ON AN ITERATIVE METHOD FOR THE APPROXIMATE SOLUTION OF AN INITIAL AND BOUNDARY-VALUE PROBLEM FOR A GENERALIZED BOUSSINESQ MODEL

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Abstract

In this work an iterative method is proposed for finding the approximate solution of an initial and boundary problem for a nonstationary Generalized Boussinesq model for thermally driven convection. The model allows temperature dependent viscosity and thermal conductivity. We give also the convergence-rate bounds for the propost method.

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1. Introduction

The purpose of this paper is to show the existence and uniqueness of a strong solution of the first initial boundary-value problem for generalized Boussinesq model of the viscous, incompressible heat conducting fluids. Let u, p, φ be the velocity, the pressure and the temperature of the fluid, respectively. The motion of the fluid is described by the initial boundary-value problem (see [2]):

$$\begin{gathered} \frac{\partial u}{\partial t} + u.\nabla u - \nabla \cdot (\nu(\varphi)\nabla u) + grad\ p = \alpha\ \varphi g + h, \\ div\ u = 0 \quad \text{in } (0,T)\times\Omega, \\ u(x,t) = 0 \text{ on } (0,T)\times\partial\Omega \quad \text{and} \quad u(x,0) = u_0(x) \quad \text{on } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded open subset of $I\!R^N, N = 2$ or 3. The conservation of internal energy is described by the initial boundary value problem

$$\begin{gathered} \frac{\partial \varphi}{\partial t} + u.\nabla\varphi - \nabla\cdot(k(\varphi)\nabla\varphi) = f \quad \text{in } (0,T)\times\Omega, \\ \varphi(x,t) = \eta \quad \text{on } (0,T)\times\partial\Omega \text{ and } \quad \varphi(x,0) = \varphi_0(x) \quad \text{on } \Omega. \end{gathered} \tag{1.2}$$

The viscosity of the fluid is $\nu(\varphi)$ and the coefficient of heat conduction is $k(\varphi), g, h$ and f are external forces, $\alpha > 0$ is a positive constant associated to the coefficient of volume expansion. The system (1.1)-(1.2) does not belong to any of the three traditional types of classification of partial differential equations. To show the existence of strong solution we will use an iterational approach and we give convergence-rates for this method in several norms. We feel that it is appropriate to cite some earlier works on the initial value problem (1.1)-(1.2) and to locate our contributions therein. For simplicity, we will consider homogeneous conditions on $\partial\Omega$; the general case can be reduced to this one by assuming suitable smoothness on the boundary data.

When $\nu(\varphi)$ and $k(\varphi)$ are a positive constants, the problem (1.1)-(1.2) is the classical Boussinesq model, this model has well studied, see for instance Morimoto [8], Hishida [3], Rojas-Medar and Lorca [11], [12], [13].

The model considered in this work was studied by Lorca and Boldrini [5], [6],

[7], they used the spectral Galerkin method as method of approximation.
Following ideas from [14], an iterational method was proposed by Zarubin [16] for finding the approximate solution of the classical Boussinesq equations. Unfortunately, although the statement of Theorem 1, p.1081 in [16] furnishes a convergence rate, the proof of this result is incorrect. In another class of fluids Ortega-Torres and Rojas-Medar [9], Ortega-Torres, Rojas-Medar and Conca [10] obtained the convergence rates for the method proposed by Zarubin.
In this paper we will combine the arguments used by Lorca and Boldrini [5] and Ortega-Torres and Rojas-Medar [9] to show the existence and uniqueness of strong solutions for problem (1.1)-(1.2) as well as the convergence-rates bound. Although this not too interesting case from the practical pointview, we hope that the techniques that we developed here could be adapted in the important case where the full discretization are used.
The paper is organized as follows: In the Section 2, we state some preliminaries results that will be useful in the rest of the paper, we described the approximation method. In the Section 3, we stablished our principal result on the existence and uniqueness of strong solution as well the convergence-rate bounds. In the Section 4, we give the results on the pressure.
Finally, we would like to say that, as it usual in this context, to simplicity the notation in the expressions we will denote by $C, C_1, \ldots$,generic positive constants depending only on the fixed data of the problem.

2. Preliminaries and Results

We begin by recalling certain definitions and facts to be used later in this paper. The $L^2(\Omega)$-product and norm are denoted by $(\,,\,)$ and $|\ \ |$, respectively; the $L^p(\Omega)$-norm by $|\ \ |_{L^p}, 1 \leq p \leq \infty$; the $H^m(\Omega)$- norm are denoted by $\|\ \ \|_{H^m}$ and the $W^{k,p}(\Omega)$-norm by $|\ \ |_{W^{k,p}}$.
Here $H^m(\Omega) = W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the H^1-norm.

If B is a Banach space, we denote $L^q(0,T;B)$ the Banach space of the B-valued functions defined in the interval (0, T) that are L^q-integrable in the sense of Bochner.
Let $C^\infty_{0,\sigma}(\Omega) = \{v \in C_0^\infty(\Omega)^N;\ \text{div } v = 0\}, V = \text{closure of } C^\infty_{0,\sigma}(\Omega) \text{ in } (H_0^1(\Omega))^N$ and $H = \text{closure of } C^\infty_{0,\sigma}(\Omega)$ in $(L^2(\Omega))^N$.
Let P be the orthogonal projection from $(L^2(\Omega))^N$ onto H obtained by the usual Helmholtz decomposition. Then, the operator $A : H \to H$ given by $A = -P\Delta$ with domain $D(A) = (H^2(\Omega))^N \cap V$ is called the Stokes operator.
In order to obtain regularity properties of the Stokes operator we will assume that Ω is of class $C^{1,1}$ [1]. This assumption implies, in particular, that when $Au \in L^2(\Omega)$, then $u \in H^2(\Omega)$ and $\|u\|_{H^2}$ and $|Au|$ are equivalent norms.
Throughout the paper, we will suppose that ν, ν', k, k' are continuous functions and

$$\begin{gathered} 0 < \nu_0 < \nu(\theta) < \nu_1 < +\infty, \quad 0 < k_0 < k(\theta) < k_1 < +\infty \\ |\nu'(\theta)| < \nu_1' < +\infty, \quad |k'(\theta)| < k_1' < +\infty, \quad \text{for all } \theta \in I\!R. \end{gathered} \tag{2.1}$$

We consider the following iterative process of the approximate solution of problem (1.1)-(1.2).
If u^n is given, we defined the following equations,

$$u_t^{n+1} - P(div(\nu(\varphi^{n+1})\nabla u^{n+1})) + P(u^n.\nabla u^{n+1}) + P(\alpha\varphi^{n+1}g) = Ph, \tag{2.2}$$

$$\varphi_t^{n+1} - (div(k(\varphi^{n+1})\nabla\varphi^{n+1})) + (u^n.\nabla\varphi^{n+1}) = f, \tag{2.3}$$

$$u^{n+1}(x,0) = 0, \quad \varphi^{n+1}(x,0) = 0, \text{ in } \Omega. \tag{2.4}$$

Where for simplicity of exposition, we have taken homogeneous boundary conditions, and $u_0 = \varphi_0 = 0$.
Combining the arguments of [5] and [9] it is possible to show the following uniform estimates in n for the approximations (u^n, φ^n).

Lemma 2.1. Let Ω be a bounded domain in $I\!R^N(N=2 \quad \text{ou} \quad 3)$ with $C^{1,1}$ boundary; we suposse ν, k satisfying (2.1), $g \in L^\infty(0,T;(L^2(\Omega))^N); f, f_t \in L^2(0,T;L^2(\Omega)); h, g_t, h_t \in L^2(0,T;(L^2(\Omega))^N)$. Let $u^1=\varphi^1=0$.
Then for each n, the problem (2.2)-(2.4), has an unique strong solution (u^n,φ^n) such that $u^n \in L^\infty(0,T;D(A)), \varphi^n \in L^\infty(0,T;H^2(\Omega))$, and $u_t^n \in L^\infty(0,T;H)\cap L^2(0,T;V), \varphi_t^n \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, for each n and the following estimates uniformly in n, are verified:

$$sup_t\{|u_t^n(t)|^2+|\varphi_t^n(t)|^2\} \le M,$$
$$\int_0^t |\nabla u_\tau^n(\tau)|^2 d\tau + \int_0^t |\nabla \varphi_\tau^n(\tau)|^2 d\tau \le M,$$
$$sup_t\{|Au^n(t)|^2+|\Delta\varphi^n(t)|^2\} \le M,$$

for all $t\in[0,T]$, where M is a positive constant independent of n.

Theorem 2.1 Under the conditions of Lemma 2.1, then the approximate solutions (u^n,φ^n) converge in the space $L^\infty(0,T;D(A))\times L^\infty(0,T;H^2(\Omega))$.
The limiting element (u,φ) is a solution of problem (1.1)-(1.2) and the solution is unique. The rate of convergence satisfies the inequalities:

$$sup_t\{|\nabla u^n(t)-\nabla u(t)|^2+|\nabla\varphi^n(t)-\nabla\varphi(t)|^2\} \le M_2\frac{(M_1T)^{n-1}}{(n-1)!},$$
$$\int_0^t |Au^n(\tau)-Au(\tau)|^2 d\tau + \int_0^t |\Delta\varphi^n(\tau)-\Delta\varphi(\tau)|^2 d\tau \le M_3\frac{(M_1T)^{n-1}}{(n-1)!},$$
$$\int_0^t |\nabla u^n(\tau)-\nabla u(\tau)|^2 d\tau + \int_0^t |\nabla\varphi^n(\tau)-\nabla\varphi(\tau)|^2 d\tau \le M_4\frac{(M_1T)^{n}}{n!},$$
$$\int_0^t |u_\tau^n(\tau)-u_\tau(\tau)|^2 d\tau + \int_0^t |\varphi_\tau^n(\tau)-\varphi_\tau(\tau)|^2 d\tau \le M_5\frac{(M_1T)^{n-1}}{(n-1)!},$$

$$sup_t\{|u_t^n(t)-u_t(t)|^2+|\varphi_t^n(t)-\varphi_t(t)|^2\}$$
$$\le M_6\left[\frac{(M_1T)^{n-2}}{(n-2)!}+\left[\frac{(M_1T)^{n-1}}{(n-1)!}\right]^{1/2}\right],$$

$$\int_0^t |\nabla u_\tau^n(\tau) - \nabla u_\tau(\tau)|^2 d\tau + \int_0^t |\nabla \varphi_\tau^n(\tau) - \nabla \varphi_\tau(\tau)|^2 d\tau$$
$$\leq M_6 \left[\frac{(M_1 T)^{n-2}}{(n-2)!} + \left[\frac{(M_1 T)^{n-1}}{(n-1)!} \right]^{1/2} \right],$$

$$sup_t\{|Au^n(t) - Au(t)|^2 + |\Delta\varphi^n(t) - \Delta\varphi(t)|^2\}$$
$$\leq M_7 \left[\frac{(M_1 T)^{n-2}}{(n-2)!} + \left[\frac{(M_1 T)^{n-1}}{(n-1)!} \right]^{1/2} \right],$$

for all $t \in [0, T]$, where the positives constants are independents of n.

3. Proof of Theorem 2.1

In this section, we prove several convergence-rates bounds for the approximate solutions.

The following lemma will be fundamental in our future arguments.

Lemma 3.1. Let $0 \leq \beta_1(t) \leq M$ for all $t \in [0, T]$ and assume that the following inequality is true for all $r \geq 2$

$$0 \leq \beta_r(t) \leq C \int_0^t \beta_{r-1}(s) ds.$$

Then,

$$\beta_r(t) \leq M \frac{(Ct)^{r-1}}{(r-1)!} \leq M \frac{(C\ T)^{r-1}}{(r-1)!} \tag{3.1}$$

for all $t \in [0, T]$ and $r \geq 2$. Therefore, $\beta_r(t) \to 0$ as $r \to \infty, \forall t \in [0, T]$.

Moreover,

$$\int_0^t \beta_r(s) ds \leq \frac{MC^{r-1}t^r}{r!} \leq \frac{M}{C} \frac{(CT)^r}{r!}.$$

(See [9]).

Let $u^{n,s}(t) = u^{n+s}(t) - u^n(t)$ and $\varphi^{n,s}(t) = \varphi^{n+s}(t) - \varphi^n(t)$, $\forall n, s \geq 1$.
Then the following equation are satisfied by $u^{n,s}$ and $\varphi^{n,s}$

$$\begin{aligned} u_t^{n,s} - P((div(\nu(\varphi^{n+s})\nabla u^{n,s}) + (u^{n-1,s}.\nabla u^n) - \alpha g \varphi^{n,s} \\ -div((\nu(\varphi^{n+s}) - \nu(\varphi^n))\nabla u^n) + (u^{n+s-1}.\nabla u^{n,s})) = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \varphi_t^{n,s} - div(k(\varphi^{n+s})\nabla\varphi^{n,s}) + (u^{n+s-1}.\nabla\varphi^{n,s}) \\ -div((k(\varphi^{n+s}) - k(\varphi^n))\nabla\varphi^n) + (u^{n-1,s}.\nabla\varphi^n) = 0. \end{aligned} \tag{3.3}$$

Lemma 3.2. Let $v \in V \cap (H^2(\Omega))^N$ and consider the Helmholtz decomposition of $-\Delta v$, that is,

$$-\Delta v = A\ v + \nabla q,$$

where $q \in H^1(\Omega)$ is taken such that $\int_\Omega q\ dx = 0$.
Then, for every $\varepsilon > 0$ there exists a positive constant C_ε independent of v; and there exists c such that, the following estimates holds

$$|q| \leq C_\varepsilon |\nabla v| + \varepsilon |Av|, \quad \|q\|_{H^1(\Omega)} \leq c|Av|. \tag{3.4}$$

(See [5]).

Lemma 3.3. There exists a positive constant $C > 0$, independent of n and s, such that:

$$\begin{aligned} |\nabla u^{n,s}(t)|^2 + |\nabla\varphi^{n,s}(t)|^2 + \textstyle\int_0^t |Au^{n,s}(\tau)|^2 d\tau + \int_0^t |\Delta\varphi^{n,s}(\tau)| d\tau \\ \leq C \textstyle\int_0^t (|\nabla u^{n-1,s}(\tau)|^2 + |\nabla\varphi^{n-1,s}(\tau)|^2 + |\nabla u^{n,s}(\tau)|^2 + |\nabla\varphi^{n,s}(\tau)|^2) d\tau \end{aligned}$$

Proof. Multiplying (3.2.) by $Au^{n,s}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla u^{n,s}|^2 - (div(\nu(\varphi^n)\nabla u^{n,s}), Au^{n,s}) = -(u^{n-1,s}.\nabla u^n, Au^{n,s}) \\ &+(div((\nu(\varphi^{n+s}) - \nu(\varphi^n))\nabla u^n, Au^{n,s}) - (u^{n+s-1}.\nabla u^{n,s}, Au^{n,s}) \\ &-\alpha(g\varphi^{n,s}, Au^{n,s}). \end{aligned}$$

Using the identity: $div(\nu(\theta)\nabla v) = \nu(\theta)\Delta v + \nu'(\theta)\nabla(\theta)\nabla v$, and Lemma 3.2, then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla u^{n,s}|^2 + (\nu(\varphi^{n+s})Au^{n,s}, Au^{n,s}) = ((\nu(\varphi^{n+s}) - \nu(\varphi^n))Au^n, Au^{n,s})\\
&+((\nu'(\varphi^{n+s})\nabla\varphi^{n+s} - \nu'(\varphi^n)\nabla\varphi^n)\nabla u^n, Au^{n,s})\\
&+(\nu'(\varphi^{n+s})\nabla\varphi^{n+s}\nabla u^{n,s}, Au^{n,s}) + (\nu(\varphi^{n+s})\nabla q^{n,s}, Au^{n,s}) \qquad (3.5)\\
&+((\nu(\varphi^{n+s}) - \nu(\varphi^n))\nabla q^n, Au^{n,s}) + (u^{n+s-1}\cdot\nabla u^{n,s}, Au^{n,s})\\
&+(u^{n-1,s}\cdot\nabla u^n, Au^{n,s}) + \alpha(g\varphi^{n,s}, Au^{n,s}).
\end{aligned}$$

Now, we estimate the right hand side terms by using Hölder's inequality, Sobolev embedding and Young's inequality, we obtain:

$$\begin{aligned}
|((\nu(\varphi^{n+s}) - \nu(\varphi^n))Au^n, Au^{n,s})| &\leq C|\nu(\varphi^{n+s}) - \nu(\varphi^n)|_{L^\infty}|Au^n||Au^{n,s}|\\
&\leq C|\varphi^{n,s}|_{L^\infty}|Au^n||Au^{n,s}|\\
&\leq C|\nabla\varphi^{n,s}|^{1/2}|\Delta\varphi^{n,s}|^{1/2}|Au^{n,s}|\\
&\leq C_\varepsilon|\nabla\varphi^{n,s}||\Delta\varphi^{n,s}| + \varepsilon|Au^{n,s}|^2\\
&\leq C_{\varepsilon,\varepsilon_1}|\nabla\varphi^{n,s}|^2 + \varepsilon_1|\Delta\varphi^{n,s}|^2 + \varepsilon|Au^{n,s}|^2,
\end{aligned}$$

$$\begin{aligned}
&|((\nu'(\varphi^{n+s})\nabla\varphi^{n+s} - \nu'(\varphi^n)\nabla\varphi^n)\nabla u^n, Au^{n,s})|\\
&\leq C|\nu'(\varphi^{n+s})\nabla\varphi^{n,s} + ((\nu'(\varphi^{n+s}) - \nu'(\varphi^n))\nabla\varphi^n|_{L^3}|\nabla u^n|_{L^6}|Au^{n,s}|\\
&\leq C(\nu_1'|\nabla\varphi^{n,s}|_{L^3} + |\varphi^{n,s}|_{L^6}|\nabla\varphi^n|_{L^6})|Au^{n,s}|\\
&\leq C_{\varepsilon,\varepsilon_1}|\nabla\varphi^{n,s}|^2 + \varepsilon_1|\Delta\varphi^{n,s}|^2 + C_\varepsilon|\nabla\varphi^{n,s}|^2 + \varepsilon|Au^{n,s}|^2,
\end{aligned}$$

$$\begin{aligned}
|(\nu'(\varphi^{n+s})\nabla\varphi^{n+s}\nabla u^{n,s}, Au^{n,s})| &\leq C\nu_1'|\nabla\varphi^{n+s}|_{L^4}|\nabla u^{n,s}|_{L^4}|Au^{n,s}|\\
&\leq C|\Delta\varphi^{n+s}||\nabla u^{n,s}|^{1/4}|Au^{n,s}|^{7/4}\\
&\leq C_\varepsilon|\nabla u^{n,s}|^2 + \varepsilon|Au^{n,s}|^2,
\end{aligned}$$

$$\begin{aligned}|(\nu(\varphi^{n+s})\nabla q^{n,s}, Au^{n,s})| &= |(q^{n,s}, div(\nu(\varphi^{n+s})Au^{n,s}))| \\ &= |(q^{n,s}, \nu'(\varphi^{n+s})\nabla\varphi^{n+s}Au^{n,s})| \\ &\leq C\nu_1'|q^{n,s}|_{L^4}|\nabla\varphi^{n+s}|_{L^4}|Au^{n,s}| \\ &\leq C|q^{n,s}|^{1/4}|\|q^{n,s}\|_{H^1}^{3/4}|\Delta\varphi^{n+s}||Au^{n,s}| \\ &\leq C_\varepsilon|\nabla u^{n,s}|^2 + \varepsilon|Au^{n,s}|^2,\end{aligned}$$

$$\begin{aligned}|((\nu(\varphi^{n+s}) - \nu(\varphi^n))\nabla q^n, Au^{n,s})| &\leq C|\varphi^{n,s}|_{L^\infty}|\nabla q^n||Au^{n,s}| \\ &\leq C|\nabla\varphi^{n,s}|^{1/2}|\Delta\varphi^{n,s}|^{1/2}|Au^n||Au^{n,s}| \\ &\leq C_{\varepsilon,\varepsilon_1}|\nabla\varphi^{n,s}|^2 + \varepsilon_1|\Delta\varphi^{n,s}|^2 + \varepsilon|Au^{n,s}|^2,\end{aligned}$$

$$\begin{aligned}|(u^{n+s-1}.\nabla u^{n,s}, Au^{n,s})| &\leq C|u^{n+s-1}|_{L^6}|\nabla u^{n,s}|_{L^3}|Au^{n,s}| \\ &\leq C|\nabla u^{n+s-1}||\nabla u^{n,s}|^{1/2}|Au^{n,s}|^{3/2} \\ &\leq C_\varepsilon|\nabla u^{n,s}|^2 + \varepsilon|Au^{n,s}|^2,\end{aligned}$$

$$\begin{aligned}|(u^{n-1,s}.\nabla u^n, Au^{n,s})| &\leq C|u^{n-1,s}|_{L^6}|\nabla u^n|_{L^3}|Au^{n,s}| \\ &\leq C|\nabla u^{n-1,s}||Au^n||Au^{n,s}| \\ &\leq C_\varepsilon|\nabla u^{n-1,s}|^2 + \varepsilon|Au^{n,s}|^2,\end{aligned}$$

$$\begin{aligned}|\alpha(g\varphi^{n,s}, Au^{n,s})| &\leq C|g|_{L^3}|\varphi^{n,s}|_{L^6}|Au^{n,s}| \\ &\leq C_\varepsilon|\nabla\varphi^{n,s}|^2 + \varepsilon|Au^{n,s}|^2.\end{aligned}$$

By taking $\varepsilon, \varepsilon_1 > 0$ sufficiently smalls in the above estimates, we obtain in (3.5) the following integral inequality:

$$\begin{aligned}&|\nabla u^{n,s}(t)|^2 + \nu_0\int_0^t |Au^{n,s}(\tau)|^2 d\tau \\ &\leq C\int_0^t(|\nabla u^{n-1,s}(\tau)|^2 + |\nabla u^{n,s}(\tau)|^2)d\tau + C\int_0^t |\nabla\varphi^{n,s}(\tau)|^2 d\tau \\ &+3\varepsilon_1\int_0^t |\Delta\varphi^{n,s}(\tau)|^2 d\tau.\end{aligned} \tag{3.6}$$

Analogously, multiplying(3.3) by $\Delta\varphi^{n,s}$, we obtain

$$|\nabla\varphi^{n,s}(t)|^2 + \frac{3}{2}k_0\int_0^t |\Delta\varphi^{n,s}(\tau)|^2 d\tau \leq C\int_0^t(|\nabla u^{n-1,s}(\tau)|^2 + |\nabla\varphi^{n,s}(\tau)|^2)d\tau. \tag{3.7}$$

Adding (3.6) and (3.7), choosing $\varepsilon = \frac{k_0}{6}$ and $min\{1, \nu_0, k_0\}$, after applying the Gronwall's inequality , we obtain

$$|\nabla u^{n,s}(t)|^2 + |\nabla \varphi^{n,s}(t)|^2 + \int_0^t (|Au^{n,s}(\tau)|^2 + |\Delta \varphi^{n,s}(\tau)|^2) d\tau \qquad (3.8)$$
$$\leq C \int_0^t (|\nabla u^{n-1,s}(\tau)|^2 + |\nabla \varphi^{n-1,s}(\tau)|^2) d\tau.$$

This complete the proof of the Lemma 3.3.

Corollary 3.1. There exists a positive constant $c > 0$, independent of n and s, such that:

$$\int_0^t |u_\tau^{n,s}(\tau)|^2 d\tau + \int_0^t |\varphi_\tau^{n,s}(\tau)|^2 d\tau \leq c \int_0^t (|\nabla u^{n-1,s}(\tau)|^2 + |\nabla \varphi^{n-1,s}(\tau)|^2) d\tau$$
$$+ c \int_0^t (|Au^{n,s}(\tau)|^2 + |\Delta \varphi^{n,s}(\tau)|^2) d\tau + c \int_0^t |\nabla \varphi^{n,s}(\tau)|^2 d\tau.$$

Proof of the Theorem 2.1.

Setting $\phi_{n,s}(t) = |\nabla u^{n,s}(t)|^2 + |\nabla \varphi^{n,s}(t)|^2$, the Lemma 3.3, implies

$$\phi_{n,s}(t) + \int_0^t (|Au^{n,s}(\tau)|^2 + |\Delta \varphi^{n,s}(\tau)|^2) d\tau \leq M_1 \int_0^t \phi_{n-1,s}(\tau) d\tau. \qquad (3.9)$$

Thus,

$$\phi_{n,s}(t) \leq M_1 \int_0^t \phi_{n-1,s}(\tau) d\tau.$$

From Lemma 3.1, since $\phi_{n,s}(t) \leq M_2$, by the estimates given in the Lemma 2.1 we get

$$\phi_{n,s}(t) \leq M_2 \frac{(M_1 t)^{n-1}}{(n-1)!}.$$

Moreover,

$$|\nabla u^{n,s}(t)|^2 + |\nabla \varphi^{n,s}(t)|^2 \leq M_2 \frac{(M_1 t)^{n-1}}{(n-1)!} \leq M_2 \frac{(M_1 T)^{n-1}}{(n-1)!}. \qquad (3.10)$$

Also, (3.9) and (3.10), imply

$$\int_0^t (|Au^{n,s}(\tau)|^2 + |\Delta \varphi^{n,s}(\tau)|^2) d\tau \leq M_3 \frac{(M_1 t)^{n-1}}{(n-1)!} \leq M_3 \frac{(M_1 T)^{n-1}}{(n-1)!}. \qquad (3.11)$$

Also, we observe that (3.10), implies

$$\int_0^t (|\nabla u^{n,s}(\tau)|^2 + |\nabla \varphi^{n,s}(\tau)|^2) d\tau \leq M_4 \frac{(M_1 t)^n}{n!}. \qquad (3.12)$$

The Corollary 3.1, together with estimate (3.10) and the estimates given in the Lemma 2.1, imply

$$\int_0^t (|u_\tau^{n,s}(\tau)|^2 + |\varphi_\tau^{n,s}(\tau)|^2)d\tau \leq M_5 \frac{(M_1 t)^{n-1}}{(n-1)!} \leq M_5 \frac{(M_1 T)^{n-1}}{(n-1)!}. \tag{3.13}$$

Differentiating (3.2) with respect to t and taking $u_t^{n,s}$ as a test function in the resulting equation, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_t^{n,s}|^2 + (\nu(\varphi^{n+s})\nabla u_t^{n,s}, \nabla u_t^{n,s}) = |(\nu'(\varphi^{n+s})\varphi_t^{n+s}\nabla u^{n,s}, \nabla u_t^{n,s}) \\
&+((\nu(\varphi^{n+s}) - \nu(\varphi^n))\nabla u_t^n, \nabla u_t^{n,s}) + (u_t^{n-1,s}.\nabla u^n, u_t^{n,s}) \\
&+((\nu'(\varphi^{n+s})\varphi_t^{n+s} - \nu'(\varphi^n)\varphi_t^n)\nabla u^n, \nabla u_t^{n,s}) + (u_t^{n+s-1}.\nabla u^{n,s}, u_t^{n,s}) \\
&+(u^{n-1,s}.\nabla u_t^n, u_t^{n,s}) + \alpha(g_t\varphi^{n,s}, u_t^{n,s}) + \alpha(g\varphi_t^{n,s}, u_t^{n,s})|.
\end{aligned} \tag{3.14}$$

We estimate the right-hand side of (3.14) as usual to obtain

$$\begin{aligned}
&\frac{d}{dt}|u_t^{n,s}|^2 + \nu_0|\nabla u_t^{n,s}|^2 \leq C[|\nabla\varphi_t^{n+s}||Au^{n,s}|^2 + |\nabla\varphi^{n,s}||\nabla u_t^n|^2 \\
&+|\varphi_t^{n,s}|^2 + |\nabla\varphi^{n,s}|^2|\nabla\varphi_t^n|^2 + |Au^{n,s}|^2 + |g_t|^2|\nabla\varphi^{n-1,s}|^2 \\
&+|u_t^{n-1,s}|^2 + |\nabla u^{n-1,s}|^2|\nabla u_t^n|^2 + |\varphi_t^{n-1,s}|^2] + \delta|\nabla\varphi_t^{n,s}|^2.
\end{aligned} \tag{3.15}$$

Analogously, we get

$$\begin{aligned}
&\frac{d}{dt}|\varphi_t^{n,s}|^2 + \frac{3}{2}k_0|\nabla\varphi_t^{n,s}|^2 \leq C[|\nabla\varphi_t^{n+s}||\Delta\varphi^{n,s}|^2 + |\nabla\varphi^{n,s}||\nabla\varphi_t^n|^2 \\
&+|u_t^{n-1,s}|^2 + |\nabla u^{n-1,s}|^2|\nabla\varphi_t^n|^2 \\
&+|\varphi_t^{n,s}|^2 + |\nabla\varphi^{n,s}|^2|\nabla\varphi_t^n|^2 + |\Delta\varphi^{n,s}|^2].
\end{aligned} \tag{3.16}$$

Adding (3.15) and (3.16), after integrate with respect to t, we have

$$\begin{aligned}
&|u_t^{n,s}(t)|^2 + |\varphi_t^{n,s}(t)|^2 + \nu_0\int_0^t |\nabla u_\tau^{n,s}(\tau)|^2 d\tau + k_0\int_0^t |\nabla\varphi_\tau^{n,s}(\tau)|^2 d\tau \\
&\leq C[\int_0^t (|Au^{n,s}(\tau)|^2 + |\Delta\varphi^{n,s}(\tau)|^2)d\tau + \int_0^t (|u_\tau^{n-1,s}(\tau)|^2 + |\varphi_\tau^{n-1,s}(\tau)|^2)d\tau \\
&+\int_0^t (|u_\tau^{n,s}(\tau)|^2 + |\varphi_\tau^{n,s}(\tau)|^2 d\tau + \int_0^t |\nabla\varphi_\tau^{n,s}(\tau)|(|Au^{n,s}(\tau)|^2 + |\Delta\varphi^{n,s}(\tau)|^2)d\tau \\
&+\int_0^t |\nabla\varphi^{n,s}(\tau)|^2|\nabla\varphi_\tau^n(\tau)|^2 d\tau + \int_0^t |\nabla u_\tau^n(\tau)|^2(|\nabla\varphi^{n,s}(\tau)| + |\nabla u^{n-1,s}(\tau)|^2)d\tau \\
&+\int_0^t |\nabla\varphi^{n-1,s}(\tau)|^2|g_\tau(\tau)|^2 d\tau + \int_0^t |\nabla u^{n-1,s}(\tau)|^2|\nabla(\varphi_\tau^n(\tau))|^2 d\tau.
\end{aligned}$$

Now, choosing $\min\{1, \nu_0, k_0\}$, from (3.10), (3.11), (3.13) and using the Gronwall's inequality, we get

$$\begin{aligned}
&|u_t^{n,s}(t)|^2 + |\varphi_t^{n,s}(t)|^2 + \int_0^t (|\nabla u_\tau^{n,s}(\tau)|^2 + |\nabla \varphi_\tau^{n,s}(\tau)|^2) d\tau \\
&\leq C[M_3 \frac{(M_1 T)^{n-1}}{(n-1)!} + M_5 \frac{(M_1 T)^{n-2}}{(n-2)!} + sup_t |\nabla \varphi^{n,s}(t)|^2 \int_0^t |\nabla \varphi_\tau^n(\tau)|^2 d\tau \\
&+ (\int_0^t |\nabla \varphi_\tau^{n+s}(\tau)|^2 d\tau)^{1/2} (\int_0^t |Au^{n,s}(\tau)|^2 + |\Delta \varphi^{n,s}(\tau)|^2) d\tau)^{1/2} \\
&+ (sup_t |\nabla \varphi^{n,s}(\tau)| + sup_t |\nabla u^{n-1,s}(\tau)|^2) \int_0^t |\nabla u_\tau^n(\tau)|^2 d\tau \\
&+ sup_t |\nabla \varphi^{n-1,s}(\tau)|^2 \int_0^t |g_\tau(\tau)|^2 d\tau + sup_t |\nabla u^{n-1,s}(\tau)|^2 \int_0^t |\nabla \varphi_\tau^n(\tau)|^2 d\tau].
\end{aligned}$$

The estimate given in the Lemma 2.1, implies

$$\begin{aligned}
&|u_t^{n,s}(t)|^2 + |\varphi_t^{n,s}(t)|^2 + \int_0^t (|\nabla u_\tau^{n,s}(\tau)|^2 + |\nabla \varphi_\tau^{n,s}(\tau)|^2) d\tau \\
&\leq C[M_3 \frac{(M_1 T)^{n-1}}{(n-1)!} + M_5 \frac{(M_1 T)^{n-2}}{(n-2)!} + \left[M_3 \frac{(M_1 T)^{n-1}}{(n-1)!} \right]^{1/2} \qquad (3.17) \\
&+ M_2 \frac{(M_1 T)^{n-1}}{(n-1)!} + \left[M_2 \frac{(M_1 T)^{n-1}}{(n-1)!} \right]^{1/2} + M_2 \frac{(M_1 T)^{n-2}}{(n-2)!} \\
&+ M_2 \frac{(M_1 T)^{n-2}}{(n-2)!} + M_2 \frac{(M_1 T)^{n-2}}{(n-2)!}] \\
&\leq M_6 \left[\frac{(M_1 T)^{n-2}}{(n-2)!} + \left[\frac{(M_1 T)^{n-1}}{(n-1)!} \right]^{1/2} \right].
\end{aligned}$$

From (3.2) and (3.3), is easily to show

$$|Au^{n,s}|^2 + |\Delta \varphi^{n,s}|^2 \leq C[|\nabla u^{n-1,s}|^2 + |\nabla u^{n,s}|^2 + |u_t^{n,s}|^2 + |\varphi_t^{n,s}|^2 + |\nabla \varphi^{n,s}|^2].$$

Then,

$$\begin{aligned}
|Au^{n,s}(t)|^2 + |\Delta \varphi^{n,s}(t)|^2 &\leq C[M_2 \frac{(M_1 T)^{n-2}}{(n-2)!} + M_2 \frac{(M_1 T)^{n-1}}{(n-1)!} \\
&+ M_6 \left[\frac{(M_1 T)^{n-2}}{(n-2)!} + \left[\frac{(M_1 T)^{n-1}}{(n-1)!} \right]^{1/2} \right]] \\
&\leq M_7 \left[\frac{(M_1 T)^{n-2}}{(n-2)!} + \left[\frac{(M_1 T)^{n-1}}{(n-1)!} \right]^{1/2} \right] \qquad (3.18)
\end{aligned}$$

by virtue of (3.10) and (3.17).
Since the spaces $L^2(0,T;V), L^2(0,T;H_0^1(\Omega)), L^\infty(0,T;D(A)), L^\infty(0,T;H^2(\Omega)), L^\infty(0,T;H), L^\infty(0,T;$
are Banach spaces, it is easily see that
$u^n \to u$ strongly in $L^\infty(0,T;D(A))$,
$u_t^n \to u_t$ *strongly in* $L^\infty(0,T;H) \cap L^2(0,T;V)$,
$\varphi^n \to \varphi$ *strongly in* $L^\infty(0,T;H^2(\Omega))$,
$\varphi_t^n \to \varphi_t$ *strongly in* $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$,
as $n \to \infty$.
Now, the next step is to take limit. But, once the above convergences has been stablished this is a standar procedure, and we obtain

$$\int_0^t \langle u_t - div(\nu(\varphi)\nabla u) + u.\nabla u - \alpha g\varphi - h, v\rangle \phi(t)dt = 0,$$

$$\int_0^t \langle \varphi_t - div(k(\varphi)\nabla\varphi) + u.\nabla\varphi - f, \psi\rangle \beta(t)dt = 0,$$

for all $v \in (L^2(\Omega))^N$, $\psi \in L^2(\Omega)$ and $\phi,\ \beta \in L^\infty(0,T)$.
These equation together with the Du Bois-Reymond's Theorem imply

$$\langle u_t - div(\nu(\varphi)\nabla u) + u.\nabla u - \alpha g\varphi - h, v\rangle = 0,$$

$$\langle \varphi_t - div(k(\varphi)\nabla\varphi) + u.\nabla\varphi - f, \psi\rangle = 0,$$

a. e. in Ω, for every $v \in (L^2(\Omega))^N, \psi \in L^2(\Omega)$.
These two last inequalities, imply

$$u_t - P(div(\nu(\varphi)\nabla u)) + P(u.\nabla u) = \alpha P(g\varphi) + P(h),$$

$$\varphi_t - div(k(\varphi)\nabla\varphi) + u.\nabla\varphi = f.$$

The convergences-rates bound of Theorem, can be obtained by taking the limit as $s \to \infty$ in the inequalities (3.10), (3.11), (3.12), (3.13), (3.17) and (3.18).This completes the proof of the Theorem.

4. Results on the Pressure

By using the Amrouche and Girault [1] results on the Stokes problem and the estimates given in the above sections, we obtain easily the following propositions:

Proposition 4.1 Under the hypotheses of Lemma 2.1 for each n, there exists $p^n \in L^\infty(0,T;H^1(\Omega)/I\!R)$ such that

$$sup_t\{\|p^n(t)\|^2_{H^1(\Omega)/I\!R}\} \leq C_0,$$

for all $t \in [0,T]$, where C_0 is a positive constant independent of n.

Proposition 4.2 Under the hypotheses of Theorem 2.1, we have that the approximate pressure p^n converge in the space $L^\infty(0,T;H^1(\Omega)/I\!R)$. The limiting element p is such that (u,φ,p) is a solution of problem (1.1)-(1.2) and the solution is unique. Moreover, the rate of convergence satisfies the inequalities:

$$\int_0^t |p^n(\tau)-p(\tau)|^2_{H^1(\Omega)/I\!R} d\tau \leq M_8 \frac{(M_1T)^{n-1}}{(n-1)!},$$

$$sup_t\{|p^n(t)-p(t)|^2_{H^1(\Omega)/I\!R}\} \leq M_9 \left[\frac{(M_1T)^{n-2}}{(n-2)!} + \left[\frac{(M_1T)^{n-1}}{(n-1)!}\right]^{1/2}\right],$$

for all $t \in [0,T]$, where the constants M_1, M_8, M_9 are independent of n.

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