Partitions with Attached Parts

by

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Abstract: *In this paper we present a combinatorial interpretation, by partitions with attached parts, for a special family of summations, wich includes very interesting particular cases.*

1. Introduction

Our main purpose in this paper is to prove the following theorem:

Theorem 1. *Let $A_{k,i}(n)$ denote the number of partitions of n with parts in $M_{k,i}$ which satisfy: (a) if "$rk-(i+1)$" and "$sk-(i+1)$" are parts then $|r-s| \geq 2$, and (b) $tk+1$ is a part (repetitions allowed) only if "$(t+1)k-(i+1)$" or "$(t+2)k-(i+1)$" occurs as a part. Then for $1 \leq i \leq k$ we have*

$$\sum_{n=0}^{\infty} A_{k,i}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{kn^2+(k-i-1)n}}{(q^k, q^k)_n(q; q^k)_n}.$$

For the cases $k = i = 1$ or $k = i = 2$ or $k > 2$ and $i = k-2$ we have to consider the parts as elements of multisets indexed as $[(2+s)k-(i+1)]_1$ and $[rk+1]_2$.

Here we use the standard notation

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1})$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

The sets $M_{k,i}$ which appear in the Theorem 1 above are defined as

$$M_{k,i} = \{rk+1 \text{ or } (2+s)k-(i+1) | r,s \geq 0\}.$$

2. Particular Cases

The set of partitions enumerated by $A_{2,1}(n)$ is the same enumerated by $\mathcal{C}_{2,2}(n)$, i.e.,

"*the number of partitions of* n *wherein:(a) 2 appears as a part at most 1 time, (b) the total number of appearences of 2j and* $2j+2$ *together is at most 1, and (c)* $2j+1$ *is allowed to appear (and may be repeated if it appears) only if the total number of appearences of* $2j$ *and* $2j+2$ *together is precisely 1*"

which is given in Theorem 1, pg. 92 of Andrews and Santos [1]. From this we get the following result:

Theorem 2: Let $\mathcal{A}_{2,2}(n)$ denote the number of partitions of n into parts that are either even but $\not\equiv 0, \pm 6 \pmod{16}$ or odd and $\equiv \pm 3 \pmod{8}$; $\mathcal{B}_{2,2}(n)$ the number of partitions of n into parts that are even but $\not\equiv 0 \pmod{8}$ or distinct, odd and $\equiv \pm 3$ (mod 8); and $A_{2,1}(n)$ the number of partitions of n with parts into $M_{2,1}$ satisfying:

(a) if $2r-2$ and $2s-1$ are parts then $|r-s| \geq 2$, and

(b) $2t+1$ is a part (repetitions allowed) only if $2t$ or $2t+2$ occurs as a part. Then, for each n,

$$\mathcal{A}_{2,2}(n) = \mathcal{B}_{2,2}(n) = A_{2,1}(n).$$

We observe, furthermore, that $A_{2,1}(n)$ is also equal to $A_2(n)$:

"*The number of partitions of* n *with parts in*

$$M_2 = \{2_2, 3_3, 4_2, 4_4, 5_3, 5_5, 6_2, 6_4, 6_6, ...\}$$

such that the difference between any consecutives parts a_i *and* $b_j (a > b)$ *satisfy* $a - b \geq i + j$"

described in Theorem 2.1 of Santos and Mondek [3].

The table below give us an illustration for Theorem 2 and the observation above for $n = 10$.

$\mathcal{A}_{2,2}(10) = 7$	$\mathcal{B}_{2,2}(10) = 7$	$A_{2,1}(10) = 7$	$A_2(10) = 7$
$8+2$	10	10	10_{10}
$5+5$	$6+4$	$8+2$	10_8
$5+3+2$	$6+2+2$	$6+2+1+1$	10_6
$4+4+2$	$5+3+2$	$4+3+3$	10_4
$4+3+3$	$4+4+2$	$3+3+2+1+1$	10_2
$4+2+2+2$	$4+2+2+2$	$3+2+1+...+1$	8_4+2_2
$2+...+2$	$2+...+2$	$2+1+...+1$	8_2+2_2

The second particular case is for $k = i = 1$.

From equation 6 of [2] which is

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q;q)_n^2} = \frac{2}{(q;q)_\infty}$$

we obtain:

Theorem 3. The number of partitions of n enumerated by $A_{1,1}(n)$ is equal to two times the number of partitions of n.

For exemple we have $A_{1,1}(4) = 10$ which enumerates the following partitions:

4_1
$4_1 + 0_1$
$3_1 + 1_1$
$3_1 + 1_2 + 0_1$
$3_2 + 1_1$
$2_1 + 2_2 + 0_1$
$2_1 + 1_2 + 1_2 + 0_1$
$2_2 + 2_2 + 0_1$
$2_2 + 1_2 + 1_2 + 0_1$
$1_2 + 1_2 + 1_2 + 1_2 + 0_1$

Our third particular case is for $k = i = 2$:

Using the identity 85 of Slater [4]

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q;q)_{2n}} = (q^6;q^8)_\infty (q^2;q^8)_\infty (q^4;q^{16})_\infty (q^8;q^8)_\infty$$

we rewrite it as

$$\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q^2;q^2)_n(;q^2)_n} = \frac{1}{(q;q^2)_\infty}.$$

From this we can see that is valid the following theorem:

Theorem 4. The number of partitions of n enumerated by $A_{2,2}(n)$ is equal to number of partitions of n in odd parts.

To illustrate this theorem we present below the partitions for $n = 8$:

partitions enumerated by $A_{2,2}(8)$	partitions of 8 in odd parts
$7_1 + 1_1$	$7 + 1$
$5_2 + 3_1$	$5 + 3$
$5_1 + 1_1 + 1_2 + 1_2$	$5 + 1 + 1 + 1$
$3_2 + 3_2 + 1_1 + 1_2$	$3 + 3 + 1 + 1$
$3_2 + 1_1 + 1_2 + ... + 1_2$	$3 + 1 + ... + 1$
$1_1 + 1_2 + ...1_2$	$1 + ... + 1$

3. The Proof of Theorem 1

Proof. First we define $A_{k,i}(m,n)$ to be the number of partitions of the type enumerated by $A_{k,i}(n)$ with the added condition that the number of parts in each partitions is exactly m.

Now our goal is to prove that

$$\cup_{k,i}(z) := \sum_{m,n\geq 0} A_{k,i}(m,n) z^m q^n = \sum_{n\geq 0} \frac{z^n q^{kn^2+(k-i-1)n}}{(q^k;q^k)_n (zq;q^k)_n} := \bigvee_{k,i}(z) \qquad (1)$$

We have the following functional equation:

$$\bigvee_{k,i}(z) - \vee_{k,i}(zq^k) = \sum_{n=0}^{\infty} \frac{z^n q^{kn^2+(k-i-1)n}}{(q^k;q^k)_n (zq^{k+1};q^k)_{n-1}} \left(\frac{1}{1-zq} - \frac{q^{kn}}{1-zq^{kn+1}}\right)$$

$$= \frac{1}{(1-zq)(1-zq^{k+1})} \sum_{n=1}^{\infty} \frac{z^n q^{kn^2+(k-i-1)n}}{(q^k;q^k)_{n-1} (zq^{2k+1};q^k)_{n-1}}$$

$$= \frac{zq^{2k-i-1}}{(1-zq)(1-zq^{k+1})} \bigvee_{k,i}(zq^{2k})$$

that is,

$$\bigvee_{k,i}(z) = \bigvee_{k,i}(zq^k) + \frac{zq^{2k-i-1}}{(1-zq)(1-zq^{k+1})} \bigvee_{k,i}(zq^{2k}) \qquad (2)$$

After this we observe that (2), together with $\bigvee_{k,i}(0) = 1$, uniquely determine $\bigvee_{k,i}(z)$ as double power series in z and q.

On the other side, due to the definition of $M_{k,i}$ and the condition (b) of the enunciate, $\bigcup_{k,i}(z) - \bigcup_{k,i}(zq^k)$ enumerates all those partitions of the type enumerated by $\bigcup_{k,i}(z)$ that contain any number of 1's attached to an appearence of "$2k-(i+1)$" or any $(k+1)$'s attached to an appearence of "$2k-(i+1)$" and not to an appearence of $3k-(i+1)$. This, together with condition (a), tell us that the partitions in $\bigcup_{k,i}(z) - \bigcup_{k,i}(zq^k)$ are generated by

$$\frac{zq^{2k-i-1}}{(1-zq)(1-zq^{k+1})} \bigcup_{k,i}(zq^{2k}).$$

Considering that $\bigcup_{k,i}(0) = 1, (1)$ is proved.

To finish the proof of the theorem is sufficient to note that the same argument used above can be used for the cases: $k = i = 1, k = i = 2, k > 2$ and $i = k - 2$.

$\blacksquare$

Remark. The natural question that arise from this work: there exist others k or i, different from the ones that we give in section 2, for which we have sums?

References

[1] Andrews, G.E. and Santos, J.P.O. (1997). Rogers-Ramanujan Type Identities for Partitions with Attached Odd Parts. The Ramanujan Journal 1, 91-99.

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[3] ________. (1998). Partitions with "$\lfloor \frac{N+1}{2} \rfloor$ copies of N", Relatório de Pesquisa 09/98, Unicamp.

[4] Slater, L.J. (1952). Further identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (2) 54, 147-167.

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