Multi-resolution Representation of Gaussian Fields with Discontinuities

By Aluísio Pinheiro

Department of Statistics, State University of Campinas - Brazil

ABSTRACT We present a theoretical view of image analysis, in particular edge detection. The problem is to find a discontinuity on a Gaussian random field driven by a Laplacian operator. The idea is first to represent the *ideal field*, i.e. the continuum field, as the limit of adequate lattice fields. Depending on the discontinuity characteristics, the problem can be solved immediately using some of the theory developed by Glimm, Jaffe, Nelson and Simon, among others. For some cases, however, the discontinuity does not allow us to apply directly the latter techniques. We present a two-step procedure that shows those limit results. The second part of the work is to provide with a theoretical reasoning for several contrast-based detection procedures found on the literature. It is shown, for the case of a closed discontinuity, that a naive contrast procedure is consistent.

1 Introduction

The research described is motivated by two related problems, one applied and one theoretical. The applied problem is to access the feasibility of using wavelets in multi-resolution representation of images containing edges. Those edges could correspond to boundaries between different objects, or between objects and background.

Noticeable works along this line include [11] and [12] which search wavelet maxima to detect edges while de-noising deterministic images. Some computationally efficient algorithms were given in the papers.

Non-parametric techniques have also been employed in the detection of boundaries on images assumed to be composed by a deterministic signal and additive noise.

Nice procedures are obtained and works in this area include the ones done by Korostelev, Müller, and Tsybakov. A good survey is provided in [9].

Since on both cases, images are not considered as random samples from a statistical distribution, these approaches did not account for the variability and structure of similar images taken on the same objects. Therefore, taking images as random samples leads naturally to the theoretical problem of representation of Gaussian random fields with discontinuities. The Gaussian assumption is mainly for simplicity. However, even in the Gaussian case, little has been known for discontinuous fields, in contrast to the extensive studies for regular (stationary, continuous, etc.) fields.

Consider a Gaussian random field $X = \{X_t,\, t \in T \subset I\!\!R^d\}$ with mean zero and covariance function C. If one assumes that C is continuous, then an interesting 0-1 law about the continuity of sample paths of X states (see [2], P66 for the discussion and further references):

Let X be a Gaussian process with continuous covariance function C. Then

(i) $P\{X_t \text{ is continuous } \forall t \in T\} = 1$ iff $P\{X_t \text{ is continuous}\} = 1$ at

each $t \in T$; and

(ii) $P\{X_t \text{ is continuous } \forall t \in T\} = 0$ or 1.

What happens when X has a discontinuous covariance function C? A short answer is that no general results are available, because discontinuities of C can be introduced in many different ways. The situation is similar to differential equations with singularities. The method in solving this singular problem will depend on the form of discontinuities and the covariance structure.

A solution of this kind was given in [4]. Using wavelet analysis, the authors studied a one-dimensional (1D) Gaussian process induced by a special elliptical differential operator $L = \frac{d}{dt}a(t)\frac{d}{dt}$ on $I = (-1,1) \setminus \{0\}$, where $a(t)$ is assumed to be continuous and positive on I, and $a(t)/|t|^{\alpha} \to c$ for some positive number c, as $t \to 0$. This defines a Gaussian process X on I and, if $\alpha \geq 1$, the point 0 becomes a singularity for X. The novelty of [4] is to use wavelets as a basis for the related reproducing kernel Hilbert space (RKHS), generated by localizing the operator L (see [14] for discussion on RKHS related to Gaussian processes). A Karhunen-Loeve-type decomposition for the process X was also obtained.

The method in [4] works only when one can *isolate* the singularity in such a way that the wavelets will *miss* it altogether; otherwise, the form of the wavelets could not be maintained. That is the case only when the singularities are regularly spaced on the nodes of a dyadic decomposition of the region where the process or field is defined. In other words, this is strictly limited to the 1D nature of the problem: singularities are finite and isolated. Such a limitation hinders the extension of this approach to 2D or 3D, in which singularities usually form uncountable sets: curves, surfaces, etc.

In this work, we study Gaussian random fields induced by Laplacian operators. This framework was adopted as a starting point in quantum field theory by Glimm, Jaffe, Nelson and Simon, among others, in the 1960s and 70s. However, no singularities were involved in their studies. An important problem they considered is lattice approximation. Consider a sequence of compatible lattice fields, where the dimension of each lattice is referred to as resolution of the lattice. Lattice approximation involves convergence in various modes of this sequence to an appropriate continuum field as the resolution tends to infinity. The compatibility is imposed on successive lattices by using dyadic partitions in the same spirit as other methods of multi-resolution analysis. There are two excellent books in quantum field theory where one can obtain the information necessary to the understanding of this problem, [8] and [18]

Our first main result is Theorem 5.1 — lattice approximation for certain fields with discontinuous covariance functions. We follow and modify Simons approach as can be seen in [18]. For simplicity and clarity, we only state our result in 2D, although the analysis applies in any higher dimension. In what follows, most of the notation agrees with [18].

An application of the lattice approximation theory is to justify a class of boundary detection algorithms in image analysis. There is an extensive literature in image analysis concerning segmentation and boundary detection. We refer the reader to [24] for a summary of recent development in the area. There are two aspects penetrating the works on boundary detection: a boundary is highlighted by the large contrast on the two opposite sides; a boundary is subject to certain regularity conditions. A common sense is that any reasonable procedure for boundary detection based on digital images is expected to locate the correct boundary as the resolution gets higher and higher. But to the best of our knowledge, no works have been done to justify this mathematically, especially when images are regarded as realizations of a random field. The main difficulty is that to deal with images of finite size with *increasing* resolution, the usual asymptotics in probability in which we assume lattices expanding to infinity but with *fixed* spacing does not work. When the resolution goes to infinity (or equivalently, the spacing goes to zero), we have a sequence of discrete random fields tending to a limiting random field on the continuum. Such a qualitative transition poses a tremendous technical difficulty. The lattice approximation turns out to be precisely the right asymptotics analysis to adopt in this project. In what follows, we will also prove the consistency of a class of contrast-based procedures in boundary detection, stated in Theorem 6.1. In order to gain good understanding, we simplify the situation and only consider boundaries consisting of piecewise smooth Jordan curves, i.e. we do not add the regularity as a factor into the boundary selection procedure, but only focus our attention on the average contrast part. Furthermore, we only consider a single closed curve as the boundary. The same argument works for the case of multiple curves with slightly more technicality involved. Our work is theoretical in spirit, and we do not discuss many important issues in computation and practical implementation.

Notation

- $I\!R$ - space of real numbers
- $\boldsymbol{Z}$ - integers
- $I\!N$ - natural numbers
- $L^2(\Omega)$ - space of square integrable functions supported on $\Omega \subseteq I\!R^2$
- l^2 - squared summable sequences
- $C_0^\infty(\Omega)$ - test functions on an open region $\Omega \subset I\!R^2$
- $\mathcal{D}'(\Omega)$ - distributions (dual of $C_0^\infty(\Omega)$)
- $\mathcal{S}(\Omega)$ - Schwartz test functions on Ω
- $\mathcal{S}'(\Omega)$ - tempered distributions (dual of $\mathcal{S}(\Omega)$)
- Δ - free Laplacian
- h - spacing of a lattice field
- Δ_Λ - Laplacian restricted to $\Lambda \subset I\!R^2$
- Δ_h - discrete Laplacian
- $<,>_{\mathcal{H}}$ or $(,)_{\mathcal{H}}$ - inner product on a Hilbert space $\mathcal{H}$
- $\|\cdot\|_{\mathcal{H}}$ - norm on $\mathcal{H}$
- $\hat{f}$ - Fourier transform of f

- $\tilde{f}$ - inverse Fourier transform of f
- δ_x - Delta-Dirac function at x
- $\bar{A}$ - closure of A
- A^o - interior of A

The lattice approximation is carried out in three steps: Step 1 is for the free field; Step 2 deals with the fields defined on a bounded region of arbitrary shape with the Dirichlet boundary condition; and Step 3 allows the bounded region in Step 2 to contain smooth curves at which the covariance functions of the random fields are discontinuous.

In Step 1, let $\mathcal{S}'(I\!R^2)$ be the Schwartz space of tempered distributions on $I\!R^2$. The free field is a $\mathcal{S}'(I\!R^2)$-valued Gaussian process ϕ satisfying

$$\left(-\Delta + a^2\right)\phi = \eta, \tag{1}$$

where Δ is the Laplacian, $a > 0$ is a constant and η is the white noise. The lattice approximation for the free field is accomplished by standard Fourier analysis.

In Step 2, let Λ be an open bounded region in $I\!R^2$ with boundary $\partial\Lambda$. We consider the Gaussian field ϕ on Λ defined by the Dirichlet problem

$$\left(-\Delta_\Lambda + a^2\right)\phi = \eta \text{ in } \Lambda, \quad \text{and} \quad \phi = 0 \text{ on } \partial\Lambda, \tag{2}$$

where Δ_Λ is the Laplacian restricted to Λ. Then the lattice approximation is obtained by carefully projecting both the continuum field and the lattice fields in Step 1 to Λ and its lattice counterpart Λ_h respectively.

In Step 3, let Λ be the open unit square containing several smooth curves in it. Without loss of generality, we consider the case with just one such curve, denoted by c. For $\epsilon > 0$ sufficiently small, let Λ_ϵ be the open region obtained by subtracting a closed ϵ-neighborhood ("sausage") of c from Λ. The lattice approximation in Step 2 applies to the Gaussian field on Λ_ϵ with the Dirichlet boundary condition on $\partial\Lambda_\epsilon$. Then a limit can be identified when letting $\epsilon \downarrow 0$, due to the monotonicity of the related covariance operators.

At first glance, use of some wavelet basis seems appropriate in lattice approximation due to the localization property of wavelets, especially for the representation of random fields with discontinuities. So far our attempt at this approach has not turned out to be successful. Our study is still inconclusive. The problem with wavelets is that they are not eigenfunctions of any well-known differential operator, including the Laplacian. The redundancy of wavelet bases creates difficulty in showing the convergence of the operators. We will continue our study in wavelet representation of Gaussian fields with discontinuities.

The three-step lattice approximation does not specify a random field on the *entire* domain Λ, since it always forces a deterministic zero configuration on the curve c. As a theoretical problem, we plan to study an alternative model which offers a possibility of defining a Gaussian field on the entire Λ, including c, and still highlighting the discontinuities along c. More details on this will be given in Section 7.

The basic framework and results for lattice approximations as stated here are drawn mainly from the works developed by Simon, Glimm and Jaffe and, therefore, we refer the readers to [18] and [8]. However, we will be working with a slight modification of the originally indexing spaced proposed in [18].

Section 2 is based on its entirety on Simon's work. On Section 3, we introduce the problem on a different way, as stated in Definition 3.3, in order to precise relate it to Definition 3.2. The results of Sections 3 and 4 are therefore slight modifications of the ones found in [18].

The rest of this text goes as follows. On Section 2, the basic framework is given for the establishment of a consistent theory of distribution-valued Gaussian random fields on the continuum. The *Free Field* is presented in Section 3, where its lattice approximation is proven to exist. Section 4 shows that this approximation is still valid when the support of the field is a region that attends some regularity conditions.

Discontinuities are introduced on the problem and, in Section 5, one shows that a direct use of the existing results is not possible but a careful two-step lattice approximation procedure is proven to be enough.

Section 6 provides a formal justification for a general contrast-based procedure for boundary detection for certain distribution-valued Gaussian random fields with discontinuities. Further research is proposed in Section 7 and proofs are given in Appendix A. A brief introduction to the theory of distributions, its properties and Fourier analysis is given in Appendix B.

2 The Basic Framework

Definition 2.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and V a real vector space. A random process indexed by V is a map $\phi : (V, \Omega) \to \mathbb{R}$ such that, for every $f, g \in V$, $a \in \mathbb{R}$:*

$$\begin{aligned} \phi(f+g) &= \phi(f) + \phi(g) \ a.e. \\ \phi(af) &= a\phi(f) \ a.e. \end{aligned}$$

Definition 2.2 *Let $\{X_\alpha, \ \alpha \in A\}$ be a set of random variables defined on $(\Omega, \mathcal{F}, P)$. This set is called full if the equivalence classes of $\{X_\alpha^{-1}(B) \ \mid \alpha \in A, B$ a Borel set in $\mathbb{R}\}$ in $\mathcal{F} \setminus Z_P$ are not all contained on a proper σ-subring of $\mathcal{F} \setminus Z_P$, where Z_P is the zero-probability class of Ω.*

Definition 2.3 *Let $\mathcal{H}$ be a Hilbert space. The Gaussian process ϕ indexed by $\mathcal{H}$ is a stochastic process satisfying:*

(i) $\{\phi(f) | f \in \mathcal{H}\}$ *is full;*

(ii) *Each $\phi(f)$ is a Gaussian random variable;*

(iii) $< \phi(f), \phi(g) >_{L^2(\Omega, P)} = 1/2 \; < f, g >_{\mathcal{H}}$.

Note: The factor 1/2 in the isometry (iii) is simply a notational convention to relate this definition to Fock spaces (see [18]).

Define $\mathcal{Q}_{\mathcal{H}}$ as the measurable space generated by such a process, i.e. the smallest measurable space with respect to which ϕ is a random process. The following questions are posed:

1. Does a Gaussian random process indexed by $\mathcal{H}$ exist?
2. If such a process exists, is it unique?
3. How to construct $\mathcal{Q}_{\mathcal{H}}$?

Beginning from the third question, one can answer all three at once. There are several choices for $\mathcal{H}$ and $\mathcal{Q}_{\mathcal{H}}$. One classical choice, whose reasoning will become clear as the text develops, is to choose $\mathcal{H}$ to be the space $\mathcal{S}$ of test functions, and $\mathcal{Q}_{\mathcal{H}}$ to be its dual, i.e. the space $\mathcal{S}'$ of tempered distributions. First, one has

Definition 2.4 *A cylinder set in $\mathcal{S}'(\mathbb{R}^2)$ is the set of distributions T such that:*

$$(T(f_1), T(f_2), \ldots, T(f_m)) \in B,$$

where B is a Borel set in $\mathbb{R}^2$ and $f_1, f_2, \ldots, f_m$ are elements in $\mathcal{S}(\mathbb{R}^2)$.

A cylinder measure μ is defined on the σ-algebra generated by the cylinder sets such that $\mu(\mathcal{S}'(\mathbb{R}^2)) = 1$.

But we know that each f in $\mathcal{S}$ defines a measurable function $\phi(f)$ on $\mathcal{S}'$ by:

$$\phi(f)(T) = T(f)$$

and that $\{\phi(f) | f \in \mathcal{S}\}$ is full. Also, $\phi(f_m) \to \phi(f)$ pointwise, whenever $f_m \to f$ weakly, which implies that:

$$\int exp\,(i\phi(f_m))\,d\mu \to \int exp\,(i\phi(f))\,d\mu.$$

Given all the above construction, properties of the cylinder sets and measure, and their relation to the test functions, one can invoke

Theorem 2.1 (Minlos' Theorem) *Let b be defined on $\mathcal{S}(\mathbb{R}^2)$. Then there is a unique cylinder measure μ on $\mathcal{S}'(\mathbb{R}^2)$ such that*

$$b(f) = \int exp\,(i\phi(f))\,d\mu$$

if and only if

(i) $b(0) = 1$;

(ii) $f \mapsto b(f)$ *is continuous in the strong topology;*

(iii) *For any $f_1, f_2, \ldots, f_m \in \mathcal{S}(\mathbb{R}^2)$ and complex numbers $z_1, z_2, \ldots, z_m$,*

$$\sum_{j,\,k=1}^{m} z_j \bar{z}_k\, b\,(f_j - f_k) \geq 0.$$

$b(\cdot)$ is called the generating function or characteristic function. In particular, the Gaussian measure μ on $\mathcal{S}'(\mathbb{R}^2)$ with mean zero and covariance C is defined via the generating function

$$b(f) = exp\left(-\frac{1}{4}\,C(f,f)\right)$$

for a positive definite quadratic form C on $\mathcal{S}$.

Note: The factor 1/4 is the result of the factor 1/2 introduced previously in Definition 2.3 (iii).

Minlos' theorem guarantees the existence and uniqueness of the Gaussian process in $\mathcal{S}'$ with a prescribed covariance operator C. Moreover, if one can construct a Hilbert space $\mathcal{H}$ whose inner-product reduces to a positive definite form on $\mathcal{S}$, and $\mathcal{H} = \bar{\mathcal{S}}$, then $\mathcal{S}'$ (or an appropriate subspace) itself can be used as a model for $\mathcal{Q}_{\mathcal{H}}$. In the next section, the denseness of test functions in a distribution space will prove very useful.

3 Lattice Approximation for the Free Field

First, note that Minlos' theorem also applies in construction of a Gaussian process indexed by a closure of $\mathcal{S}$ (with respect to an appropriate norm).

Second, to define a Gaussian process induced by a linear operator, it suffices to identify an appropriate Hilbert space and check that the quadratic form associated to this operator is positive definite.

There are essentially two different but equivalent ways to define the free field as a distribution-valued Gaussian process: one is via weak solutions of a SPDE driven by the white noise; the other specifies the covariance function (assuming mean zero) and uses Minlos' theorem.

Definition 3.1 *Let Λ be a regular region in $I\!R^2$ (see Definition 4.3). The white noise $\eta \in \mathcal{S}'(\Lambda)$ is defined by*

$$\eta(\psi) = \sum_k \langle \psi, e_k \rangle \, \xi_k, \quad \psi \in \mathcal{S}(\Lambda) \subset L^2(\Lambda),$$

where $\{e_k\}$ is an orthonormal basis of $L^2(\Lambda)$ with respect to the inner product $\langle \;,\; \rangle$, and $\{\xi_k\}$ is a collection of iid N(0,1) random variables. Note that $\eta(\psi)$ is *a* $N(0,\ \|\psi\|_2^2)$ *random variable with the* $L^2(\Lambda)$*-norm* $\|\cdot\|_2$.

Note: The white noise η in Λ is unique with respect to the equivalence relation "$\simeq$": $\eta_1, \eta_2 \in \mathcal{S}'(\Lambda)$ are said to satisfy $\eta_1 \simeq \eta_2$, if $\eta_1(\psi)$ and $\eta_2(\psi)$ have the same probability distribution for every $\psi \in \mathcal{S}(\Lambda)$.

Definition 3.2 *The free field* ϕ *is defined as a solution of the equation:*

$$\left(-\Delta + a^2\right) \phi = \eta, \tag{3}$$

where $\eta \in \mathcal{S}'(I\!R^2)$ *is the white noise.*

Note: Let $\mathcal{N}^2$ be given by Definition 3.3. Then, $\mathcal{N}^2 = \overline{C_0^\infty(I\!R^2)}$ and the embedding theorem can be used (as in [7] Theorem 11.1) so the equation (3) can be understood as

$$\int_{I\!R^2} \phi(x)(-\Delta + a^2)\, \psi(x)\ dx = \eta(\psi) \quad \forall \psi \in C_0^\infty(I\!R^2), \tag{4}$$

and there is a continuous version ϕ satisfying (4), which we adopt as the free field. The free field is the Gaussian random process with mean zero and the covariance function $C(x, y)$ which satisfies the equation

$$(-\Delta + a^2)^2\, C(x, y) = \delta_{x-y}, \quad x, y \in I\!R^2.$$

Consider the following index space:

Definition 3.3 *Let* $\mathcal{N}^2$ *be the Hilbert space of all (real) distributions* $f \in \mathcal{S}'(I\!R^2)$ *whose Fourier transforms are functions that satisfy*

$$\|f\|^2_{\mathcal{N}^2} = \int_{I\!R^2} \frac{|\hat{f}(k)|^2}{\left(k^2 + a^2\right)^2}\, dk < \infty,$$

where $k = (k_1,\ k_2) \in I\!R^2$ *and* $k^2 = k_1^2 + k_2^2$. *Hence we equip* $\mathcal{N}^2$ *with the norm* $\|.\|_{\mathcal{N}^2}$. *In general, for* $D \subset I\!R^2$, *we let*

$$\mathcal{N}^2_D = \overline{\mathcal{N}^2 \cap \mathcal{S}'(D)}.$$

This norm is related to the operator $-\Delta + a^2$ via the isometry

$$< f, g >_{\mathcal{N}^2} = 2 < f, \left(-\Delta + a^2\right)^{-2} g >_{L^2},$$

following the fact that $-\Delta$ has eigenvalues k^2 and associated eigenfunctions $e^{i<k,x>}$, $k \in \boldsymbol{Z}^2$, $x \in I\!R^2$.

Using the notation introduced in Section 2, set $\mathcal{H} = \mathcal{N}^2$ and $\mathcal{Q}_{\mathcal{H}} = \mathcal{Q}_{\mathcal{N}^2}$. Then the free field can also be characterized as a Gaussian process indexed by $\mathcal{N}^2$, i.e. the Gaussian process whose realizations are elements in $\mathcal{Q}_{\mathcal{N}^2}$. We denote the corresponding Gaussian measure by μ_0. Although this characterization of the free field is not as direct as Definition 3.2, it turns out to be convenient when dealing with lattice approximations. This will become clear in the proof of Theorem 3.1.

Note: A slightly different Hilbert space $\mathcal{N}$ is used in [18] with the squared norm

$$\|f\|^2_{\mathcal{N}} = \int \frac{|\hat{f}(k)|^2}{k^2 + a^2}\, dk.$$

We use $\mathcal{N}^2$ instead of $\mathcal{N}$ to keep the consistency with Definition 3.2.

Now we define the lattice Gaussian field with a given resolution. Let $h > 0$ and $L_h = \{nh \,|\, n \in \boldsymbol{Z}^2\}$.

Define the discrete (negative) Laplacian $-\Delta_h$ by

Definition 3.4

$$(-\Delta_h f)(nh) = h^{-2}\left[4f(nh) - \sum_{|m-n|=1} f(mh)\right].$$

The lattice field can be thought as a digital image with resolution h^{-1}. For coherence among different resolutions, the dyadic lattice will be used, i.e. $h = 2^{-j},\ j \in I\!N$.

Definition 3.5 *For $h > 0$, let ϕ_h be the Gaussian random field with mean zero and covariance function*

$$E\left[\phi_h(n)\,\phi_h(m)\right] = h^{-2}\left(-\Delta_h + a^2\right)^{-2}(n, m), \quad n, m \in \boldsymbol{Z}^2,$$

where $(-\Delta_h + a^2)^{-2}$ *is the inverse (countable) matrix of* $(-\Delta_h + a^2)^2$.

This classical definition is not appropriate to tie the lattice free fields with the continuum free field in the same space for the study of lattice approximations. Our next step is to realize the lattice free fields also as distribution-valued Gaussian random variables, and then identify the right indices in $\mathcal{N}^2$ for them, as defined in Definition 3.3.

Definition 3.6 *Define* $\phi_h(g)$ *by*

$$\phi_h(g) = \sum_{n \in \mathbf{Z}^2} h^2 g(nh)\phi_h(n), \quad g \in C_0^\infty\left(\mathbb{R}^2\right).$$

For notational simplification, we use ϕ_h both as a function on L_h and as a functional evaluated clearly at some element of $\mathcal{N}^2$.

To realize $\phi_h(n)$ on the continuum, let $f_{n,h}$ be the unique function in $\mathcal{N}^2$ with the Fourier transform

$$\hat{f}_{n,h} = \frac{exp\left(-iknh\right)\mu^2(k)}{2\pi\mu_h^2(k)} 1(h|k_1| \leq \pi,\ h|k_2| \leq \pi),$$

where $1(A)$ is the indicator of event A,

$$\mu^2(k) = k^2 + a^2;$$

and observe that

$$\mu_h^2(k) = h^{-2}(4 - 2\cos(k_1 h) - 2\cos(k_2 h)) + a^2$$

are the eigenvalues of $-\Delta_h + a^2$ with respect to the Fourier exponentials.

Note that

$$E\left[\phi\left(f_{n,h}\right)\phi\left(f_{m,h}\right)\right] = \left(\frac{\hat{f}_{n,h}}{k^2 + a^2}, \frac{\overline{\hat{f}}_{m,h}}{k^2 + a^2}\right)_{L^2}$$

$$
\begin{aligned}
&= \int_{h|k_1|\le\pi,\, h|k_2|\le\pi} \frac{exp\,(ik(n-m)h)\,(k^2+a^2)^2}{(2\pi)^2\mu_h^4(k)(k^2+a^2)^2}\,dk \\
&= (2\pi)^{-2}\int_{h|k_1|\le\pi,\, h|k_2|\le\pi} exp\,(ik(n-m)h)\,\mu_h^{-4}(k)\,dk \\
&= E\left[\phi_h(n)\,\phi_h(m)\right]
\end{aligned}
$$

is exactly the covariance function of ϕ_h. With that, one can realize $\phi_h(n)$ as an element in $\mathcal{Q}_{\mathcal{N}^2}$ by

$$\phi_h(n) = \phi(f_{n,h}).$$

The next lemma indicates that an important step in lattice approximation for the free field is the convergence of lattice Laplacians to the continuum Laplacian.

Lemma 3.1 (Simon) *For $\mu^2(k)$ and $\mu_h^2(k)$, we have*

(i) *For each $k \in I\!R^2$, $\mu_h^2(k) \to \mu^2(k)$ as $h \to 0$.*

(ii) *If $\max(|k_1|, |k_2|) \le \pi/h$, then $\mu_h^{-1}(k) \le \frac{\pi}{2}\mu^{-1}(k)$.*

(iii) $\mu_h^{-2}(k)\ 1(\max(|k_1|, |k_2|) \le \pi/h) \to \mu^{-2}(k)$ *as $h \to 0$ in each $L^p(I\!R^2)$, $p > 1$.*

For every $g \in C_0^\infty(I\!R^2)$, define

$$\hat{g}_h(k) = \left[\sum_{n\in\mathbf{Z}^2} \frac{h^2}{2\pi}\, g(nh)\, \exp(-iknh)\right] 1(\max(|k_1|, |k_2|) \le \pi/h).$$

Then we have

Lemma 3.2 (Simon) *$\hat{g}_h \to \hat{g}$ in each $L^p(I\!R^2)$, as $h \to 0$, for every $p \ge 2$.*

Theorem 3.1 *With the realization $\phi_h(n) = \phi(f_{n,h})$, for every $g \in C_0^\infty(I\!R^2)$ we have*

$$\phi_h(g) \to \phi(g),$$

in $L^2(\mathcal{Q}_{\mathcal{N}^2}, \mu_0)$, as $h \to 0$.

4 Lattice Approximation for Gaussian Fields with Dirichlet Boundary Conditions

Definition 4.1 *The Dirichlet field ϕ_Λ is defined as a solution of the following Dirichlet problem:*

$$\left(-\Delta_\Lambda + a^2\right)\phi = \eta \quad \text{in } \Lambda, \quad \text{with} \quad \phi = 0 \quad \text{on } \partial\Lambda, \tag{5}$$

where Δ_Λ is the Laplacian restricted to Λ.

In the same spirit as in Definition 3.2, ϕ_Λ is taken as a continuous version of ϕ that satisfies

$$\int_\Lambda \phi(x)\,(-\Delta_\Lambda + a^2)\,\psi(x)\,dx = \eta(\psi) \quad \forall\psi \in C_0^\infty(\Lambda). \tag{6}$$

Hence ϕ_Λ is a $\mathcal{S}'(\Lambda)$-valued Gaussian field with mean zero and the covariance function $C_\Lambda(x,y)$ which satisfies the equation

$$(-\Delta_\Lambda + a^2)^2\, C_\Lambda(x,y) = \delta_{x-y}, \quad x, y \in \Lambda. \tag{7}$$

Definition 4.2 *Given $\Lambda \subset \mathbb{R}^2$, one defines:*

$$\begin{aligned}
&\Lambda_h = \Lambda \cap L_h \\
&\Lambda_h^{ext} = L_h \setminus \Lambda_h \\
&\Lambda_h^{int} = \{nh \in \Lambda_h \,|\, mh \in \Lambda_h \text{ if } |n-m| \leq 2\} \\
&\partial\Lambda_h = \Lambda_h \setminus \Lambda_h^{int} \\
&\partial\Lambda_h^{ext} = \{nh \in \Lambda_h^{ext} \,|\, mh \in \Lambda_h \text{ if } |n-m| \leq 2\}
\end{aligned}$$

Let $\mathcal{N}_h^2$ be the space of real-valued sequences on L_h such that:

$$\|f\|_h^2 = \sum \left(-\Delta_h + a^2\right)^{-2}(n,m)\, f(n)\, f(m) < \infty.$$

Given $\Lambda \subset I\!R^2$, *let* e_{Λ_h} *be the projection in* $\mathcal{N}_h^2$ *onto those sequences with support in* Λ_h *and* $p_{\Lambda_h} = 1 - e_{\Lambda_h^{ext}}$. *Finally, define the field* $\phi_{\Lambda,h}$ *by*

$$\phi_{\Lambda,h}(n) = \phi_h\left(p_{\Lambda_h} e_n\right), \quad n \in \boldsymbol{Z}^2,$$

where e_n *is defined by* $e_n(m) = \delta_{n-m}$, $m \in \boldsymbol{Z}^2$.

Lemma 4.1 (Simon)

$$p_{\Lambda_h} e_n = \begin{cases} e_n - \sum_{m \in \partial\Lambda_h^{ext}} a_n(m)\, e_m, & \text{if } n \in \Lambda_h, \\ 0, & \text{if } n \in \Lambda_h^{ext}, \end{cases} \tag{8}$$

with some coefficients $a_n(\cdot)$.

Definition 4.3 *An open region* $\Lambda \subset I\!R^2$ *is said to be regular if* $C_0^\infty(I\!R^2 \setminus \Lambda)$ *is dense in* $\mathcal{N}^2_{I\!R^2\setminus\Lambda}$ *with respect to the* $\mathcal{N}^2$*-topology.*

The following counterexample due to Simon illustrates the necessity of imposing the regularity condition on region Λ:

Let Λ_1 *be the unit disc and* $\Lambda_2 = \Lambda_1 \setminus \{(x, \pi x)\}$. *When using the lattice approximation, one gets* $(\Lambda_1)_h = (\Lambda_2)_h$ *but, since one can easily find distributions in* $\mathcal{N}^2$ *supported by* $\Lambda_1 \setminus \Lambda_2$, *the Dirichlet lattice theory cannot converge for both domains.*

Lemma 4.2 *For a regular region* Λ, *let* $e = e_{I\!R^2\setminus\Lambda}$ *be an orthogonal projection onto* $\mathcal{N}^2_{I\!R^2\setminus\Lambda}$ *in* $\mathcal{N}^2$*-norm and define* e_h *to be the projection onto the span of* $\{f_{n,h} \,|\, n \in \Lambda_h^{ext}\}$. *Then* e_h *converges strongly to* e.

Corollary 4.1 *Using notation in the previous lemma and defining* $p = 1 - e$ *and* $p_h = 1 - e_h$, *one immediately has the strong convergence of* p_h *to* p.

Finally, we state

Theorem 4.1 *Let Λ be a regular region. For $g \in C_0^\infty(I\!R^2)$, define*

$$\phi_{\Lambda,h}(g) = \sum_{n \in \boldsymbol{Z}^2} g(nh)\, \phi_{\Lambda,h}(n).$$

Then, as $h \downarrow 0$,

$$\phi_{\Lambda,h}(g) \to \phi_\Lambda(g),$$

in $L^2(\mathcal{Q}_{\mathcal{N}^2_\Lambda}, \mu_0)$.

5 Lattice Approximation for Certain Random Fields with Discontinuities

Let $\Lambda = (0,1)^2$ and $c \in \Lambda$ be a smooth open curve, as in figure 1. The question of interest is: does Theorem 4.1 hold on the region $\Lambda_0 = \Lambda \setminus c$? The answer is *no* due to the *irregularity* of Λ_0.

In fact, observe that $I\!R^2 \setminus \Lambda_0 = (I\!R^2 \setminus \Lambda) \bigcup c$. If $f \in \mathcal{N}^2_{I\!R^2 \setminus \Lambda}$, then both f and $f + \delta_x$ are elements of $\mathcal{N}^2_{I\!R^2 \setminus \Lambda_0}$, with $x \in c$. Consequently,

$$\mathcal{N}^2_{I\!R^2 \setminus \Lambda} \subset \mathcal{N}^2_{I\!R^2 \setminus \Lambda_0}, \tag{9}$$

and the inclusion $\subset$ is proper. On the other hand,

$$C_0^\infty\left(I\!R^2 \setminus \Lambda_0\right) = C_0^\infty\left(I\!R^2 \setminus \Lambda\right),$$

because c is a lower-dimensional sub-manifold in $I\!R^2$, and the $I\!R^2$-continuity of a function at any point on c must involve some part of a neighborhood of c. This, together with (9), shows that Λ_0 is not regular.

Therefore, Barry Simon's approach in [18] cannot be applied directly. Nevertheless, the following two-step procedure is a remedy.

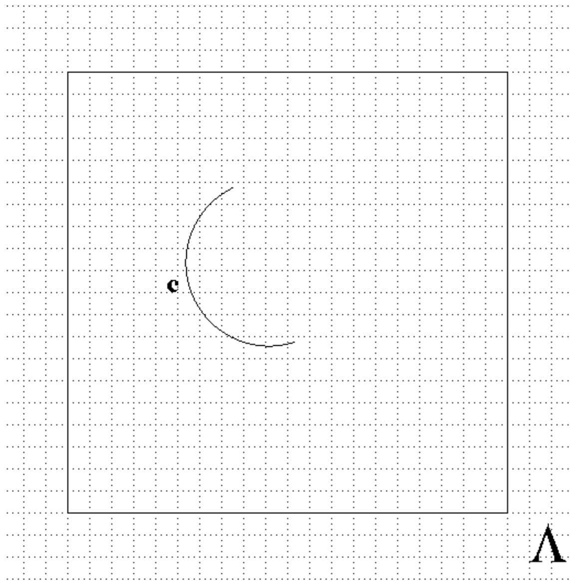

Figure 1: *Space* Λ *on which* ϕ_{Λ_c} *is defined*

Let $\epsilon > 0$ be sufficiently small such that $c_\epsilon \subset \Lambda$, where

$$c_\epsilon = \{x \in I\!R^2 : \; |x-y| \leq \epsilon \text{ for some } y \in c\},$$

and let

$$\Lambda_\epsilon = \Lambda \setminus c_\epsilon .$$

Using the following lemmas, and projecting the elements of $\mathcal{N}^2_{I\!R^2\setminus\Lambda_\epsilon}$ onto $\mathcal{N}^2_{c_\epsilon}$ and $\mathcal{N}^2_{I\!R^2\setminus\Lambda}$, we can show that Λ_ϵ is a regular region.

Lemma 5.1 *Any open convex region* $\Lambda_\star$ *in* $I\!R^2$ *is regular.*

Lemma 5.2 *Let* $\Lambda_\star$ *be an open region in* $I\!R^2$*. Then* $C_0^\infty(\Lambda_\star)$ *is dense in* $\mathcal{N}^2_{\Lambda_\star}$ *with the* $\mathcal{N}^2$*-topology.*

Now, following the notation in previous section, let $\Lambda_{h,\epsilon}$ be the lattice associated to Λ_ϵ. Then we have

Theorem 5.1 *Let Λ_0 and Λ_ϵ be given as above. Define ϕ_c to be the Gaussian field indexed by $\mathcal{N}^2_{\Lambda_0}$, and $\phi_{h,\epsilon}$ to be the Gaussian field indexed by $\mathcal{N}^2_{\Lambda_{h,\epsilon}}$. Then for $g \in C_0^\infty(\mathbb{R}^2)$, as $h \downarrow 0$ and subsequently $\epsilon \downarrow 0$,*

$$\phi_{h,\epsilon}(g) \to \phi_c(g),$$

in $L^2(\mathcal{Q}_{\mathcal{N}^2_{\Lambda_0}}, \mu_0)$.

Note: The order here between convergences of h and ϵ is necessary since lattice approximation for the Dirichlet field should be used for fields indexed by regular regions, which is not true if one takes $\epsilon \downarrow 0$ before $h \downarrow 0$.

6 Boundary Detection in Gaussian Random Fields with Discontinuities

As mentioned in the introduction, we assume that c is a closed Jordan curve. Define Λ_1 to be the open region enclosed in c and Λ_2 to be the interior of its complement with respect to Λ, as shown in figure 2. Given the nature of the Laplacian operator, the fields in Λ_1 and Λ_2 must be defined independently.

One way of doing this is as follows:

Definition 6.1 *Let ϕ_1 and ϕ_2 be two independent free fields indexed by $\mathcal{N}^2(\mathbb{R}^2)$. Take Λ to be the unit cube $(0,1)^2$ and c a continuous closed curve such that $c \subset \Lambda$. Define Λ_1 as the open region enclosed in c and $\Lambda_2 = (\Lambda \setminus \Lambda_1)^o$. Then, the Gaussian random field driven by the Laplacian restricted to $\Lambda_c = \Lambda \setminus c$ is defined as:*

$$\phi_{\Lambda_c}(f) = \phi_{\Lambda_1}(p_{\Lambda_1}(f)) + \phi_{\Lambda_2}(p_{\Lambda_2}(f)),$$

for every $f \in \mathcal{N}^2(\mathbb{R}^2)$.

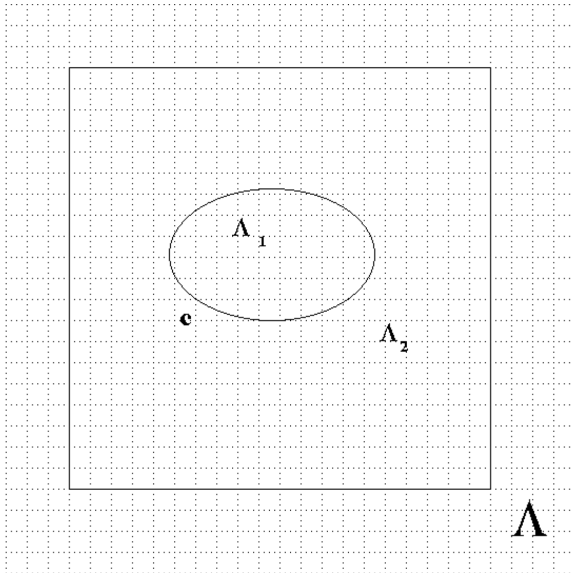

Figure 2: *Space Λ on which ϕ_{Λ_c} is defined*

A fact clearly established in the definition of ϕ_{Λ_c} is that $\phi_{\Lambda_c}(f)$ and $\phi_{\Lambda_c}(g)$ are independent, whenever $f \in \mathcal{N}^2(\Lambda_1)$ and $g \in \mathcal{N}^2(\Lambda_2)$.

The interest of this work is to estimate the curve c given a realization of ϕ_{Λ_c}. In practice, a continuum field is not observable. Let $\phi_{\Lambda_c,h}$ be the corresponding lattice field, where $h = 2^{-n}$, for some $n \in I\!N$. This dyadic nature of h provides a sequence of nested lattices as resolution increases, i.e. each lattice is a refinement of the previous one by a factor of two. It was already established in subsection 4 that the lattice field is a good approximation for the continuum one in $L^2(\mathcal{Q}_{\mathcal{N}^2_{\Lambda_c}}, \mu_0)$ sense.

A procedure for detection of c will be defined on the lattice taking into account its embedding in the continuum. The complex structure of the lattice makes the characterization a very hard task. Moreover, since c is a curve that can not be consistently defined on any finite lattice, a complete answer can only exist asymptotically. Therefore, we will be studying the estimator when h is small.

The detection is done by maximizing the average contrast around any curve defined on the interior of Λ. For the reason of the numerical complexity of this operation, some stronger results on the convergence of the lattice to the continuum field are necessary. Simple L^2 convergence is not enough: a rate of order $O\left(h^{-\alpha}\right)$ for some positive α is essential to ensure smoothness within each of the regions Λ_1 and Λ_2. Also, we will be dealing with double arrays, which significantly increases the level of difficulty of the problem. As a last remark, the Gaussian assumption is crucial for the argument.

In order to properly define the procedure, we will need the following definition of a sequence of test functions:

Definition 6.2 *Let f be a test function; then, $f_{r,x,h}$ is a sequence of test functions defined as follows:*

$$f_{r,x,h}(y) = r^2 f(r(y-x)).$$

Note: On the previous definition, both r and x can be taken as depending on h. In the case where r increases as h decreases and x is fixed, the sequence will converge to the δ-Dirac distribution centered at x as $h \to 0$. For that reason, we will be calling it a delta sequence.

For computing the contrast *around* a curve $c_\star$, one will need a clear definition of the set that is around $c_\star$. Its elements will be called bounds and their lengths must clearly go to zero as h gets small. It is preferable that each of their lengths is of the same order of h. In general, each of these bounds is taken to have length h. In our case, however, because of the discontinuity on c, the bounds will be defined to be $2h$ apart. Note that this doesn't change the order of their lengths while preserves independence between the sides of the bounds under the true curve c. A formal definition of the set of bounds is given as follows.

Definition 6.3 *Let $c_\star$ be a continuous curve defined in Λ. The set of all bounds in the lattice Λ_h around $c_\star$ of distance $2h$ is defined as:*

$$B_{h,\star} = \{\overline{x_i, x_i + he}\},$$

where $e = (0,2)$ or $e = (2,0)$

Note: The set of bounds around the true curve c is defined on the same way and will be denoted by B_h. With that setup, one define the average contrast around a curve $c_\star$ as:

Definition 6.4 *Define the average contrast around $c_\star$, under lattice of spacing h as:*

$$AC_{c_\star,h}\left(f\right) = \frac{1}{n_{h,\star}} \sum_{i=1}^{n_{h,\star}} \left(\phi_{\Lambda_c,h}\left(f_{r,x_i,h}\right) - \phi_{\Lambda_c,h}\left(f_{r_h,x_i+he,h}\right)\right)^2,$$

where $n_{h,\star}$ is the cardinality of $B_{h,\star}$, $\overline{x_i, x_i + he}$ are, for $i = 1, 2, \ldots, n_{h,\star}$, the elements of $B_{h_\star}$ and $e = (0,2)$ or $e = (2,0)$.

Note: The definition of $AC_{c,h}\left(f\right)$ is completely analogous. In practice, this procedure is taken for a fixed f and, for reasons of notational simplification, we will be writing $AC_{c,h}$ and $AC_{c_\star,h}$. Notice, however, that such a result will still be valid for a whole class of test functions.

A technical issue is that of the possible values for e. The strict positiveness of at least one of its elements is an obvious necessity in order to meaningfully define a contrast. The fact that one *jumps* over the nearest neighbor is only to avoid delicate situations where a lattice point is too close to the boundary c. As will be seen in theorem 6.1, r_h will be very large when h gets small. That means that the actual smearing that is being performed on the lattice field is restricted to a very small neighborhood. Such a restriction is necessary for the independence of any evaluations of the fields close to the true boundary c.

The following result establishes that the procedure of choosing the maximum contrast eventually leads to c.

Theorem 6.1 *Take f to be a test function that has Hölder continuity of some positive order, $\alpha \geq 3+\gamma$. Let $r_h = h^{-1-\gamma}$, for some $0<\gamma<1/2$. Then, as $h \to 0$,*

$$P\left(AC_{c,h} \leq \max_{\{c_\star : c \cap c_\star = \emptyset\}} AC_{c_\star,h}\right) \to 0.$$

A series of results will be established in order to prove theorem 6.1. A very general result, that is a simple consequence of bounds for normal tail probabilities is the following:

Lemma 6.1 *Let $\{X_{n,i},\ n=1,2,\ldots,\ i=1,2,\ldots,n\}$ be a double sequence of row-wise identically distributed normal random variables with mean zero and variance σ_n^2. Then,*

$$\max_{1 \leq i \leq n} X_{n,i}^2 \xrightarrow{p} 0, \tag{10}$$

whenever there exists a positive constant α such that $\sigma_n^2 < n^{-\alpha}$.

Although very simple, one can immediately see how powerful lemma 6.1 is, most importantly when dealing with double arrays instead of simple sequence of random variables. A first application of this lemma is for the curves within boundaries. The lemma will be stated for Λ_1, with no loss of generality.

Lemma 6.2 *Let x and $x+he$ be a pair of neighboring points within $\Lambda_{h,1} = \Lambda_1 \bigcap L_h$. Let σ_h^2 be defined as:*

$$\sigma_h^2 = E\left(\phi_{\Lambda_1}\left(f_{r_h,x,h}\right) - \phi_{\Lambda_1}\left(f_{r_h,x+he,h}\right)\right)^2,$$

where $r_h = h^{-1-\gamma}$, for some $0 < \gamma < 1/2$.

Then, $\sigma_h^2 = O\left(h^2\right)$.

All the results established so far concern average contrast around *wrong* curves on the continuum field. Since AC is defined on the lattice, rates of the L^2 convergence from the lattice to the continuum are needed.

Theorem 6.2 *Let ϕ_{Λ_c} and $\phi_{\Lambda_c,h}$ be the restrictions to Λ_c of, respectively, the free field and the lattice approximation to the free field, h being the spacing parameter for the lattice. Then, the L^2 convergence from theorem 4.1 is of order h^2 for any Hölder continuous test function f of some positive order $\alpha \geq 2$.*

Two results follow immediately theorem 6.2.

Corollary 6.1 *Let ϕ_{Λ_c} and $\phi_{\Lambda_c,h}$ be the restrictions to Λ_c of, respectively, the free field and the lattice approximation to the free field, h being the spacing parameter for the lattice. Let $f_{r_h,x,h}$ be a sequence of translations of f that preserves L^1 norm and such that $r_h = h^{-1-\gamma}$, for some positive γ, as in definition 6.2. Take $f_{h,r_h,x,h}$ to be its lattice counterpart. Then,*

$$\|f_{h,r_h,x,h} - f_{r_h,x,h}\|^2_{\mathcal{N}^2} = O\left(h^2\right),$$

for each Hölder continuous test function f of some positive order $\alpha \geq 3 + \gamma$.

Note: Because the delta sequence preserves only L^1 norm, Hölder continuity of higher order is required to obtain the same order of L^2 convergence as the ones for fixed test functions.

Corollary 6.2 *Take f to be a test function that has Hölder continuity of some positive order $\alpha \geq 3 + \gamma$. Define ϕ_{Λ_c} and $\phi_{\Lambda_c,h}$ as in theorem 6.2. Then, as $h \to 0$,*

$$\max_{\{c_\star : c_\star \cap c = \emptyset\}} \max_{x_i \in B_{h,\star}} \left(\phi_{\Lambda_c}(f_{h,r_h,x_i,h}) - \phi_{\Lambda_c,h}(f_{h,r_h,x_i,h})\right)^2 \xrightarrow{p} 0.$$

The last result needed for the *wrong* curve is the following:

Lemma 6.3 *Let $B_{h,\star} \subset \Lambda_{c,h,1}$. Take $r_h = h^{-1-\gamma}$, for some $0 < \gamma < 1/2$. Then,*

$$\max_{\{c_\star \cap c = \emptyset\}} \max_{x_i \in B_{h,\star}} \left(\phi_{\Lambda_c,h}(f_{r,x_i,h}) - \phi_{\Lambda_c,h}(f_{r,x_i+he,h})\right)^2 \xrightarrow{p} 0,$$

as $h \to 0$.

With lemma 6.3, it is proved that the average contrast around any curve disjoint from the true boundary c will be uniformly small. This result is possible only because each individual bound is negligible in L^2 sense when h is small. As will be seen in the next statements, the story for the contrast around c is completely different.

Lemma 6.4 *Let x and $x + he$ be a pair of neighboring points such that $x \in \Lambda_{h,1} = \Lambda_1 \bigcap L_h$ and $x + he \in \Lambda_{h,2} = \Lambda_2 \bigcap L_h$. Let σ_h^2 be defined as:*

$$\sigma_h^2 = E\left(\phi_{\Lambda_c}\left(f_{r_h,x,h}\right) - \phi_{\Lambda_c}\left(f_{r_h,x+he,h}\right)\right)^2,$$

where $r_h = h^{-1-\gamma}$, for some $0 < \gamma < 1/2$.

Then, as $h \to 0$,

$$\sigma_h^2 \to 2\sigma_c^2 > 0,$$

where $\sigma_c^2 = \|\phi_\Lambda\left(\delta_y\right)\|^2_{\mathcal{N}^2(\Lambda_c)}$, for $y \in \Lambda_c$.

Note: Although Theorem 6.1 is stated for curves $c_\star$ that are disjoint from c, a contrast-based procedure can still be justified. Clearly, if the intersection set between $c_\star$ and c is finite, the theorem can be applied directly. In the case when $c_\star$ and c intersection is another curve, a multi-step procedure would be enough. On the first step, all the *bad candidates* (curves with at most a finite number of points in common with c) would be eliminated. One would either come up with c or curves $c_\star$ such that $c_\star \cap c = c'$, where c' is an open curve in Λ. Instead of a single curve of highest contrast, suppose one takes the k curves of highest contrasts, where k is supposed to be fixed. The next step would be to clean the k curves and that would be done simply by finding which pieces of these curves are in c.

That procedure would be asymptotically consistent by virtue of Theorem 6.1 if one compares different pieces of the curve. Finally, one would have several *pieces* of c and the task now is to compare the contrast around the possible curves that can put them together, again using theorem 6.1. Some further iterations may be needed but theorem 6.1 ensures this procedure would eventually yield the right curve c.

7 Future Research: Wavelets and Random Fields Driven by Other Operators

The area of random fields in the presence of singularities is quite unexplored. One natural way of continuing the research done in this work would be extending these results for operators more general than the Laplacian. One example of such a class of operators would be $\sum_{\alpha<\beta} D^{\alpha}$, as studied in detail by Benassi and Jaffard in [4] for the case without discontinuities.

One question that still deserves an answer is whether wavelets can be successfully applied in this context. Although it is clear that one-dimensional methods cannot be directly extended to higher dimensions, all the results obtained in the deterministic problems create a positive expectation. Moreover, we have just become aware of some theoretical development in the wavelets construction that doesn't make use of dyadic cubes, done by, among others, Sweldens and Daubechies. A good review of this new approach is given in [22]. Since the *regularity* of the dyadic cubes is the main obstacle in applying wavelet bases in *irregular* multidimensional setups, it looks like some progress can be made through the application of *second generation* wavelets to problems of Gaussian random fields with discontinuities.

Appendix A: Proofs

Proofs for Lemmas 3.1 and 3.2 can also be found in [18]. The proofs for Theorems 3.1 and 4.1 and Lemmas 4.1, 4.2 and 5.1 are slight modifications of the ones found in [18], in order to accommodate the change from $\mathcal{N}$ to $\mathcal{N}^2$ spaces.

Lemma 3.1

For $\mu^2(k)$ and $\mu_h^2(k)$, we have

(i) *For each $k \in I\!R^2$, $\mu_h^2(k) \to \mu^2(k)$ as $h \to 0$.*

(ii) *If $\max(|k_1|, |k_2|) \leq \pi/h$, then $\mu_h^{-1}(k) \leq \frac{\pi}{2}\mu^{-1}(k)$.*

(iii) $\mu_h^{-2}(k)\ 1(\max(|k_1|, |k_2|) \leq \pi/h) \to \mu^{-2}(k)$ *as $h \to 0$ in each $L^p(I\!R^2)$, $p > 1$.*

Proof

(i) Note that

$$\begin{aligned}
\mu_h^2(k) &= h^{-2}\left(4 - 2\cos(k_1 h) - 2\cos(k_2 h)\right) + a^2 \\
&= 2k_1^2 (k_1 h)^{-2}\left(1 - \cos(k_1 h)\right) + 2k_2^2 (k_2 h)^{-2}\left(1 - \cos(k_2 h)\right) + a^2 \\
&\to k^2 + a^2 = \mu^2(k),
\end{aligned}$$

as $h \downarrow 0$, using $\lim_{x \to 0} x^{-2}(1 - \cos(x)) = 1/2$.

(ii) Use $\pi^2(1 - \cos(x)) \geq 2x^2$, for $x \in [-\pi, \pi]$.

(iii) This follows directly from (i)-(ii), using the dominated convergence theorem.

□

Lemma 3.2

$\hat{g}_h \to \hat{g}$ in each $L^p(I\!R^2)$, as $h \to 0$, for every $p \geq 2$.

Proof

In order to prove convergence for any L^p, it is sufficient to prove that $\sup_{h\leq 1} ||\hat{g}_h||_\infty < \infty$ and convergence in L^2. Suppose, with no loss of generality, that g is supported on the unit square circle. Then,

$$||\hat{g}_h||_\infty \leq ||\hat{g}||_\infty (2\pi)^{-1} h^2 \left(h^{-1} + 2\right)^2 ,$$

i.e. $\sup_{h\leq 1} ||\hat{g}_h||_\infty < \infty$.

Also,

$$\begin{aligned} ||\hat{g}_h||^2_{L^2} &= h^2 \sum |g_h|^2 \\ &\to ||g||^2_{L^2} \\ &= ||\hat{g}||^2_{L^2}, \end{aligned}$$

i.e. $||\hat{g}_h||^2_{L^2} \to ||\hat{g}||^2_{L^2}$ as $h \to 0$.

Take any $f \in \mathcal{S}(I\!R^2)$ and let $\tilde{f}$ denote the inverse Fourier transform:

$$\begin{aligned} \int \bar{f}(k)\hat{g}_h(k)dk &\to \int g(x)\overline{\tilde{f}}(x)dx \\ &= \int \hat{g}(k)\bar{f}(k)dk. \end{aligned}$$

That means that $\hat{g}_h$ converges weakly in L^2 to $\hat{g}$ which, plus the convergence of its norm, implies the convergence in L^2. $\square$

Theorem 3.1

With the realization $\phi_h(n) = \phi(f_{n,h})$, *for every* $g \in C_0^\infty(\mathbb{R}^2)$ *we have*

$$\phi_h(g) \to \phi(g),$$

in $L^2(\mathcal{Q}_{\mathcal{N}^2}, \mu_0)$, *as* $h \to 0$.

Proof

$$\|\phi_h(g) - \phi(g)\|^2_{L^2(Q_{N^2},\mu_0)} = \int \left| \frac{\hat{g}_h(k)}{\mu_h^2(k)} - \frac{\hat{g}(k)}{\mu^2(k)} \right|^2 dk \to 0,$$

because $\hat{g}_h \to \hat{g}$ and $\mu_h^{-2}(k) 1(h|k_i| \leq \pi) \to \mu^{-2}(k)$ in any L^p, in particular L^4. $\square$

Lemma 4.1

$$p_{\Lambda_h} e_n = \begin{cases} e_n - \sum_{m \in \partial\Lambda_h^{ext}} a_n(m)\, e_m, & \text{if } n \in \Lambda_h, \\ 0, & \text{if } n \in \Lambda_h^{ext}, \end{cases} \tag{G.1}$$

with some coefficients $a_n(\cdot)$.

Proof If $n \in \Lambda_h^{ext}$, it comes directly from the definition of projection. For $n \in \Lambda_h$, let $k \in \Lambda_h^{ext} \setminus \partial\Lambda_h^{ext}$:

$$\begin{aligned} (e_k, p_{\Lambda_h} e_n)_{l^2(L_h)} &= h^2 \left(\left(-\Delta_h + a^2 \right)^2 e_k, p_{\Lambda_h} e_n \right)_{\mathcal{N}_h^2} \\ &= h^2 \left(p_{\Lambda_h} \left(-\Delta_h + a^2 \right)^2 e_k, e_n \right)_{\mathcal{N}_h^2} \\ &= (e_n, e_k)_{l^2(L_h)} \\ &= 0, \end{aligned}$$

since $(-\Delta_h + a^2)^2 e_k$ has support in Λ_h^{ext} by definition of $\partial\Lambda_h^{ext}$. Thus, $e_n - p_{\Lambda_h} e_n$ is supported on Λ_h^{ext} and vanishes on $\Lambda_h^{ext} \setminus \partial\Lambda_h^{ext}$, i.e. it has support in $\partial\Lambda_h^{ext}$.

□

Lemma 4.2

For a regular region Λ, let $e = e_{I\!R^2\setminus\Lambda}$ be an orthogonal projection onto $\mathcal{N}^2_{I\!R^2\setminus\Lambda}$ in $\mathcal{N}^2$-norm and define e_h to be the projection onto the span of $\{f_{n,h} \,|\, n \in \Lambda_h^{ext}\}$. Then e_h converges strongly to e.

Proof

Given $g \in C_0^\infty(I\!R^2)$, let:

$$g_h(x) = h^2 \sum_n g(nh) f_{nh}(x), \;\; x \in I\!R^2$$

and

$$(-\Delta_h g)(x) = h^2 \sum_n \left(h^{-2} \left(4g(nh) - \sum_{|n-m|=1} g(mh) \right) \right).$$

Then, in $\mathcal{N}^2$-topology, as $h \to 0$,

$$\begin{aligned} g_h &\to g \\ -\Delta_h g_h &\to -\Delta g \end{aligned}$$

Now, suppose that $g \in C_0^\infty(I\!R^2 \setminus \Lambda)$. Note that:

$$\begin{aligned} \|g_h - g\|^2 &\geq |\langle e_h(g_h - g), g_h - g \rangle| \\ &= |\langle e_h(g_h - g), e_h(g_h - g) \rangle| \\ &= \|e_h(g_h - g)\|^2 \end{aligned}$$

and $\|g_h - g\| \to 0$. So $e_h \to e$ strongly, for all $g \in C_0^\infty(I\!R^2 \setminus \Lambda)$, and consequently for all g in the range of p, because Λ is regular.

Let $g \in \mathcal{N}^2$. Suppose that $e_h g \to f$ weakly. Let $h \in C_0^\infty(\Lambda)$:

$$\begin{aligned}
\int h(x) f(x) dx &= \left(\left(-\Delta + a^2\right)^2 h, f \right)_{\mathcal{N}^2} \\
&= \lim_h \left(\left(-\Delta_h + a^2\right)^2 h_h, f \right)_{\mathcal{N}^2} \\
&= \lim_h \left(\left(-\Delta_h + a^2\right)^2 h_h, p_h g \right)_{\mathcal{N}^2} \\
&= 0,
\end{aligned}$$

because f has support on $\mathbb{R}^2 \setminus \Lambda$ and, for h small enough, $(-\Delta_h + a^2)$ has support in Λ_h. So, f is in the range of p and, then for $h \in \mathcal{N}^2$,

$$\begin{aligned}
(h, f) &= (eh, f) \\
&= \lim_h (eh, e_h g) \\
&= \lim_h (e_h eh, g) \\
&= (eh, g).
\end{aligned}$$

So $e_h g \to eg$ weakly. But

$$\begin{aligned}
\|e_h g - eg\|^2 &= \langle e_h g - eg, e_h g - eg \rangle \\
&= \langle g, e_h g \rangle + \langle g, eg \rangle \\
&\quad - \langle eg, e_h g \rangle - \langle e_h g, eg \rangle \\
&\to 0,
\end{aligned}$$

by the weak convergence above. □

Theorem 4.1

Let Λ be a regular region. For $g \in C_0^\infty(\mathbb{R}^2)$, define

$$\phi_{\Lambda,h}(g) = \sum_{n \in \mathbf{Z}^2} g(nh)\, \phi_{\Lambda,h}(n).$$

Then, as $h \downarrow 0$,

$$\phi_{\Lambda,h}(g) \to \phi_\Lambda(g),$$

in $L^2(\mathcal{Q}_{\mathcal{N}^2_\Lambda}, \mu_0)$.

Proof

We need a definition of an isometric projection in $L^2(\mathcal{Q}_{\mathcal{N}^2}, d\mu_0)$ to the projections defined in $\mathcal{N}^2$. This is given by $\Gamma(.)$. For the projections e_h and e in $\mathcal{N}^2$, there exist respective unique induced projections $\Gamma(e_h)$ and $\Gamma(e)$ in $L^2(\mathcal{Q}_{\mathcal{N}^2}, d\mu_0)$ such that the isometries

$$(\Gamma(e)\phi, f) = (\phi, ef)$$

and

$$(\Gamma(e_h)\phi, f) = (\phi, e_h f)$$

hold for every $\phi \in L^2(\mathcal{Q}_{\mathcal{N}^2}, d\mu_0)$ and $f \in C_0^\infty(I\!R^2)$. Then, let $\Gamma(p) = \Gamma(1-e)$ and $\Gamma(p_h) = \Gamma(1-e_h)$.

Since $\phi_h(g) \to \phi(g)$ in $L^2(\mathcal{Q}_{\mathcal{N}}, d\mu_0)$ and $p_h \to p$ strongly,

$$\begin{aligned} \|\Gamma(p_h)\phi_h(g) - \Gamma(p)\phi(g)\| &\leq \|\Gamma(p_h)\|\|\phi_h(g) - \phi(g)\| \\ &+ \|\Gamma(p - p_h)\phi(g)\| \\ &\to 0. \end{aligned}$$

□

Lemma 5.1

Any open convex region $\Lambda_\star$ in $I\!R^2$ is regular.

Proof

Suppose, with no loss of generality, that $0 \in \Lambda_\star$. Then, for any $\lambda > 1$, $\{\lambda y | y \in I\!R^2 \setminus \Lambda_\star\} \subset S^o \subset I\!R^2 \setminus \Lambda_\star$, for some S. Let $f \in \mathcal{N}^2$ and define:

$$f_\lambda(y) = f\left(\lambda^{-1} y\right).$$

f_λ converges to f in $\mathcal{N}^2$-norm. Moreover, defining appropriate unitary and multiplier functions in $C_0^\infty\left(I\!R^2 \setminus \Lambda_\star\right)$, one can successfully approximate f_λ. □

Lemma 5.2

Let $\Lambda_\star$ be an open region in $I\!R^2$. Then $C_0^\infty\left(\Lambda_\star\right)$ is dense in $\mathcal{N}^2_{\Lambda_\star}$ with the $\mathcal{N}^2$-topology.

Proof

One knows that, for any $f \in \mathcal{N}^2_{\Lambda_\star}$, there are two sequences of elements on $C_0^\infty\left(I\!R^2\right)$, called unitary and multiplier functions, that approximate f. The task, therefore, is to prove that such sequences are actually elements of $C_0^\infty\left(\Lambda_\star\right)$, which can be easily done given that $\Lambda_\star$ is open and the convolution and multiplication supports can be taken as narrow as one wants. □

Theorem 5.1

Let Λ_0 and Λ_ϵ be given as above. Define ϕ_c to be the Gaussian field indexed by $\mathcal{N}^2_{\Lambda_0}$, and $\phi_{h,\epsilon}$ to be the Gaussian field indexed by $\mathcal{N}^2_{\Lambda_{h,\epsilon}}$. Then for $g \in C_0^\infty(I\!R^2)$, as $h \downarrow 0$ and subsequently $\epsilon \downarrow 0$,

$$\phi_{h,\epsilon}(g) \to \phi_c(g),$$

in $L^2(\mathcal{Q}_{\mathcal{N}^2_{\Lambda_0}}, \mu_0)$.

Proof

Define for each $\epsilon > 0$, ϕ_ϵ to be the Gaussian random field indexed by $\mathcal{N}^2_{\Lambda_\epsilon}$ and, since Λ_ϵ is a regular region, it follows from Theorem 4.1 that, for $g \in C_0^\infty(\mathbb{R}^2)$ and fixed $\epsilon > 0$, $\phi_{h,\epsilon}(g) \to \phi_\epsilon(g)$ as $h \downarrow 0$. Then, letting $\epsilon \downarrow 0$, the operators $\Gamma_{\Lambda_\epsilon}$ are monotonically increasing with respect to the $\mathcal{N}^2_{\Lambda_0}$-norm, so the sequence of fields $\{\phi_\epsilon, \epsilon > 0\}$ converges to ϕ_0. $\Box$

Lemma 6.1

Let $\{X_{n,i},\ n = 1, 2, \ldots,\ i = 1, 2, \ldots, n\}$ be a double sequence of row-wise identically distributed normal random variables with mean zero and variance σ_n^2. Then,

$$\max_{1 \le i \le n} X_{n,i}^2 \xrightarrow{p} 0, \tag{G.2}$$

whenever there exists a positive constant α such that $\sigma_n^2 < n^{-\alpha}$.

Proof

As $n \to \infty$,

$$P(X_{n,i}^2 > \epsilon) \sim \frac{2\sigma_n}{\sqrt{\epsilon}} exp(-\frac{\epsilon}{2\sigma_n^2})$$

implies

$$\begin{aligned} P\left(\max_{1 \le i \le n} X_{n,i}^2 > \epsilon\right) &= P\left(\bigcup_{1 \le i \le n} \left[X_{n,i}^2 > \epsilon\right]\right) \\ &\le nP\left(X_{n,1}^2 > \epsilon\right) \\ &\sim \frac{n2\sigma_n}{\sqrt{\epsilon}} exp(-\frac{\epsilon}{2\sigma_n^2}) \\ &\to 0, \end{aligned}$$

if $\sigma_n < n^{-\alpha}$, for some positive α. $\square$

Theorem 6.2

Let ϕ_{Λ_c} and $\phi_{\Lambda_c,h}$ be the restrictions to Λ_c of, respectively, the free field and the lattice approximation to the free field, h being the spacing parameter for the lattice. Then, the L^2 convergence from theorem 4.1 is of order h^2 for any Hölder continuous test function f of some positive order $\alpha \geq 2$.

Proof

Notice that $1 - \cos(x) = O\left(x^2\right)$ as $x \to 0$. Moreover, since

$$\hat{f}_h(k) = \left[\sum_{n \in \boldsymbol{Z}^2} \frac{h^2}{2\pi} f(nh) \exp(-iknh)\right] 1(\max(|k_1|, |k_2|) \leq \pi/h),$$

the Fourier transformation of f_h is equivalent to the Fourier transform of a step approximation of f, where h represents the length of each step. Therefore, given the Hölder continuity of f, one can approximate $\hat{f}$ by $\hat{f}_h$ in L^2 with appropriate order.

So,

$$\begin{aligned} \|f_h - f\|^2_{\mathcal{N}^2} &= \int \left| \frac{\hat{f}_h(k)}{\mu_h^2(k)} - \frac{\hat{f}(k)}{\mu^2(k)} \right|^2 \\ &\leq 2\left[\int \left| \frac{1}{\mu_h^2(k)} - \frac{1}{\mu^2(k)} \right|^2 \left|\hat{f}^2\right|^2 \right. \\ &+ \left. \int \frac{1}{\mu_h^4(k)} \left|\hat{f}_h(k) - \hat{f}(k)\right|^2 d^2k\right] \\ &\leq Mh^2, \end{aligned}$$

the first integral due to the bound for $1 - cos(x)$ and the fact that g is a test function (and, consequentially, its Fourier transform lives in Schwartz space) and the second integral by the lower bound for $1/\mu$ and the Hölder bound for f. □

Corollary 6.1

Let ϕ_{Λ_c} and $\phi_{\Lambda_c,h}$ be the restrictions to Λ_c of, respectively, the free field and the lattice approximation to the free field, h being the spacing parameter for the lattice. Let $f_{r_h,x,h}$ be a sequence of translations of f that preserves L^1 norm and such that $r_h = h^{-1-\gamma}$, for some positive γ, as in definition 6.2. Take $f_{h,r_h,x,h}$ to be its lattice counterpart. Then,

$$\|f_{h,r_h,x,h} - f_{r_h,x,h}\|^2_{\mathcal{N}^2} = O\left(h^2\right),$$

for each Hölder continuous test function f of some positive order $\alpha \geq 3+\gamma$.

Proof

The proof is the same as for theorem 6.2. □

Corollary 6.2

Take f to be a test function that has Hölder continuity of some positive order $\alpha \geq 3+\gamma$. Define ϕ_{Λ_c} and $\phi_{\Lambda_c,h}$ as in theorem 6.2. Then, as $h \to 0$,

$$\max_{\{c_\star : c_\star \cap c = \emptyset\}} \max_{x_i \in B_{h,\star}} \left(\phi_{\Lambda_c}(f_{h,r_h,x_i,h}) - \phi_{\Lambda_c,h}(f_{h,r_h,x_i,h})\right)^2 \xrightarrow{p} 0.$$

Proof

Notice that, defining σ_h^2 as the variance of the process above and using theorem 6.2 one can immediately apply lemma 6.1. □

Lemma 6.2

Let x and $x+he$ be a pair of neighboring points within $\Lambda_{h,1} = \Lambda_1 \bigcap L_h$. Let σ_h^2 be defined as:

$$\sigma_h^2 = E\left(\phi_{\Lambda_1}\left(f_{r_h,x,h}\right) - \phi_{\Lambda_1}\left(f_{r_h,x+he,h}\right)\right)^2,$$

where $r_h = h^{-1-\gamma}$, for some $0 < \gamma < 1/2$.

Then, $\sigma_h^2 = O\left(h^2\right)$.

Proof

With no loss of generality, suppose e=(2,0). Let $\beta > 1$.

$$\begin{aligned}
\sigma_h^2 &= \|\phi_{\Lambda_1}(f_2) - \phi_{\Lambda_1}(f_1)\|^2_{\mathcal{N}^2} \\
&= \int \frac{|1-exp(2ik_1.h)|^2|\hat{f}_1(\frac{k}{r})|^2}{(k^2+m^2)^2} d^2k \\
&= \int \frac{\sin^2(2k_1.h)}{(k^2+m^2)^2}|\hat{f}_1(\frac{k}{r})|^2 d^2k \\
&\leq \int \frac{\sin^2(2k_1.h)}{(k_1^2+m^2)^2(k_2^2+m^2)^2}(\frac{k_1^2}{r^2}+m^2)^{-p_1}(\frac{k_2^2}{r^2}+m^2)^{-p_2} d^2k \\
&\leq M\left[\int_{|k_1|\leq h^{-1+\gamma}} k_1^2 h^2 \frac{(\frac{k_1^2}{r^2}+m^2)^{-p_1}}{(k_1^2+m^2)^2} dk_1\right. \\
&+ \int_{h^{-1+\gamma}<|k_1|\leq h^{\beta(-1-\gamma)}} \frac{(\frac{k_1^2}{r^2}+m^2)^{-p_1}}{(k_1^2+m^2)^2} dk_1
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{|k_1|>h^{\beta(-1-\gamma)}} \frac{(\frac{k_1^2}{r^2}+m^2)^{-p_1}}{(k_1^2+m^2)^2} dk_1 \Bigg] \\
&\leq M\left[M_1 h^2 + M_2 h^{4(1-\gamma)} + M_3 h^{4\beta(1+\gamma)}\right] \\
&= M_\star h^2
\end{aligned}
$$

□

Lemma 6.3

Let $B_{h,\star} \subset \Lambda_{c,h,1}$. Take $r_h = h^{-1-\gamma}$, for some $0 < \gamma < 1/2$. Then,

$$
\max_{\{c_\star \cap c = \emptyset\}} \max_{x_i \in B_{h,\star}} \left(\phi_{\Lambda_c,h}(f_{r,x_i,h}) - \phi_{\Lambda_c,h}(f_{r,x_i+he,h})\right)^2 \xrightarrow{p} 0,
$$

as $h \to 0$.

Proof

For notational purposes let's write $\max_{c_\star:\{c_\star \cap c = \emptyset\}} \max_{x_i \in B_{h,\star}}$ as $\max_{c_\star}$. Also, one writes, with no loss of generality, f_2 and f_1 for the test functions defined by the different bounds. Note that:

$$
\begin{aligned}
\max_{c_\star} \left(\phi_{\Lambda_c,h}(f_2) - \phi_{\Lambda_c,h}(f_1)\right)^2 &\leq \max_{c_\star} \left(\phi_{\Lambda_c,h}(f_2) - \phi_{\Lambda_c}(f_2)\right)^2 \\
&+ \max_{c_\star} \left(\phi_{\Lambda_c,h}(f_1) - \phi_{\Lambda_c}(f_1)\right)^2 \\
&+ \max_{c_\star} \left(\phi_{\Lambda_c}(f_2) - \phi_{\Lambda_c}(f_1)\right)^2 .
\end{aligned}
$$

One, then, uses the convergence of the first and second summands by Corollary 6.2 and the third by Lemma 6.2 and theorem 6.1 to get the desired result. □

Theorem 6.1

Take f to be a test function that has Hölder continuity of some positive order, $\alpha \geq 3+\gamma$. Let $r_h = h^{-1-\gamma}$, for some $0<\gamma<1/2$. Then, as $h \to 0$,

$$P\left(AC_{c,h} \leq \max_{\{c_\star : c \cap c_\star = \emptyset\}} AC_{c_\star,h}\right) \to 0.$$

Proof

Since $\max_{\{c_\star : c\cap c_\star=\emptyset\}} AC_{c_\star,h} \to 0$ in probability as $h \to 0$, one concludes that, $\forall \epsilon, \gamma$, $\exists h_0$ such that:

$$P\left(\max_{c_\star:\{c\cap c_\star=\emptyset\}} AC_{c_\star,h} > \epsilon\right) < \gamma/2,$$

$\forall h \leq h_0$.

Following lemma 6.4, one knows that the contrast of each bound in B_h behaves in such a way that, $\forall \gamma > 0$, $\exists \epsilon$ and h_2 such that

$$P\left(\left|\phi_{\Lambda_c}\left(f_{r_h,x,h}\right) - \phi_{\Lambda_c}\left(f_{r_h,x,h}\right)\right|^2 > \epsilon\right) < \gamma/2$$

$\forall h \geq h_2$.

Since the variance for the average contrast is no larger than the average of each of its components (that are asymptotically equally distributed), $\forall \gamma$, $\exists \epsilon$ and h_1 such that:

$$P\left(AC_{c,h} < \epsilon\right) < \gamma/2,$$

$\forall h \leq h_1$.

Therefore, for every γ, one gets:

$$P\left(AC_c \leq \max_{\{c_\star : c\cap c_\star=\emptyset\}} AC_{c_\star,h}\right) < 1-\gamma,$$

$\forall h \leq \min(h_0, h_1)$. □

Lemma 6.4

Let x and $x+he$ be a pair of neighboring points such that $x \in \Lambda_{h,1} = \Lambda_1 \bigcap L_h$ and $x+he \in \Lambda_{h,2} = \Lambda_2 \bigcap L_h$. Let σ_h^2 be defined as:

$$\sigma_h^2 = E\left(\phi_{\Lambda_c}\left(f_{r_h,x,h}\right) - \phi_{\Lambda_c}\left(f_{r_h,x+he,h}\right)\right)^2,$$

where $r_h = h^{-1-\gamma}$, for some $0 < \gamma < 1/2$.

Then, as $h \to 0$,

$$\sigma_h^2 \to 2\sigma_c^2 > 0,$$

where $\sigma_c^2 = \|\phi_\Lambda(\delta_y)\|^2_{\mathcal{N}^2(\Lambda_c)}$, for $y \in \Lambda_c$.

Proof

Because $r_h = h^{-1-\gamma}$, $0 < \gamma < 1/2$ and the distance between points equals $2h$, $f_{r_h,x,h} = p_{\Lambda_1}\left(f_{r_h,x,h}\right)$ and $f_{r_h,x+he,h} = p_{\Lambda_2}\left(f_{r_h,x+he,h}\right)$. Therefore, $\phi_{\Lambda_c}\left(f_{r_h,x,h}\right) = \phi_{\Lambda_1}\left(f_{r_h,x,h}\right)$ and $\phi_{\Lambda_c}\left(f_{r_h,x+he,h}\right) = \phi_{\Lambda_2}\left(f_{r_h,x+he,h}\right)$ and each of these fields can be treated separately. Let y_1 and y_2 be two fixed points in the sequence of lattices for every $h \le h_0$ contained respectively in Λ_1 and Λ_2. Consider $\phi_{\Lambda_c}\left(f_{r_h,y_1,h}\right)$ and $\phi_{\Lambda_c}\left(f_{r_h,y_2,h}\right)$. By Corollary 6.1, each of these fields can be approximated in L^2 sense by the equivalent continuum fields. Notice that each of this random variables has a limiting variance σ_c^2. However:

$$\begin{aligned} \|\phi_{\Lambda_c}\left(f_{r_h,x,h}\right) - \phi_{\Lambda_c}\left(f_{r_h,x,h}\right)\|^2_{\mathcal{N}^2(\Lambda_c)} &= \|\phi_{\Lambda_c}\left(f_{r_h,y_1,h}\right) - \phi_{\Lambda_c}\left(f_{r_h,y_1}\right)\|^2_{\mathcal{N}^2(\Lambda_c)} \\ &\to 0 \end{aligned}$$

when $h \to 0$.

Therefore,

$$\left(\phi_{\Lambda_c}\left(f_{r_h,x,h}\right) - \phi_{\Lambda_c}\left(f_{r_h,x+he,h}\right)\right)^2 \xrightarrow{L^1} Y^2,$$

as $h \to 0$, where $Y \sim N(0, 2\sigma_c^2)$.

□

Appendix B: Distributions and Fourier Theory

For a more comprehensive view on distributions, we refer to [3] and [19]. For a better understanding of why and how these techniques apply to Laplacian and random fields, the collection by Reed and Simon is a good reference, specially [17], [16] and [15].

Let Ω be an open subset of $I\!R^2$ and $C^\infty(\Omega)$ be the space of all functions defined on Ω with continuous derivatives of all orders. Take K as a compact subset of Ω: $C_K^\infty(\Omega)$ will be the subspace of functions in $C^\infty(\Omega)$ supported on K.

Then, the space of test functions (supported in Ω) can be defined as

$$C_0^\infty(\Omega) = \bigcup_{K \in \Omega} C_K^\infty(\Omega).$$

Note that a test function is simply a compactly supported function, for which derivatives of all degrees exist continuously. Another notation for the space of test functions is $\mathcal{D}(\Omega)$, which relates to the definition of the space of distributions as its dual, i.e. $\mathcal{D}'(\Omega)$.

In Fourier analysis, however, $\mathcal{D}'$ does not have all the desirable properties. For that reason, Schwartz developed a smaller class, called tempered distributions, the dual of the so-called Schwartz class of functions. This class of test functions, larger than $\mathcal{D}$, is denoted by $\mathcal{S}$ and defined as the class of functions for which derivatives of all orders exists continuously and vanish at infinity more rapidly than any power of $|x|$. Following the usual dual notation, the tempered distributions class is called $\mathcal{S}'$.

In general, pointwise properties do not make sense for distributions, so it is usual to define and prove everything in an *inner product* format. Unless otherwise specified, f and g will be used for representing test functions, Greek alphabet for distributions, a for scalars and upper-case Roman letters for operators. Some of the most useful properties of the distributions are:

- the k-th distributional derivative of a distribution ϕ will be called $\phi^{(k)}(\cdot)$ and defined by:
$$\left(\phi^{(k)}, f\right) = \left(\phi, (-1)^k f^{(k)}\right);$$
- the partial derivative of degree k is given by:
$$\left(\partial^k \phi, f\right) = \left(\phi, (-1)^{|k|} \partial^k f\right);$$
- a sequence of distributions $\{\phi_n, n \in I\!N\}$ converges weakly to ϕ if
$$(\phi_n, f) \to (\phi, f),$$
for each test function f;
- If $\{\phi_n, n \in I\!N\}$ converges weakly to ϕ, then the same convergence applies to partial derivatives of all degrees.
- $\phi * f \in \mathcal{S}$;
- $((\phi * f), g) = \left(\phi, (f' * g)\right)$,
where $f'(x) = f(-x)$; and
- Suppose that $\int f = 1$ and define $f_\epsilon(x) = \epsilon^{-2} f(\epsilon^{-1} x)$. Then, $\phi * f_\epsilon \to \phi$ as $\epsilon \to 0$, i.e, distributions can be approximated by sequences of Schwartz functions;

Some of the most important properties of the Schwartz class are:

- $\mathcal{S}$ is closed under derivation, multiplication by polynomials of any degree, product, convolution and Fourier transform; and
- Because Fourier transform is an operation from $\mathcal{S}$ to $\mathcal{S}$, it is well defined for tempered distributions, on the following form:

$$\left(\hat{\phi}, f\right) = \left(\phi, \hat{f}\right)$$

and its basics properties are still valid.

References

[1] R. Adam. *Sobolev Spaces.* Academic Press, 1975.

[2] R. J. Adler. *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, volume 12 of *Lecture Notes - Monograph Series.* Institute of Mathematical Statistics, 1990.

[3] M. Al-Gwaiz. *Theory of Distributions.* Marcel Dekker, 1992.

[4] A. Benassi and S. Jaffard. Wavelet decomposition of one and several dimensional Gaussian processes. In L. Schumaker and G. Webb, editors, *Recent Advances in Wavelet Analysis*, pages 119–154. Academic Press, 1994.

[5] K. Chui. *An introduction to wavelets.* Academic Press, 1992.

[6] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. John Wiley and Sons, third edition, 1968.

[7] A. Friedman. *Partial Differential Equations.* Holt, Rinehart and Winston, 1969.

[8] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View.* Springer-Verlag, second edition, 1987.

[9] A. P. Korostelev and A. B. Tsybakov. *Minimax Theory of Image Reconstruction*, volume 82 of *Lecture Notes in Statistics*. Springer-Verlag, 1993.

[10] S. Mallat. A theory for multi-resolution signal decomposition: the wavelet representation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11(7):674–693, 1989.

[11] S. Mallat and W. L. Hwang. Singularity detection and processing with wavelets. *IEEE Transactions on Information Theory*, 38(2):617–643, 1992.

[12] S. Mallat and S. Zhong. Characterization of signals from multi-scale edges. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14(7):710–732, 1992.

[13] Y. Meyer. *Wavelets and Operators*. Cambridge University Press, 1992.

[14] L. Pitt. A Markov property for Gaussian processes with a multidimensional parameter. *Arch. Rational Mech.*, 43:367–391, 1971.

[15] M. Reed and B. Simon. *Analysis of Operators*, volume 4 of *Methods of Modern Mathematical Physics*. Academic Press, 1972.

[16] M. Reed and B. Simon. *Fourier Analysis, Self-Adjointness*, volume 2 of *Methods of Modern Mathematical Physics*. Academic Press, 1972.

[17] M. Reed and B. Simon. *Functional Analysis*, volume 1 of *Methods of Modern Mathematical Physics*. Academic Press, second edition, 1980.

[18] B. Simon. *The* $P(\Phi_2)$ *Euclidean (Quantum) Field Theory*. Princeton University Press, 1974.

[19] R. Strichartz. *A Guide to Distribution Theory and Fourier Analysis*. CRC Press, 1994.

[20] W. Sweldens. *Construction and Applications of Wavelets in Numerical Analysis*. PhD thesis, Department of Computer Science , Katholieke Universiteit Leuven, Belgium, 1994.

[21] W. Sweldens. The lifting scheme: a new philosophy in biorthogonal wavelet constructions. In *SPIE Proceedings, Signal and Image Processing III*, volume 2569, pages 69–79. 1995.

[22] W. Sweldens. The lifting scheme: A construction of second generation wavelets. *SIAM Journal of Mathematical Analysis*, 29(2):511–546, 1997.

[23] G. Walter. Wavelets and generalized functions. In C K Chui, editor, *Wavelets - A Tutorial in Theory and Applications*, pages 51–70. Academic Press, 1992.

[24] S. C. Zhu and A. Yuille. Region competition: Unifying snakes, region growing and Bayes / MDL for multi-band segmentation. *IEEE Transactions on Pattern Recognition and Machine Intelligence*, 18(9):884–900, 1996.