complete system in the spaces H, V and $H^2(\Omega) \cap V$, with their usual inner products (u,v), $(\nabla u, \nabla v)$ and (Au, Av) respectively.

For each $n \in I\!N$, we denote by P_n the orthogonal projections from $L^2(\Omega)$ onto $V_n = span\{\varphi^1(x), ..., \varphi^n(x)\}$. To more details on the Stokes operator see Temam [33].

We observe that for the regularity of the Stokes operator, it is usually assumed that Ω is of class C^3; this being in order to use Cattabriga's results[9].We use instead the stronger results of Amrouche and Girault[1] which implies, in particular, that when $Au \in L^2(\Omega)$, then $u \in H^2(\Omega)$ and $||u||_{H^2}$ and $||Au||$ are equivalent norms when Ω is of Class $C^{1,1}$. Here $||\cdot||$ denotes the L^2-norm; also in this paper we will denote the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$.

The following assumptions on the initial data was employed by Lemoine[19] :
$(A1)$ The initial value ρ_0 belongs to $C^1(\overline{\Omega})$ and satisfies

$$\min_{x\in\overline{\Omega}} \rho_0(x) > 0.$$

Hereafter he put $M_1 = \min_{x\in\overline{\Omega}} \rho_0(x)$ and $M_2 = \max_{x\in\overline{\Omega}} \rho_0(x)$.
$(A2)$ The initial value u_0 belongs to $W^{2-2/q,q}(\Omega)$, where $\nabla.u_0 = 0, u_0|_{\partial\Omega} = 0$ with $q > 3$.
$(A3)$ $\mu \in C^1(]0,\infty[), \mu(a) \geq \mu_1 > 0$ for all $a > 0$,
$(A4)$ $f \in L^q(Q_T)$.

Remark 1. *Since $\mu \in C^1(]0,\infty[)$,we have that for all $T > 0$ finite, there exist positives constants $\mu_{2,}\mu_1', \mu_2'$ such that*

$$\begin{aligned} 0 &< \mu_1 \leq \mu(a) \leq \mu_2, \\ 0 &< \mu_1' \leq \mu'(a) \leq \mu_2' \end{aligned}$$

Under these hypotheses, he proved the following result:

Theorem 2.1. *([19], p.698). There exists $T_0 \leq T$ such that the equations (2.1) have a solution $(u, \nabla p, \rho)$ wich satisfies*

$$u \in \mathcal{W}_q^{2,1}(Q_{T_0}), \nabla p \in L^q(Q_{T_0}), \rho \in C^1(\overline{Q_{T_0}}).$$

Moreover, there exists $R > 0$ *depending on* Ω, μ, T, ρ_0 *such that if*

$$\|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-2/q,q}(\Omega)} \leq R,$$

then $(u, \nabla p, \rho)$ *is a solution of (2.1) for* $T_0 = T$.

Here $\mathcal{W}_q^{2,1}(Q_t)$ denote the space of distributions $u \in L^q(0,T; W^{2,q}(\Omega))$ such that $\partial_t u \in L^q(Q_t)$ with their natural norm.

We can rewrite the problem (2.1), by using the orthogonal projection P, as follows

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + u.\nabla \rho &= 0 \quad (\text{a.e.} \quad (x,t) \in \Omega \times]0,T[\,), \\
P(\rho(t)\frac{\partial u}{\partial t} - \nabla.((\mu(\rho(t))(\nabla u +^t \nabla u))) &= P(-\rho(t)u.\nabla u + \rho(t) f) \\
(0 &< t < T), \\
u(0) = u_0(x), \qquad & \rho(x,0) = \rho_0(x).
\end{aligned}
\tag{2.2}
$$

or, equivalently

$$
\begin{cases}
\rho_t + u \cdot \nabla \rho = 0 \quad \text{for} \quad (x,t) \in \Omega \times (0,T), \\
\\
(\rho u_t, v) + (\rho u \cdot \nabla u, v) + (\mu(\rho)(\nabla u +^t \nabla u), \nabla v) \\
\qquad = (\rho f, v) \quad \text{for} \quad 0 < t < T, \ \forall\, v \in V, \\
\\
u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x,0) = \rho_0(x).
\end{cases}
\tag{2.3}
$$

The spectral semi-Galerkin approximations for (u, ρ) are defined for each $n \in \mathbf{N}$ as the solution $(u^n, \rho^n) \in C^2([0,T]; V_n) \times C^1(\overline{Q_{T_0}})$ of

$$
\begin{cases}
\rho_t^n + u^n \cdot \nabla \rho^n = 0 \quad \text{for} \quad (x,t) \in \Omega \times (0,T), \\
\\
(\rho^n u_t^n, v) + (\rho^n u^n \cdot \nabla u^n, v) + (\mu(\rho^n)(\nabla u^n +^t \nabla u^n), \nabla v) \\
\qquad = (\rho^n f, v) \quad \text{for} \quad 0 < t < T, \ \forall\, v \in V_n, \\
\\
u^n(x,0) = P_n u_0(x), \quad \rho^n(x,0) = \rho_0(x), x \in \Omega.
\end{cases}
\tag{2.4}
$$

Applying the analogous arguments of Lemoine, we obtain

Corollary 2.2. *The approximations of spectral semi-Galerkin satisfy the following estimates uniform in n*

$$
\|\nabla u^n(t)\|^2 + \int_0^t \|u_\tau^n(\tau)\|^2 d\tau \ \ \leq \ \ C, \tag{2.5}
$$

$$
\begin{aligned}
\int_0^t \|Au^n(t)\|^2 &\leq C, & (2.6)\\
\|\rho^n\|_{L^\infty} &\leq C, & (2.7)\\
\|\nabla\rho^n\|_{L^\infty} &\leq C, & (2.8)\\
\|\rho^n_t\|_{L^\infty} &\leq C. & (2.9)
\end{aligned}
$$

We note that whence ρ, u satifies (2.1)i, we have $\rho_t = -\mathrm{div}(\rho u)$, then for $v, \omega \in H^1(\Omega)$, using integration by parts, result

$$
(\rho_t v, \omega) = -(\mathrm{div}(\rho u)v, \omega) = (\rho u.\nabla v, \omega) + (\rho u.\nabla\omega, v). \qquad (2.10)
$$

Also,

$$
(\rho\omega_t, \omega) = \frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\omega\|^2 - \frac{1}{2}(\rho_t\omega, \omega). \qquad (2.11)
$$

To prove our results we use the following results.

Lemma 2.3. *Let $u \in V$ then the following estimate holds (see[25])*

$$
\|u\|_{L^3} \leq C\|u\|^{1/2}\|\nabla u\|^{1/2}.
$$

Moreover, if $u \in V \cap H^2(\Omega)$ (see [8])

$$
\|u\|_{L^\infty} \leq C\|\nabla u\|^{1/2}\|Au\|^{1/2}.
$$

Lemma 2.4. *Let $v \in V \cap H^2(\Omega)$ and consider the helmoltz decomposition of $-\triangle v$, that is,*

$$
-\triangle v = Av + \nabla q,
$$

where $q \in H^1(\Omega)$ is taken such that $\int_\Omega q dx = 0$ (recall that A is the Stokes operator). Then, for every $\varepsilon > 0$ there exists a positive constant C_ε independent of v such that the following estimate holds

$$
\|q\| \leq C\varepsilon\|\nabla v\| + \varepsilon\|Av\|.
$$

To prove our results we will precise more regularity on the initial data. We assume the following

$(A1')$ $\rho_0 \in C^1(\overline{\Omega})$

$(A2')$ $u_0 \in D(A)$

$(A3')$ $f \in L^2(0,T;H^1(\Omega))$

$(A4')$ $f_t \in L^2(Q_T)$.

With these regularity, we will prove the following more strong result.

Proposition 2.5. *We assume $(A1')$, $(A2'),(A3')$ and $(A4')$. The dimension of Ω may be two or three. The unique solution given by the Theorem 2.1 satisfies*

$$\begin{aligned} \rho &\in C^1(\overline{\Omega}\times[0,T_0]), \\ u &\in L^\infty([0,T_0];\ D(A)), \\ u_t &\in L^\infty([0,T_0];H)\cap L^2(0,T_0;V). \end{aligned}$$

Moreover, there hold the following estimates uniformly in n for the approximations

$$\|u_t^n(t)\|^2 + \int_0^t \|\nabla u_\tau^n(\tau)\|^2 d\tau \leq C, \tag{2.12}$$

$$\sup_t \|Au^n(t)\|^2 \leq C. \tag{2.13}$$

Proof.

Proof the estimate (2.12):

Differentiating the velocity equation (2.4) with respect to t, we have

$$\begin{aligned} &(\rho_t^n u_t^n + \rho u_{tt}^n, v) - (\nabla((\mu'(\rho^n)\rho_t^n(\nabla u^n +^t \nabla u^n), v) - (\nabla((\mu(\rho^n)(\nabla u_t^n +^t \nabla u_t^n)), v) \\ = &(\rho_t^n f, v) + (\rho^n f_t, v) - ((\rho^n u^n.\nabla u^n)_t, v) \end{aligned}$$

and setting $v = u_t^n(t) \in V$ and using (2.11) with $\omega = u_t^n$,we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(\rho^n)^{1/2}u_t^n\|^2 + \|(\mu(\rho^n)^{1/2}\nabla u_t^n\|^2 &= (\rho_t^n f, u_t^n) + (\rho^n f_t, u_t^n) \\ &+\frac{1}{2}(\rho_t^n u_t^n, u_t^n) - ((\rho^n u^n.\nabla u^n)_t, u_t^n) + ((\mu'(\rho^n)\rho_t^n\nabla u^n, \nabla u_t^n). \end{aligned}$$

On the other hand,

$$((\rho^n u^n.\nabla u^n)_t, u_t^n) = (\rho_t^n u^n.\nabla u^n, u_t^n) + (\rho^n u_t^n.\nabla u^n, u_t^n) + (\rho^n u^n.\nabla u_t^n, u_t^n).$$

Consequently,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\rho^n)^{1/2}u_t^n\|^2 + \|(\mu(\rho^n)^{1/2}\nabla u_t^n\|^2 \;=\;& \frac{1}{2}(\rho_t^n u_t^n, u_t^n) - (\rho_t^n u^n.\nabla u^n, u_t^n) \\ & -(\rho^n u_t^n.\nabla u^n, u_t^n) + (\rho^n u^n.\nabla u_t^n, u_t^n) \\ & +(\rho_t^n f_t, u_t^n) + (\rho^n f_{t,} u_t^n) \qquad (2.14) \\ & +((\mu'(\rho^n)\rho_t^n \nabla u^n, \nabla u_t^n).\end{aligned}$$

On the other hand, recalling the estimates (2.7) and (2.9), we have

$$\begin{aligned}|(\rho_t^n u_t^n, u_t^n)| \;&\leq\; \|\rho_t^n\|_{L^\infty}\|u_t^n\|^2 \leq C\|u_t^n\|^2, \\ |(\rho^n f_t, u_t^n)| \;&\leq\; \|\rho^n\|_{L^\infty}\|u_t^n\|\|f_t\| \leq C\|u_t^n\|^2 + C\|f_t\|^2.\end{aligned}$$

Also, using the Lemma 2.3

$$\begin{aligned}|(\rho^n u^n.\nabla u_t^n, u_t^n)| \;&\leq\; \|\rho^n\|_{L^\infty}\|u^n\|_{L^6}\|\nabla u_t^n\|\|u_t^n\|_{L^3} \\ &\leq\; C\|\nabla u^n\|\|u_t^n\|^{1/2}\|\nabla u_t^n\|^{3/2} \\ &\leq\; C_\varepsilon\|\nabla u^n\|^4\|u_t^n\|^2 + \varepsilon\|\nabla u_t^n\|^2,\end{aligned}$$

$$\begin{aligned}|(\rho^n u_t^n.\nabla u^n, u_t^n)| \;&\leq\; \|\rho^n\|_{L^\infty}\|u_t^n\|_{L^3}\|\nabla u^n\|\|u_t^n\|_{L^6} \\ &\leq\; C\|u_t^n\|^{1/2}\|\nabla u^n\|\|\nabla u_t^n\|^{3/2} \\ &\leq\; C_\varepsilon\|\nabla u^n\|^4\|u_t^n\|^2 + \varepsilon\|\nabla u_t^n\|^2,\end{aligned}$$

and

$$\begin{aligned}|(\rho_t^n u^n.\nabla u^n, u_t^n)| \;&\leq\; \|\rho_t^n\|_{L^\infty}\|u^n\|_{L^3}\|\nabla u^n\|\|u_t^n\|_{L^6} \\ &\leq\; C_\varepsilon\|\nabla u^n\|^4 + \varepsilon\|\nabla u_t^n\|^2.\end{aligned}$$

Choosing $\varepsilon = \dfrac{\mu_1}{6}$ and in (2.13) integrating of 0 to t, we obtain

$$\|(\rho^n(t))^{1/2}u_t^n(t)\|^2 + \mu_1\int_0^t \|\nabla u_\tau^n(\tau)\|^2 d\tau$$

$$\leq \|(\rho_0^n)^{1/2} u_t^n(0)\|^2 + C\int_0^t (\|u_\tau^n(\tau)\|^2) d\tau$$
$$+C + C\int_0^t \|\nabla u^n(\tau)\|^4 \|u_\tau^n(\tau)\|^2 d\tau$$
$$\leq C\|u_t^n(0)\|^2 + C + C\int_0^t \|\nabla u^n(\tau)\|^4 \|u_\tau^n(\tau)\|^2 d\tau$$

by using (2.5) and the hypotheses on f. From (2.2), easily we deduce

$$\begin{aligned} \|u_t^n(0)\| &\leq C\|Au^n(0)\| + C\|Au^n(0)\|\|\nabla u^n(0)\| + C\|\rho_0^n f(0)\| \\ &\leq C. \end{aligned}$$

Consequently, since $\|(\rho^n)^{1/2} u_t^n(t)\|^2 \geq M_1 \|u_t^n(t)\|^2$, we have

$$M_1\|u_t^n(t)\|^2 + \mu_1 \int_0^t \|\nabla u_t^n(\tau)\|^2 d\tau \leq C + C\int_0^t \|\nabla u^n(\tau)\|^4 \|u_\tau^n(\tau)\|^2 d\tau$$

and using the Gronwall's inequality

$$\begin{aligned} M_1\|u_t^n(t)\|^2 + \mu_1 \int_0^t \|\nabla u_\tau^n(\tau)\|^2 d\tau & \\ &\leq C \times \exp(\int_0^t \|\nabla u^n(\tau)\|^4 d\tau) \\ &\leq C \qquad (2.15) \end{aligned}$$

by virtue (2.5).

Proof the estimate (2.13):

Taking $v = Au^n$ in (2.4), we find

$$-(\nabla.((\mu(\rho^n(t))(\nabla u^n +^t \nabla u^n)), Au^n) = (\rho^n f, Au^n) - (\rho^n u_t^n, Au^n) - (\rho^n u^n.\nabla u^n, Au^n)$$

Using the identity

$$\nabla.((\mu(\rho^n(t))(\nabla u^n +^t \nabla u^n)) = \mu'(\rho^n)\nabla\rho^n(\nabla u^n +^t \nabla u^n) + \mu(\rho^n)\triangle u^n. \quad (2.16)$$

where $\nabla\rho^n(\nabla u^n +^t \nabla u^n)$ denotes the vector field whose ith component is given by $[\nabla\rho^n(\nabla u^n +^t \nabla u^n)]_i = (\nabla\rho^n, \nabla u_i^n +^t \nabla u_i^n)_{I\!R^N}$, we find

$$\begin{aligned} -(\mu(\rho^n)\triangle u^n, Au^n) &= (\rho^n f, Au^n) - (\rho^n u_t^n, Au^n) - (\rho^n u^n.\nabla u^n, Au^n) \\ &\quad +(\mu'(\rho^n)\nabla\rho^n(\nabla u^n +^t \nabla u^n), Au^n) \end{aligned}$$

Since $Au^n \neq -\Delta u^n$, we need the Helmholtz decomposition $-\Delta u^n = Au^n + \nabla q^n$, for some q^n with $\int_\Omega q^n = 0$, and the following estimates given by Lemma 2.4: there exist $C > 0$ and, for any $\varepsilon > 0$, $C_\varepsilon > 0$, constants independent of n and such that

$$\|q^n\| \leq C_\varepsilon\|\nabla u^n\| + \varepsilon\|Au^n\| \ \text{ and } \ ||q^n||_1 \leq C\|Au^n\|. \tag{2.17}$$

Consequently,

$$\begin{aligned} (\mu(\rho^n)Au^n, Au^n) &= (\rho^n f, Au^n) - (\rho^n u_t^n, Au^n) - (\rho^n u^n.\nabla u^n, Au^n) \\ &\quad +(\mu'(\rho^n)\nabla\rho^n(\nabla u^n +^t \nabla u^n), Au^n) - (\mu(\rho^n)\nabla q^n, Au^n) \end{aligned}$$

We observe that, because $\text{div}Au^n = 0$,

$$(\mu(\rho^n)\nabla q^n, Au^n) = -(q^n, \text{div}\mu(\rho^n)Au^n) = -(q^n, \mu'(\rho^n)\nabla\rho^n Au^n).$$

Consequently, by using (2.16)

$$\begin{aligned} |(\mu(\rho^n)\nabla q^n, Au^n)| &\leq \mu_1'\|q^n\|\|\nabla\rho^n\|_{L^\infty}\|Au^n\| \\ &\leq C\|q^n\|\nabla\rho^n\|_{L^\infty}\|Au^n\| \\ &\leq C(C_\varepsilon\|\nabla u^n\| + \varepsilon\|Au^n\|)\|\nabla\rho^n\|_{L^\infty}\|Au^n\| \\ &\leq C_\varepsilon\|\nabla\rho^n\|_{L^\infty}^2\|\nabla u^n\|^2 + \varepsilon C\|\nabla\rho^n\|_{L^\infty}^2\|Au^n\|^2 \end{aligned}$$

Thus, we have

$$\begin{aligned} \|Au^n\| &\leq \frac{1}{\mu_1}(\|\rho^n\|_{L^\infty}\|f\| + C\|\rho^n\|_{L^\infty}\|u_t^n\| + \mu_1'\|\nabla\rho^n\|_{L^\infty}\|\nabla u^n\|^2) \\ &\quad +\frac{C}{\mu_1}\|\rho^n\|_{L^\infty}\|u^n\|_{L^6}\|\nabla u^n\|_{L^3} + \frac{C}{\mu_1}\|\nabla\rho^n\|_{L^\infty}^2\|\nabla u^n\|^2. \end{aligned}$$

Now, by using the Lemma 2.3,we have

$$\begin{aligned}\frac{C}{\mu_1}\|\rho^n\|_{L^\infty}\|u^n\|_{L^6}\|\nabla u^n\|_{L^3} &\leq \frac{C}{\mu_1}\|\rho^n\|_{L^\infty}\|\nabla u^n\|^{3/2}\|Au^n\|^{1/2} \\ &\leq \frac{C}{2\mu_1}\|\rho^n\|_{L^\infty}^2\|\nabla u^n\|^3 + \frac{1}{2}\|Au^n\|.\end{aligned}$$

Using the above estimate together with Corollary 2.2 , we obtain the estimate (2.13).

From the above estimates follows the regularity of the solution (u, ρ) give by the Proposition 2.5.

The following results can be found in Rautmann [26].

Lemma 2.6. *If $u \in V$ and $w \in H_0^1(\Omega)$, then there holds*

$$\|u - P_n u\|^2 \leq \frac{1}{\lambda_{n+1}}\|\nabla u\|^2.$$

Also, if $u \in V \cap H^2(\Omega)$, we have

$$\begin{aligned}\|u - P_k u\|^2 &\leq \frac{1}{\lambda_{k+1}^2}\|Au\|^2, \\ \|\nabla u - \nabla P_k u\|^2 &\leq \frac{1}{\lambda_{k+1}}\|Au\|^2.\end{aligned}$$

3. L^2-error estimates for velocity and L^r-error estimates for density

From now on, for simplicity of notation, we will write $T_0 \equiv T$.

In this section we give the H^1-error estimate for the velocity and L^r-error estimate for density. The analysis is more difficult that in the case of constant viscosity
.

Let (u,ρ) be the strong solution of problem (2.2) (or (2.3)) given by PROPOSITION 2.5 and (u^n,ρ^n) the approximate solution of problem (2.4).

We define

$$w^n = P_n u - u^n, \quad \sigma^n = \rho - \rho^n, \quad \eta^n = u - P_n u.$$

With these notations, we observe that w^n and σ^n satisfy the following equations

$$\begin{aligned}&(\rho w^n_t, v) + (\mu(\rho)\nabla w^n, \nabla v) = (\sigma^n f, v) + (\sigma^n u^n_t, v) - (\sigma^n u.\nabla u, v)\\ &-(\rho^n w^n.\nabla u, v) - (\rho^n u^n.\nabla w^n, v) - ((\mu(\rho) - \mu(\rho^n))\nabla u^n, \nabla v)\\ &-(\rho\eta^n_t, v) - (\rho^n\eta^n\nabla u, v) - (\rho^n u^n\nabla\eta^n, v) - (\mu(\rho)\nabla\eta^n, \nabla v)\end{aligned} \tag{3.1}$$

$$\sigma^n_t + u^n.\nabla\sigma^n = -w^n.\nabla\rho - \eta^n\nabla\rho. \tag{3.2}$$

To obtain the H^1-estimate, we will need the following Lemma.

Lemma 3.1. *Under the conditions of Proposition 2.5, for $2 \le r \le 6$, we have*

$$\|\sigma^n(t)\|^2_{L^r} \le C\int_0^t (\|w^n(\tau)\|^2_{L^r} + \|\eta^n(\tau)\|^2_{L^r})d\tau.$$

Proof. Since Ω is bounded, we have $L^6 \hookrightarrow L^r$ for $2 \le r \le 6$, moreover it is sufficient to prove the Lemma for $r = 6$.

Multiplying (3.2) by $|\sigma^n|^5$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{6}\frac{d}{dt}\int_\Omega |\sigma^n|^6 dx &= -\int_\Omega w^n.\nabla\rho|\sigma^n|^5 dx - \frac{1}{6}\int_\Omega u^n.\nabla|\sigma^n|^6 dx - \int_\Omega \eta^n.\nabla\rho|\sigma^n|^5 dx\\ &\le \int_\Omega |w^n||\nabla\rho||\sigma^n|^5 dx + \frac{1}{6}\int_\Omega \text{ div } u^n|\sigma^n|^6 dx + \int_\Omega |\eta^n||\nabla\rho||\sigma^n|^5 dx\\ &\le \|\nabla\rho\|_{L^\infty}\int_\Omega |w^n||\sigma^n|^5 dx + \|\nabla\rho\|_{L^\infty}\int_\Omega |\eta^n||\sigma^n|^5 dx\\ &\le C\{\Big(\int_\Omega |w^n|^6 dx\Big)^{1/6} + \Big(\int_\Omega |\eta^n|^6 dx\Big)^{1/6}\}\Big(\int_\Omega |\sigma^n|^6 dx\Big)^{5/6}\end{aligned}$$

where we used the estimate (2.8). This implies

$$\frac{1}{6}\frac{d}{dt}\|\sigma^n\|^6_{L^6} \le C(\|w^n\|_{L^6} + \|\eta^n\|_{L^6})\|\sigma^n\|^5_{L6}$$

but,

$$\frac{1}{6}\frac{d}{dt}\|\sigma^n\|_{L^6}^6 = \|\sigma^n\|_{L^6}^5 \frac{d}{dt}\|\sigma^n\|_{L^6},$$

then, since $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\frac{d}{dt}\|\sigma^n\|_{L^6} \leq C(\|w^n\|_{L^6} + \|\eta^n\|_{L^6}).$$

Integrating from 0 to t the last inequality and applying the Cauchy-Schwartz inequality, we have

$$\|\sigma^n(t)\|_{L^6} \leq C\int_0^t \|w^n(s)\|_{L^6} ds + C\int_0^t \|\eta^n(s)\|_{L^6} ds.$$

Proposition 3.2. *Under the hypotheses of Proposition 2.5, we have*

$$\|w^n(t)\|^2 + \int_0^t \|\nabla w^n(s)\|^2 ds \leq \frac{G_1(t)}{\lambda_{n+1}^2} + \frac{G_1(t)}{\lambda_{n+1}} \tag{3.3}$$

Proof. Setting $v = w^n(t)$ in (3.1), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(\rho)^{1/2}w^n\|^2 + \|(\mu(\rho))^{1/2}\nabla w^n\|^2 \quad = \quad & \frac{1}{2}(\rho_t w^n, w^n) + (\sigma^n f, w^n) - (\sigma^n u.\nabla u, w^n) \\
& -(\rho^n w^n.\nabla u, w^n) + (\sigma^n u_t^n, w^n) \\
& -(\rho^n u^n.\nabla w^n, w^n) - (\rho^n \eta^n \nabla u, w^n) \\
& -(\rho^n u^n \nabla \eta^n, w^n) - (\mu(\rho)\nabla \eta^n, \nabla w^n) \\
& -(((\mu(\rho) - \mu(\rho^n))\nabla u^n, \nabla w^n) \\
& -(\rho \eta_t^n, w^n).
\end{aligned} \tag{3.4}$$

By using the Hölder's inequality and Sobolev imbedding $H^2 \hookrightarrow L^\infty$, $H^1 \hookrightarrow L^6$, we obtain the following estimates

$$\begin{aligned}
|\frac{1}{2}(\rho_t w^m, w^n)| \quad & \leq \quad C\|\rho_t^n\|_{L^\infty}\|w^n\|^2, \\
|(\sigma^n f, w^n)| \quad & \leq \quad C_\varepsilon \|\sigma^n\|_{L^2}^2 \|f\|_{L^6}^2 + \varepsilon\|\nabla w^n\|^2,
\end{aligned}$$

$$
\begin{aligned}
|(\sigma^n u.\nabla u, w^n)| &\leq C_\varepsilon \|\sigma^n\|^2 \|Au\|^4 + \varepsilon \|\nabla w^n\|^2, \\
|(\rho^n w^n.\nabla u, w^n)| &\leq \|\rho^n\|_{L^\infty} \|w^n\| \|\nabla u\|_{L^3} \|w^n\|_{L^6} \\
&\leq C_\varepsilon \|\rho^n\|^2_{L^\infty} \|Au\|^2 \|w^n\|^2 + \varepsilon \|\nabla w^n\|^2, \\
|(\rho^n u^n.\nabla w^n, w^n)| &\leq C_\varepsilon \|\rho^n\|^2_{L^\infty} \|Au^n\|^2 \|w^n\|^2 + \varepsilon \|\nabla w^n\|^2, \\
|(((\mu(\rho) - \mu(\rho^n))\nabla u^n, \nabla w^n)| &\leq C_\varepsilon \|\sigma^n\|^2_{L^3} \|Au^n\|^2 + \varepsilon \|\nabla w^n\|^2, \\
|(\rho \eta^n_t, w^n)| &\leq C_\varepsilon \|\rho\|^2_{L^\infty} \|\eta^n_t\|^2 + \varepsilon \|\nabla w^n\|^2, \\
|(\sigma^n u^n_t, w^n)| &\leq C_\varepsilon \|\sigma^n\|^2 \|u^n_t\|^2 + \varepsilon \|\nabla w^n\|^2, \\
|(\rho^n \eta^n \nabla u, w^n)| &\leq C_\varepsilon \|\rho^n\|^2_{L^\infty} \|Au\|^2 \|\eta^n\|^2 + \varepsilon \|\nabla w^n\|^2, \\
|(\rho^n u^n \nabla \eta^n, w^n)| &= |(u^n \nabla \rho^n \eta^n, w^n) + (\rho^n u^n \nabla w^n, \eta^n)| \\
&\leq C_\varepsilon (\|\rho^n\|^2 + \|\nabla \rho^n\|^2) \|Au\|^2 \|\eta^n\|^2 + \varepsilon \|\nabla w^n\|^2, \\
|(\mu(\rho) \nabla \eta^n, \nabla w^n) &\leq C_\varepsilon \|\nabla \eta^n\|^2 + \varepsilon \|\nabla w^n\|^2.
\end{aligned}
$$

By using the above estimates, Corollary 2.2 and the Proposition 2.5, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(\rho)^{1/2} w^n\|^2 + \frac{\mu_0}{2}\|\nabla w^n\|^2 &\leq C\|w^n\|^2 + C(1 + \|u^n_t\|^2)\|\sigma^n\|^2 + C\|\sigma^n\|^2_{L^3} \\
&\quad + C\|\eta^n\|^2 + C\|\nabla \eta^n\|^2 + C\|\eta^n_t\|^2.
\end{aligned}
$$

Integrating in t the last inequality, we get

$$
\begin{aligned}
\|(\rho(t))^{1/2} w^n(t)\|^2 + \mu_0 \int_0^t \|\nabla w^n(s)\|^2 ds &\leq \|(\rho(0))^{1/2} w^n(0)\|^2 + C\int_0^t \|w^n(s)\|^2 ds \\
&\quad + C\int_0^t \|\sigma^n(s)\|^2_{L^3}\ ds + C\int_0^t \|\eta^n\|^2 ds \\
&\quad + C\int_0^t \|\nabla \eta^n\|^2 ds + C\int_0^t \|\eta^n_t(s)\|^2 ds \\
&\quad + C\int_0^t \|\sigma^n(s)\|^2 (1 + \|u^n_t\|^2_{L^6}) ds \qquad (3.5)
\end{aligned}
$$

By other hand, using the Lemma 3.1, Young inequality and Lemma 2.6, we have

$$
\begin{aligned}
C\int_0^t \|\sigma^n(s)\|^2_{L^3} ds &\leq C\int_0^t \{\int_0^s (\|w^n(\tau)\|^2_{L^3} + \|\eta^n(\tau)\|^2_{L^3}) d\tau\} ds. \\
&\leq CT\int_0^t (\|w^n(\tau)\|^2_{L^3} + \|\eta^n(\tau)\|^2_{L^3}) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq CT\int_0^t \|w^n(\tau)\|\|\nabla w^n(\tau)\|d\tau + CT\int_0^t \|\eta^n(\tau)\|\|\nabla \eta^n(\tau)\|d\tau \\
&\leq C_\varepsilon T\int_0^t \|w^n(\tau)\|^2 d\tau + \varepsilon CT\int_0^t \|\nabla w^n(\tau)\|^2 d\tau \\
&\quad + CT\int_0^t \|\eta^n(\tau)\|^2 + CT\int_0^t \|\nabla \eta^n(\tau)\|^2 d\tau \\
&\leq C_\varepsilon T\int_0^t \|w^n(\tau)\|^2 d\tau + \varepsilon CT\int_0^t \|\nabla w^n(\tau)\|^2 d\tau \\
&\quad + \frac{CT}{\lambda_{n+1}^2}\int_0^t \|Au(\tau)\|^2 d\tau + \frac{CT}{\lambda_{n+1}}\int_0^t \|Au(\tau)\|^2 d\tau
\end{aligned}
$$

Also, the estimates given in Corollary 2.2 imply

$$
C\int_0^t \|\sigma^n(s)\|_{L^3}^2 ds \leq C_\varepsilon T\int_0^t \|w^n(s)\|^2 ds + \frac{CT}{\lambda_{n+1}^2} + \frac{CT^2}{\lambda_{n+1}} + \varepsilon T\int_0^t \|\nabla w^n(s)\|^2 ds. \quad (3.6)
$$

This last estimate when is used in (3.5) furnishes

$$
\begin{aligned}
M_1\|w^n(t)\|^2 + \mu_0\int_0^t \|\nabla w^n(s)\|^2 ds \;\leq\; & C\int_0^t \|w^n(s)\|^2 ds + \frac{C}{\lambda_{n+1}^2} + \frac{C}{\lambda_{n+1}} \qquad (3.7) \\
& + \varepsilon T\int_0^t \|\nabla w^n(s)\|^2 ds + \frac{C}{\lambda_{n+1}}\int_0^t \|\nabla u_t^n(s)\|^2 ds.
\end{aligned}
$$

Taking $\varepsilon = \mu_0/2T$, after of use Gronwall's inequality we obtain the estimate (3.3).

Theorem 3.3. *Suppose the assumptions of Proposition 2.5 hold. Then, the approximations* u^n*satisfies*

$$
||u(t) - u^n(t)||^2 + \int_0^t ||\nabla u(s) - \nabla u^n(s)||^2 ds \leq \frac{G_1(t)}{\lambda_{n+1}} + \frac{1}{\lambda_{n+1}^2}\|Au\|^2 \qquad (3.8)
$$

for any $t \in [0, T]$. The continuous function $G_1(t)$ depend on t .
Proof. We have from the Lemma 2.6 and Proposition 3.2

$$
\begin{aligned}
||u(t)-u^n(t)||^2+\int_0^t ||\nabla u(s)-\nabla u^n(s)||^2 ds &\leq ||w^n(t)||^2+\int_0^t(\|\nabla w^n(s)\|^2 ds \\
&\quad +||\eta^n(t)||^2+\int_0^t ||\nabla \eta^n(s)||^2) ds \\
&\leq (G_1(t)+\|Au\|^2)(\frac{1}{\lambda_{n+1}^2}+\frac{1}{\lambda_{n+1}}).
\end{aligned}
$$

Corollary 3.4. *Under the hypotheses of the Proposition 2.5, we have for any* $2 \leq r \leq 6$

$$
\|\sigma^n(t)\|_{L^r}^2 \leq \frac{G_1(t)}{\lambda_{n+1}}+\frac{1}{\lambda_{n+1}^2}\|Au\|^2.
$$

4. $\mathbf{H}^1$-error estimates for velocity.

Proposition 4.1. *Under the hypotheses of Proposition 2.5, we have*

$$
\|\nabla w^n(t)\|^2+M_1\int_0^t \|w_\tau^n(\tau)\|^2 d\tau \leq \frac{G_2(t)}{\lambda_{n+1}}+\frac{G_3(t)}{\sqrt{\lambda_{n+1}}}, \tag{4.1}
$$

Proof. Setting $v=w_t^n(t)$ in (3.1), we obtain

$$
\begin{aligned}
\|(\rho)^{1/2}w_t^n\|^2+(\mu(\rho)\nabla w^n,\nabla w_t^n) &= \frac{1}{2}(\rho_t w^n,w_t^n)+(\sigma^n f,w_t^n)-(\sigma^n u.\nabla u,w_t^n) \\
&\quad -(\rho^n w^n.\nabla u,w_t^n)+(\sigma^n u_t^n,w_t^n) \\
&\quad -(\rho^n u^n.\nabla w^n,w_t^n)-(((\mu(\rho)-\mu(\rho^n))\nabla u^n,\nabla w_t^n).
\end{aligned} \tag{4.2}
$$

We observe that $(\mu(\rho)\nabla w^n,\nabla w_t^n)=\frac{1}{2}\frac{d}{dt}\|(\mu(\rho))^{1/2}\nabla w^n\|^2-\frac{1}{2}(\mu'(\rho)\rho_t\nabla w^n,\nabla w^n)$, consequently in the last inequality we have

$$\begin{aligned}
\|(\rho)^{1/2}w_t^n\|^2 + \frac{1}{2}\frac{d}{dt}\|(\mu(\rho))^{1/2}\nabla w^n\|^2 &= \frac{1}{2}(\rho_t w^n, w_t^n) + (\sigma^n f, w_t^n) - (\sigma^n u.\nabla u, w_t^n) \\
&\quad -(((\mu(\rho) - \mu(\rho^n))\nabla u^n, \nabla w_t^n) + (\sigma^n u_t^n, w_t^n) \\
&\quad -(\rho^n u^n.\nabla w^n, w_t^n) - (\rho^n w^n.\nabla u, w_t^n) \qquad (4.3) \\
&\quad -\frac{1}{2}(\mu'(\rho)\rho_t \nabla w^n, \nabla w^n).
\end{aligned}$$

Now, we estimate the hand-rihgt side of the above inequality of the following manner

$$\begin{aligned}
|\frac{1}{2}(\rho_t w^n, w_t^n)| &\leq C_\varepsilon \|\rho_t\|_{L^\infty}\|w^n\|^2 + \varepsilon\|w_t^n\|^2, \\
|(\sigma^n u.\nabla u, w_t^n)| &\leq \|\sigma^n\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6}\|w_t^n\| \leq C_\varepsilon\|\sigma^n\|_{L^3}\|Au\|^4 + \varepsilon\|w_t^n\|^2, \\
|(\sigma^n f, w_t^n)| &\leq \|\sigma^n\|_{L^3}\|f_t\|_{L^6}^2\|w_t^n\| \leq C_\varepsilon\|\sigma^n\|_{L^3}^2\|f\|_{L^6} + \varepsilon\|w_t^n\|^2.
\end{aligned}$$

Analogously,

$$\begin{aligned}
|(\rho^n w^n.\nabla u, w_t^n)| &\leq C_\varepsilon\|\rho^n\|_{L^\infty}^2\|Au\|^2\|\nabla w^n\|^2 + \varepsilon\|w_t^n\|^2, \\
|(\sigma^n u_t^n, w_t^n)| &\leq C_\varepsilon\|\sigma^n\|_{L^3}\|\nabla u_t^n\|^2 + \varepsilon\|w_t^n\|^2, \\
|(\rho^n u^n.\nabla w^n, w_t^n)| &\leq C_\varepsilon\|\rho^n\|_{L^\infty}^2\|Au^n\|^2\|\nabla w^n\|^2 + \varepsilon\|w_t^n\|^2.
\end{aligned}$$

The other terms are estimate as follows

$$\begin{aligned}
|\frac{1}{2}((\mu'(\rho)\nabla w^n, \nabla w^n)| &\leq \frac{1}{2}\mu_1'\|\nabla w^n\|^2, \\
|((\mu(\rho) - \mu(\rho^n))\nabla u^n, \nabla w_t^n)| &\leq |\ \mu(\rho) - \mu(\rho^n)\ |_{L^3}\ \|\nabla u^n\|_{L^6}\|\nabla w_t^n\| \\
&\leq C\|\sigma^n\|_{L^3}\|Au^n\|\|\nabla w_t^n\| \\
&\leq C\|\sigma^n\|_{L^3}\|Au^n\|(\|\nabla u_t\| + \|\nabla u_t^n\|) \\
|(\mu(\rho)\nabla\eta^n, \nabla w_t)| &\leq \mu_1\|\nabla\eta^n\|\|\nabla w_t\| \\
&\leq \mu_1\|\nabla\eta^n\|(\|\nabla u_t\| + \|\nabla u_t^n\|).
\end{aligned}$$

By using the estimates from Proposition 2.5 , the above estimates together with (4.3) imply the following integral inequality

$$\mu_0\|\nabla w^n(t)\|^2 + M_1\int_0^t \|w_t^n(\tau)\|^2 d\tau$$
$$\leq C\int_0^t (\|w^n(\tau)\|^2 + \|\nabla w^n(\tau)\|^2 + \|\sigma^n(\tau)\|_{L^3}^2) d\tau$$
$$+C\int_0^t (\|\sigma^n(\tau)\|_{L^3} + \|\nabla \eta^n(\tau)\|)\phi(\tau) d\tau$$
$$+C\int_0^t \|\nabla \eta^n(\tau)\|^2 d\tau + C\int_0^t \|\eta^n(\tau)\|^2 d\tau$$

where $\phi(t) = \|\nabla u_t\| + \|\nabla u_t^n\|$, we observe that the estimate given in Proposition 2.5 implies $\phi(.) \in L^2(0,T)$.

Applying the Proposition 3.2 and Corollary 3.4 we obtain the desired result with $G_2(t) = TG_1(t) + CG_1(t) + CT$ and $G_3(t) = (CG_1(t) + C)^{1/2}\Phi(t)$, where $\Phi(t) = (\int_0^t (\phi(\tau))^2 d\tau)^{1/2}$.

Theorem 4.2. *Under the hypotheses of Proposition 2.5, we have*

$$||\nabla(u - u^n)(t)||^2 + \int_0^t ||u_t(s) - u_t^n(s)||^2 ds \tag{4.4}$$
$$\leq \frac{G_3(t)}{\lambda_{n+1}} + \frac{G_3(t)}{\sqrt{\lambda_{n+1}}}.$$

for any $t \in [0,T]$. *The continuous function* $G_3(t)$ *depends on* t.

Proof. *We have from the Lemma 2.6 and Proposition 4.1*

$$\begin{aligned} ||\nabla u(t) - \nabla u^n(t)||^2 + \int_0^t ||u_t(s) - u_t^n(s)||^2 ds &\leq ||\nabla w^n(t)||^2 + \int_0^t \|w_\tau^n(\tau)\|^2 d\tau \\ &+||\nabla \eta^n(t)||^2 + \int_0^t \|\eta_\tau^n(\tau)\|^2 d\tau \\ &\leq (G_2(t) + \|Au\|^2 + \int_0^t \|u_\tau(\tau)\|^2 d\tau)\frac{1}{\lambda_{n+1}} \\ &+\frac{G_3(t)}{\sqrt{\lambda_{n+1}}}. \end{aligned}$$

Corollary 4.3. *Under the hypotheses of the Proposition 2.5, we have*

$$\int_0^t \|Au(\tau) - Au^n(\tau)\|^2 d\tau \leq \frac{G_4(t)}{\lambda_{n+1}} + \frac{G_5(t)}{\sqrt{\lambda_{n+1}}}, \tag{4.5}$$

for any $t \in [0,T]$. *The continuous function* $G_4(t)$ *and* $G_5(t)$ *depends on* t.

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