Quasilinear Dirichlet problems in $I\!R^N$ with critical growth

Elves A. de B. e Silva*

Departamento de Matemática, Universidade de Brasília, 70910-900 Brasília, DF, Brazil

Sérgio H. M. Soares**

Departamento de Matemática, Universidade Estadual Paulista, 15054-000 S.J. do Rio Preto, SP, Brazil

Key words and phrases: Critical exponent, unbounded domain, Mountain Pass Theorem.

Abstract

The objetive of this article is to establish the existence of nontrivial solution for a class of quasilinear elliptic problems with the nonlinearity satisfying the critical growth condition. Our proof combines perturbation arguments, the concentration-compactness principle, appropriate estimates for the levels associated with the Mountain Pass theorem, and the argument employed by Brezis and Nirenberg to study semilinear elliptic problems with critical growth.

1 Introduction

In this article, we use variational methods to study the following quasilinear problem:

$$(GP) \qquad \begin{cases} -\Delta_p u = u^{p^*-1} + \lambda f(x,u) \quad \text{in } I\!R^N, \\ \quad u \geq 0 \text{ in } I\!R^N, \\ \quad \int_{I\!R^N} |\nabla u|^p \, dx < \infty, \end{cases}$$

where $\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian of u, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $1 < p < N$, $\lambda > 0$ is a real parameter and $f : I\!R^N \times I\!R \longrightarrow I\!R$ satisfies the following conditions:

(f_1) $f \in C(I\!R^N \times I\!R, I\!R)$ and $f(x,0) \equiv 0$.

(f_2) Given $R > 0$ there exist $\theta_R \in [p, p^*)$ and positive constants $a_R, b_R > 0$ such that

$$|f(x,s)| \leq a_R s^{\theta_R - 1} + b_R, \quad \forall\, |x| \leq R, \ \forall\, s \geq 0.$$

* Research partially supported by CNPq under grant n^0 307014/89-4

** Research partially supported by PICD/CAPES

(f_3) There exist $r_1, r_2, q \in (1, p^*)$, with $r_1 \leq q \leq r_2$, an open subset $\Omega_0 \subset I\!R^N, c_i \in L^{\frac{p^*}{p^*-r_i}}(I\!R^N)$, $i = 1, 2$, and a positive constant a such that

$$\begin{cases} f(x,s) \leq c_1(x)s^{r_1-1} + c_2(x)s^{r_2-1}, & \forall\, x \in I\!R^N,\ s \geq 0, \\ F(x,s) \geq as^q, & \forall\, x \in \Omega_0,\ s \geq 0, \end{cases}$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

We also assume a version of the famous Ambrosetti–Rabinowitz condition [3],

(f_4) There exist $p < \tau < p^*$, $1 < \mu < p^*$, and $c_3 \in L^{\frac{p^*}{p^*-\mu}}(I\!R^N)$ such that

$$\frac{1}{\tau} f(x,s)s - F(x,s) \geq -c_3(x)s^\mu,\ \forall\, x \in I\!R^N,\ s \geq 0.$$

Observing that $u \equiv 0$ is a (trivial) solution of (GP), our objective in this article is to apply minimax methods to study the existence of nontrivial solutions for (GP). However, it should be pointed out that we may not apply directly such methods since, under conditions $(f_1) - (f_4)$, the associated functional is not well defined in general. We also note that we look for weak solution $u \in D^{1,p}(I\!R^N)$ in the sense of distributions (See definition in Section 2).

Our technique combines pertubation arguments, the concentration-compactness principle [1, 2], appropriate estimates for the levels associated with the Mountain Pass Theorem [3], and the argument employed by Brezis and Nirenberg [4] to study semilinear elliptic problems with critical growth.

Considering $q \in I\!R$ given by condition (f_3), in our first result we also suppose the following technical condition:

(H) $q \in (1, p^*)$ satisfies $\hat{p} = p^* - \frac{p}{p-1} < q$.

Note that $\hat{p} < p$, $\hat{p} = p$ and $\hat{p} > p$ for $p^2 < N$, $p^2 = N$ and $p^2 > N$, respectively. We can now state our main theorem on the existence of a nontrivial solution for (GP):

Theorem 1.1 *Suppose f satisfies $(f_1) - (f_4)$, with q, r_1 given by (f_3) and q satisfying condition (H). Then,*

1. *If $1 < r_1 \leq p$, there exists $\lambda^* > 0$ such that problem (GP) possesses a nontrivial solution for every $\lambda \in (0, \lambda^*)$.*

2. *If $p < r_1 < p^*$, then problem (GP) possesses a nontrivial solution for every $\lambda > 0$.*

We observe that a particular and relevant case associated with problem (GP) is given by

$$(P)\qquad \begin{cases} -\Delta_p u = u^{p^*-1} + \lambda a(x) u^{q-1} \text{ in } I\!R^N, \\ \quad u \geq 0 \text{ in } I\!R^N, \\ \quad \int_{I\!R^N} |\nabla u|^p \, dx < \infty, \end{cases}$$

where $q \in (1, p^*)$ satisfies (H) and $a : I\!R^N \longrightarrow I\!R$ is a continuous function satisfying the condition

$$(a_0) \quad a^+ = max(a, 0) \in L^{\frac{p^*}{p^*-q}}(I\!R^N) \text{ and } \exists \; x_0 \in I\!R^N \text{ such that } a(x_0) > 0.$$

The following result is a direct consequence of Theorem 1.1,

Theorem 1.2 *Suppose q satisfies (H) and a satisfies (a_0). Then,*

1. *If $1 < q \leq p$, there exists $\lambda^* > 0$ such that problem (P) possesses a nontrivial solution for every $\lambda \in (0, \lambda^*)$.*
2. *If $p < q < p^*$, then problem (P) possesses a nontrivial solution for every $\lambda > 0$.*

We observe that $f(x, s) = a(x)s^{q-1}$ satisfies $(f_1) - (f_2)$, (f_3) with $r_1 = q = r_2$, $c_1 = a^+$ and $c_2 \equiv 0$, and (f_4) with $\tau = q = \mu$ and $c_3 \equiv 0$ if $q > p$, and $\tau \in (p, p^*)$, $\mu = q$ and $c_3 = (\frac{1}{q} - \frac{1}{\tau})a^+$ if $1 < q \leq p$.

Assuming the positivity of the primitive of the nonlinearity, we do not need to consider condition (H). More specifically, supposing

(f_5) $F(x, s) = \int_0^s f(x, t) \, dt \geq 0 \quad \forall x \in I\!R^N, \; s \geq 0,$

we obtain

Theorem 1.3 *Suppose f satisfies $(f_1) - (f_5)$, with r_1 given by condition (f_3). Then,*

1. *If $1 < r_1 \leq p$, there exists $\lambda^* > 0$ such that problem (GP) possesses a nontrivial solution for every $\lambda \in (0, \lambda^*)$.*
2. *If $p < r_1 < p^*$, then problem (GP) possesses a nontrivial solution for every $\lambda > 0$.*

It is worthwhile to mention that Theorem 1.3 provides a version of Theorem 1.2 when $a \geq 0$, without assuming that q satisfies condition (H).

Problems involving critical Sobolev exponents have been considered by several authors since the seminal work of Brezis and Nirenberg [4], mainly when the domain is bounded. In recent years, the related problem for unbounded domain has been intensively studied (See, e.g., [5, 6, 7, 8, 9, 10] and their references).

In [5], Ben-Naoum, Troestler and Willem proved the existence of a nontrivial solution for (P), defined on a domain $\Omega \subset I\!R^N$, by considering the problem:

$$(P') \qquad \begin{cases} \text{minimize} & E(u) = \int_\Omega (|\nabla u|^p + a(x)|u|^q)\, dx, \\ \text{on the constraint} & u \in D^{1,p}(\Omega),\ \int_\Omega |u|^{p^*}\, dx = 1, \end{cases}$$

where $a \in L^{\frac{p^*}{p^*-q}}(\Omega)$, $a < 0$ on some subset of Ω with positive measure and $q > p^* - \frac{p}{p-1}$ when $p^2 > N$.

A recent result by Alves and Gonçalves [6] (See also [7]) establishes the existence of a nontrivial solution for (P), with $h(x)$ replacing $\lambda a(x)$ and satisfying $h(x) \geq 0$ and $h \in L^{\frac{p^*}{p^*-q}}$. In [6], it is supposed that either $1 < q < p$ and h is small, or $p < q < p^*$.

In [8], Benci and Cerami considered the case $p = q = 2$ and proved that problem (P) has at least one solution if $a(x)$ is a negative function, strictly negative somewhere, having $L^{N/2}$ norm bounded and belonging to $L^p(I\!R^N)$, for every p in a suitable neighbourhood of $\frac{N}{2}$.

Our theorems may be seen as a complement for the above mentioned results. We observe that in Theorems 1.1 and 1.3 a more general class of nonlinearity is considered. We also note that condition (f_3) provides only a local growth restriction on $f^-(x,s) \equiv \max\{-f(x,s), 0\}$. For example, we do not assume $a^- \in L^{\frac{p^*}{p^*-q}}(I\!R^N)$ in Theorem 1.2. Finally, we should mention that our argument also holds for quasilinear equations defined on bounded or unbounded domains $\Omega \subset I\!R^N$ with Dirichlet boundary conditions.

To prove Theorems 1.1 and 1.3, we first provide a technical result that establishes the existence of a weak solution in the sense of distributions for a class of quasilinear problems which may not have the associated functional well defined. In this technical result we assume the existence of a bounded sequence in $D^{1,p}$ of almost critical points for a sequence of functionals of class C^1. The main tool for our proof of this result is the concentration-compactness principle [1, 2]. To apply such result, we modify the nonlinearity, obtaining a family of functionals. Employing conditions $(f_2) - (f_3)$, we show that these functionals satisfy the geometric hypotheses of the Mountain Pass Theorem in a uniform way. Using this fact, (f_4) and our technical result, we are able to verify the existence of a sequence in $D^{1,p}(I\!R^N)$ converging weakly to a solution of (GP). Finally, we argue by contradiction, assuming that (GP) possesses only the trivial solution. This allows us to employ an argument similar to the one used by Brezis and Nirenberg in [4], deriving a contradiction.

The article is organized in the following way: Section 2 contains some preliminary materials, including the version of the Mountain Pass Theorem used in this article. In Section 3, we establish the above mentioned technical result. In Section 4, the estimates for the geometric hypotheses of the Mountain Pass Theorem are verified. Section 5 is devoted to prove the estimates from above for the critical levels. In Section 6, we prove

Theorem 1.1. In Section 7, we establish the estimates when conditions (f_3) and (f_5) are assumed. There, we also present a proof of Theorem 1.3.

2 Preliminaries

Motivated by the Sobolev embedding $W^{1,p}(I\!\!R^N) \hookrightarrow L^{p^*}(I\!\!R^N)$, for $1 < p < N$, and $p^* = \frac{Np}{N-p}$, we define $D^{1,p} \equiv D^{1,p}(I\!\!R^N)$ as the closure of $D(I\!\!R^N)$, the space of C^∞-functions with compact support, with respect to norm given by

$$\|\phi\| = \left(\int_{I\!\!R^N} |\nabla \phi|^p \, dx\right)^{1/p}.$$

Inspired by the work of Brezis and Nirenberg, [4], we make use in our argument of the extremal functions associated with the above embedding. For this purpose, we denote by S the best Sobolev constant, that is,

$$S = \inf_{u \in D^{1,p}\setminus\{0\}} \left\{ \frac{\int_{I\!\!R^N} |\nabla u|^p \, dx}{(\int_{I\!\!R^N} |u|^{p^*} \, dx)^{p/p^*}} \right\}. \tag{2.1}$$

The infimum in (2.1) is achieved by the functions (See Talenti [11], Egnell [12]),

$$w_\varepsilon(x) = \frac{\left\{ N\varepsilon[(N-p)/(p-1)]^{p-1} \right\}^{\frac{N-p}{p^2}}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}, \quad \forall\, x \in I\!\!R^N, \ \varepsilon > 0, \tag{2.2}$$

with

$$\|w_\varepsilon\|^p = \|w_\varepsilon\|^{p^*}_{L^{p^*}} = S^{N/p}, \ \forall\, \varepsilon > 0.$$

By weak solution of (GP), we mean a function $u \in D^{1,p}$ such that $u \geq 0$ $a.e.$ in $I\!\!R^N$ and the following identity holds:

$$\int_{I\!\!R^N} |\nabla u|^{N-2} \nabla u . \nabla \phi \, dx - \int_{I\!\!R^N} |u|^{p^*-1} \phi \, dx - \lambda \int_{I\!\!R^N} f(x,u)\phi \, dx = 0,$$

for every $\phi \in D(I\!\!R^N)$.

Following a well known device used to obtain a solution for (GP), we let $f(x,s) = f(x,0) = 0$, for every $x \in I\!\!R^N$ and $s < 0$.

To modify the nonlinearity, we choose $\phi \in D(I\!\!R^N)$ satisfying $0 \leq \phi(x) \leq 1$, $\phi \equiv 1$ on the ball $B(0,1)$, and $\phi \equiv 0$ on $I\!\!R^N \setminus B(0,2)$. Let $n \in I\!\!N$ and $\phi_n(x) = \phi(\frac{x}{n})$. Define

$f_n(x,s)=\phi_n(x)f(x,s)$, and consider the sequence of problems:

$$(GP)_n \qquad \begin{cases} -\Delta_p u = u^{p^*-1}+\lambda f_n(x,u), \text{ in } I\!R^N, \\ u\geq 0, \quad u\in D^{1,p}. \end{cases}$$

We now recall the variational framework associated with problem $(GP)_n$. Considering $D^{1,p}$ endowed with norm $\|u\|=\|\nabla u\|_{L^p}$, the functional associated with $(GP)_n$ is given by

$$I_{\lambda,n}(u)=\frac{1}{p}\int_{I\!R^N}|\nabla u|^p\,dx-\frac{1}{p^*}\int_{I\!R^N}(u^+)^{p^*}\,dx-\lambda\int_{I\!R^N}F_n(x,u)\,dx,$$

where $u^+=\max\{u,0\}$ and $F_n(x,s)=\int_0^s f_n(x,t)\,dt$. By hypothesis (f_2) and our construction, the functional $I_{\lambda,n}$ is well defined and belongs to $C^1(D^{1,p},I\!R)$ (See[13]). Furthermore,

$$I'_{\lambda,n}(u)\phi=\int_{I\!R^N}|\nabla u|^{p-2}\nabla u.\nabla\phi\,dx-\int_{I\!R^N}(u^+)^{p^*-1}\phi\,dx-\lambda\int_{I\!R^N}f_n(x,u)\phi\,dx.$$

for every u and $\phi\in D^{1,p}$.

Now, for the sake of completeness, we state a basic compactness result (See [5] for a proof),

Proposition 2.1 *Let Ω be a domain, not necessarily bounded, of $I\!R^N$, $1\leq p<N$, $1\leq q<p^*$, and $a\in L^{\frac{p^*}{p^*-q}}(\Omega)$. Then, the functional*

$$D^{1,p}(\Omega)\;\rightarrow\;I\!R\,:\,u\;\longmapsto\;\int_\Omega a|u|^q\,dx,$$

is well defined and weakly continuous.

Finally, we state the version of the Mountain Pass Theorem of Ambrosetti-Rabinowitz [3] used in this work. Given E a real Banach space, $\Phi\in C^1(E,I\!R)$ and $c\in I\!R$, we recall that $(u_n)\subset E$ is a Palais-Smale $(PS)_c$ sequence associated with functional Φ if $\Phi(u_n)\rightarrow c$, and $\Phi'(u_n)\rightarrow 0$, as $n\rightarrow\infty$.

Theorem 2.2 *Let E be a real Banach space and suposse $\Phi\in C^1(E,I\!R)$, with $\Phi(0)=0$, satisfies*
(Φ_1) There exist positive constants β, ρ such that $\Phi(u)\geq\beta$, $\|u\|=\rho$,
(Φ_2) There exists $e\in E$, $\|e\|>\rho$, such that $\Phi(e)\leq 0$.
Then, for the constant

$$c=\inf_{\gamma\in\Gamma}\sup_{u\in\gamma}\Phi(u)\geq\beta,$$

where $\Gamma=\{\gamma\in C([0,1],E);\;\gamma(0)=0,\gamma(1)=e\}$, there exists a $(PS)_c$ sequence (u_j) in E associated with Φ.

3 Technical result

In this section we study the existence of a weak solution in the sense of distributions for the p-Laplacian in $\mathbb{R}^N$. Consider $g(x,s) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying

(g_1) Given $R > 0$ there exist positive constants a_R, b_R such that for every $x \in \mathbb{R}^N$ with $|x| \leq R$, and $s \in \mathbb{R}$,

$$|g(x,s)| \leq a_R |s|^{p^*-1} + b_R.$$

The associated functional I in $D^{1,p}$ is defined by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \, dx - \int_{\mathbb{R}^N} G(x,u)\, dx, \tag{3.1}$$

where $G(x,s) = \int_0^s g(x,t)\, dt$. It is clear that, under condition (g_1), I may assume the values $\pm\infty$. However, if we assume the following stronger version of condition (g_1),

(g_2) There exist $a > 0$, $b \in C_0(\mathbb{R}^N)$, the space of continuous functions with compact support in $\mathbb{R}^N$, such that, for every $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$|g(x,s)| \leq a|s|^{p^*-1} + b(x),$$

then, I belongs to $C^1(D^{1,p}, \mathbb{R})$ and critical points of I are weak solutions of the associated quasilinear equation in $\mathbb{R}^N$. To establish the existence of a solution for the associated equation when (g_2) does not hold, we suppose the existence of a sequence of functions $\{g_n\} \subset C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying (g_2) and converging to g. More specifically, we assume

(g_3) Given $n \in \mathbb{N}$ there exists $g_n \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying (g_2) and

$$g(x,s) = g_n(x,s), \ \forall\ |x| \leq n, \ s \in \mathbb{R}^N.$$

Let I_n be the sequence of functionals in $D^{1,p}$ associated with g_n via (3.1). We can now state our main result in this section,

Proposition 3.1 *Suppose $g(x,s) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies (g_1) and (g_3). Then, any bounded sequence $(u_n) \subset D^{1,p}$ such that $I_n'(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a solution of*

$$\begin{cases} -\Delta_p u = g(x,u), \ \text{in} \ \mathbb{R}^N, \\ u \in D^{1,p}. \end{cases}$$

Remark 3.2 *We observe that in [14], we prove a related result for the N-Laplacian on bounded domain of $\mathbb{R}^N$ when the nonlinearity possesses exponential growth. But, unlike what happens in [14], here the functional is not of class C^1.*

The proof of Proposition 3.1 will be carried out through a series of steps. First, by Sobolev embedding and the principle of concentration-compactness [1, 2], we may assume that there exist $u \in D^{1,p}$, a nonnegative measure ν on $I\!R^N$, and sequences $(x_i) \in I\!R^N, \nu_i > 0$ and Dirac measures δ_{x_i} such that

$$\left\{ \begin{array}{l} u_n \rightharpoonup u, \text{ weakly in } D^{1,p}, \\ u_n \to u, \text{ strongly in } L^s_{loc}(I\!R^N), \; 1 \le s < p^*, \\ u_n(x) \to u(x), \text{ a.e. in } I\!R^N, \\ |u_n|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \sum_i \nu_i \delta_{x_i}, \text{ weakly* in } \mathcal{M}(I\!R^N), \\ |\nabla u_n|^p \rightharpoonup \mu, \text{ weakly* in } \mathcal{M}(I\!R^N), \\ \sum_i \nu_i{}^{p/p^*} < \infty. \end{array} \right. \tag{3.2}$$

Lemma 3.3 *There exists at most a finite number of points x_i on bounded subsets of $I\!R^N$.*

Proof: First, we note that it suffices to prove that there exists at most a finite number of points x_i on $B(0,r)$ for every $r > 0$. From (2.1) and Lemma 1.2 in [1], we obtain

$$\mu(\{x_i\}) \ge S \nu_i^{\frac{p}{p^*}}. \tag{3.3}$$

Now, for every $\varepsilon > 0$, we set $\psi_\varepsilon(x) = \psi(\frac{x-x_i}{\varepsilon})$, $x \in I\!R^N$, where $\psi \in D(I\!R^N), 0 \le \psi(x) \le 1, \psi(x) \equiv 1$ on $B(0,1)$, and $\psi(x) \equiv 0$ on $I\!R^N \setminus B(0,2)$. Since $I_n'(u_n) \to 0$, as $n \to \infty$, and $(\psi_\varepsilon u_n)$ is a bounded sequence, we have

$$\int_{I\!R^N} |\nabla u_n|^{p-2} \nabla u_n . \nabla(\psi_\varepsilon u_n)\, dx = \int_{I\!R^N} g_n(x, u_n) \psi_\varepsilon u_n \, dx + o(1).$$

By conditions (g_1), with $R > 2r$, and (g_3), for n sufficiently large, we get

$$\int_{I\!R^N} |\nabla u_n|^{p-2} \nabla u_n . \nabla(\psi_\varepsilon u_n)\, dx \le a_R \int_{I\!R^N} |u_n|^{p^*} \psi_\varepsilon \, dx + b_R \int_{I\!R^N} |u_n| \psi_\varepsilon \, dx + o(1).$$

Now, from (3.2), taking $n \to \infty$, we have

$$\lim_{n \to \infty} \int_{I\!R^N} |\nabla u_n|^{p-2} \nabla u_n . \nabla(\psi_\varepsilon u_n)\, dx \le a_R \int_{I\!R^N} \psi_\varepsilon \, d\nu + b_R \int_{I\!R^N} |u| \psi_\varepsilon \, dx.$$

Invoking Lemma 1.2 in [1] again and taking $\varepsilon \to 0$, we obtain

$$\mu(\{x_i\}) \le a_R \nu(\{x_i\}) = a_R \nu_i.$$

Thus, from (3.3), we get $a_R \nu_i \ge S \nu_i{}^{\frac{p}{p^*}}$ and, consequently, $\nu_i \ge \frac{S^{\frac{N}{p}}}{a_R}$. Since $\sum_i \nu_i{}^{\frac{p}{p^*}} < \infty$, we conclude the proof of Lemma 3.3. ∎

Lemma 3.4 *Let $K \subset I\!\!R^N$ be a compact set. Then, there exist $n_0 \in I\!\!N$ and $M = M(K) > 0$ such that*

$$\int_K |g_n(x, u_n(x))|^{\frac{p^*}{p^*-1}}\, dx \le M, \qquad \forall\, n \ge n_0.$$

<u>Proof:</u> Take $n_0 \in I\!\!N$ such that $K \subset B(0, n_0)$. From (g_3), we have $g_n(x, u_n(x)) = g(x, u_n(x))$, for every $x \in K$, and $n \ge n_0$. Now, by condition (g_1) with $R = n_0$,

$$\int_K |g_n(x, u_n(x))|^{\frac{p^*}{p^*-1}}\, dx \le (2a_{n_0})^{\frac{p^*}{p^*-1}} \|u_n\|_{L^{p^*}}^{p^*} + (2b_{n_0})^{\frac{p^*}{p^*-1}} |K|, \qquad \forall\, n \ge n_0.$$

The lemma follows by the Sobolev embedding and the hypothesis that (u_n) is a bounded sequence. ■

Lemma 3.5 *Let $K \subset (I\!\!R^N \setminus \{x_i\})$ be a compact set. Then $u_n \to u$ strongly in $L^{p^*}(K)$, as $n \to \infty$.*

<u>Proof:</u> Let $r > 0$ such that $K \subset B(0, r)$. By Lemma 3.3, there exists at most a finite number of points x_i on $B(0, r)$. Since K is a compact set and $K \cap \{x_i\} = \emptyset$, $\delta = d(K, \{x_i\})$, the distance between K and $\{x_i\}$, with $x_i \in B(0, r)$, is positive. Let $0 < \varepsilon < \delta$ and define $A_\varepsilon = \{x \in B(0, r) \mid d(x, K) < \varepsilon\}$. Choose $\psi \in D(I\!\!R^N), 0 \le \psi(x) \le 1, \psi \equiv 1$ on $A_{\frac{\varepsilon}{2}}$, and $\psi \equiv 0$ on $I\!\!R^N \setminus A_\varepsilon$. By construction, we have

$$\int_K |u_n|^{p^*}\, dx \le \int_{A_\varepsilon} \psi |u_n|^{p^*}\, dx = \int_{I\!\!R^N} \psi |u_n|^{p^*}\, dx.$$

Since $supp(\psi) \subset A_\varepsilon$ and $A_\varepsilon \cap \{x_i\} = \emptyset$, with $x_i \in B(0, r)$, from (3.2), we obtain

$$\begin{aligned} \lim_{n\to\infty} \sup \int_K |u_n|^{p^*}\, dx &\le \int_{I\!\!R^N} \psi\, d\nu = \int_{I\!\!R^N} \psi |u|^{p^*}\, dx = \\ &= \int_{A_\varepsilon} \psi |u|^{p^*}\, dx \le \int_{A_\varepsilon} |u|^{p^*}\, dx. \end{aligned}$$

Now, taking $\varepsilon \to 0$ and applying the Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \sup \int_K |u_n|^{p^*}\, dx \le \int_K |u|^{p^*}\, dx.$$

On the other hand, since $u_n \rightharpoonup u$ weakly in $L^{p^*}(K)$, it follows that

$$\|u\|_{L^{p^*}(K)} \le \lim_{n\to\infty} \inf \|u_n\|_{L^{p^*}(K)}.$$

Consequently, as $L^{p^*}(K)$ is uniformly convex, $u_n \to u$ strongly in $L^{p^*}(K)$. Lemma 3.5 is proved. ■

Lemma 3.6 *Let $K \subset I\!R^N \setminus \{x_i\}$ be a compact set. Then, $\nabla u_n \to \nabla u$ strongly in $(L^p(K))^N$, as $n \to \infty$.*

Proof: Let $\psi \in C_0^\infty(I\!R^N \setminus \{x_i\})$ such that $\psi = 1$ on K and $0 \leq \psi \leq 1$. Using that the function $h : I\!R^N \to I\!R,\ h(x) = |x|^p$ is strictly convex, we have

$$0 \leq \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u).$$

Consequently,

$$\begin{aligned} 0 &\leq \int_K \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u)\, dx \leq \\ &\leq \int_{I\!R^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u)\psi\, dx, \end{aligned}$$

and

$$\begin{aligned} &\textstyle\int_K \left[(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u).(\nabla(u_n - u))\right] dx \leq \\ &\leq \textstyle\int_{I\!R^N} \left[|\nabla u_n|^p\psi - |\nabla u_n|^{p-2}(\nabla u_n.\nabla u)\psi - \right. \\ &\left. - |\nabla u|^{p-2}(\nabla u.\nabla(u_n - u))\psi\right] dx. \end{aligned} \tag{3.4}$$

On the other hand, since $I_n^{'}(u_n) \to 0$, as $n \to \infty$, we also have

$$\int_{I\!R^N} \left[|\nabla u_n|^{p-2}\left((\nabla u_n \nabla u)\psi + (\nabla u_n \nabla \psi)u\right) - \psi g_n(x, u_n)u\right] dx = o(1), \tag{3.5}$$

as $n \to \infty$. Moreover, since (ψu_n) is a bounded sequence in $D^{1,p}$, we get

$$\int_{I\!R^N} \left[|\nabla u_n|^p\psi + |\nabla u_n|^{p-2}(\nabla u_n \nabla \psi)u_n - \psi g_n(x, u_n)u\right] dx = o(1), \tag{3.6}$$

as $n \to \infty$. Combining (3.4)-(3.6), we obtain

$$\begin{aligned} &0 \leq \textstyle\int_K \left[(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u).\nabla(u_n - u)\right] dx \leq \\ &\leq \textstyle\int_{I\!R^N} \psi g_n(x, u_n)(u_n - u)\, dx + \int_{I\!R^N} |\nabla u_n|^{p-2}(\nabla u_n.\nabla\psi)(u_n - u)\, dx + \\ &+ \textstyle\int_{I\!R^N} |\nabla u|^{p-2}\nabla u.\nabla(u - u_n)\psi\, dx + o(1),\ \text{as } n \to \infty. \end{aligned}$$

Applying Lemma 3.4 for the compact set $\Omega = supp(\psi)$, and using Holder's inequality, we get

$$\begin{aligned} &0 \leq \textstyle\int_K \left[(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u).\nabla(u_n - u)\right] dx \leq \\ &\leq M^{\frac{p^*}{p^*-1}}\|u_n - u\|_{L^{p^*}(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)}\|u_n\|^{p-1}\|u - u_n\|_{L^p(\Omega)} + \\ &+ \textstyle\int_{I\!R^N} |\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)\psi\, dx + o(1),\ \text{as } n \to \infty. \end{aligned}$$

Now, applying Lemma 3.5 for the compact set $\Omega = supp(\psi) \subset (I\!R^N \setminus \{x_i\})$, from (3.2) and boundedness of (u_n), we have

$$\int_K \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u)\, dx \to 0,\ \text{as } n \to \infty.$$

Considering that

$$(|a|^{p-2}a - |b|^{p-2}b, a-b) \geq \begin{cases} C_p|a-b|^p & \text{if } p \geq 2, \\ C_p\frac{|a-b|^2}{(|a|+|b|)^{2-p}} & \text{if } 1 < p < 2, \end{cases}$$

for every a, $b \in I\!R^N$ (See [15]), if $p \geq 2$, we get

$$\lim_{n\to\infty} C_p \int_K |\nabla u_n - \nabla u|^p \, dx = 0.$$

Furthermore, when $1 < p < 2$, we have

$$\lim_{n\to\infty} C_p \int_K \frac{|\nabla u_n - \nabla u|^2}{(|\nabla u| + |\nabla u_n|)^{2-p}} \, dx = 0. \tag{3.7}$$

Thus, by Holder's inequality,

$$\begin{aligned} \textstyle\int_K |\nabla(u_n - u)|^p \, dx &= \textstyle\int_K \frac{|\nabla(u_n-u)|^p}{(|\nabla u_n|+|\nabla u|)^{\frac{p(p-2)}{2}}} (|\nabla u_n| + |\nabla u|)^{\frac{p(p-2)}{2}} \, dx \leq \\ &\leq \left(\textstyle\int_K \frac{|\nabla(u_n-u)|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}} \, dx\right)^{\frac{p}{2}} \left(\textstyle\int_K (|\nabla u_n| + |\nabla u|)^p \, dx\right)^{\frac{2-p}{2}}. \end{aligned}$$

Finally, from this last inequality, (3.7), and the boundedness of (u_n), we have

$$\lim_{n\to\infty} \int_K |\nabla u_n - \nabla u|^p \, dx = 0.$$

Lemma 3.6 is proved. ■

As a direct consequence of Lemma 3.6, we have

Corollary 3.7 *The sequence $(u_n) \subset D^{1,p}$ possesses a subsequence (u_{n_j}) satisfying $\nabla u_{n_j}(x) \to \nabla u(x)$, for almost every $x \in I\!R^N$.*

Finally, we conclude the proof of Proposition 3.1: Given $\phi \in D(I\!R^N)$, take $n_0 > 0$ such that $supp(\phi) \subset B(0, n_0)$. From (g_3), we have

$$g_n(x,s) = g(x,s), \ \forall\, x \in supp(\phi), \text{ and } n \geq n_0. \tag{3.8}$$

Condition (g_1), with $R > n_0$, and (3.8) provide

$$|g_n(x,s)\phi(x)| \leq (a_R s^{p^*-1} + b_R)|\phi(x)|, \ \forall\, x \in supp(\phi), \ s \in I\!R^N, \ n \geq n_0. \tag{3.9}$$

Invoking (3.2), (3.9) and the fact that $(u_n) \subset D^{1,p}$ is a bounded sequence, it follows that $(g_n(x,u_n)\phi)$ and $(|\nabla u_n|^{p-2}\nabla u_n \nabla\phi)$ are uniformly integrable families in $L^1(I\!R^N)$. Thus, by Vitali's Theorem and Corollary 3.7, we get

$$\begin{cases} \lim_{n\to\infty} \int_{I\!R^N} g_n(x,u_n(x))\phi(x)\, dx = \int_{I\!R^N} g(x,u(x))\phi(x)\, dx, \ \forall\, \phi \in D(I\!R^N), \\ \lim_{n\to\infty} \int_{I\!R^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\phi \, dx = \int_{I\!R^N} |\nabla u|^{p-2}\nabla u \nabla\phi \, dx, \ \forall\, \phi \in D(I\!R^N). \end{cases} \tag{3.10}$$

Consequently, from (3.10) and $I_n'(u_n) \to 0$, as $n \to \infty$, we have

$$\int_{I\!R^N} |\nabla u|^{p-2}\nabla u \nabla \phi \, dx - \int_{I\!R^N} g(x, u(x))\phi(x)\, dx = 0, \ \forall\, \phi \in D(I\!R^N).$$

Proposition 3.1 is proved. ■

4 Mountain pass geometry

In this section, we prove that the family of functionals $I_{\lambda,n}$ satisfies conditions (Φ_1) and (Φ_2) of Theorem 2.2 in a uniform way.

Lemma 4.1 *Suppose f satisfies (f_2) and (f_3). Then,*

1. *If $1 < r_1 \leq p$, there exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, $I_{\lambda,n}$ satisfies (Φ_1), with β and ρ independent of n.*
2. *If $p < r_1 < p^*$, then for every $\lambda > 0$, $I_{\lambda,n}$ satisfies (Φ_1), with β and ρ independent of n.*

Proof: Let $u \in D^{1,p}$, and $u \neq 0$. Using Holder's inequality with exponents $\frac{p^*}{p^*-r_i}$ and $\frac{p^*}{r_i}$, $i = 1, 2$, we have

$$\int_{I\!R^N} c_i(x)(u^+)^{r_i}\, dx \leq \|c_i\|_{L^{\frac{p^*}{p^*-r_i}}} \|u^+\|^{r_i}_{L^{p^*}}. \tag{4.1}$$

Now, from the definition of ϕ_n, (f_3), (2.1) and (4.1), we get

$$\begin{aligned} I_{\lambda,n}(u) &= \frac{1}{p}\|u\|^p - \frac{1}{p^*}\|u\|^{p^*}_{L^{p^*}} - \lambda \int_{I\!R^N} F_n(x,u)\, dx \geq \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{p^*}\|u\|^{p^*}_{L^{p^*}} - \lambda \|c_1\|_{L^{\frac{p^*}{p^*-r_1}}} \|u\|^{r_1}_{L^{p^*}} - \lambda \|c_2\|_{L^{\frac{p^*}{p^*-r_2}}} \|u\|^{r_2}_{L^{p^*}} \geq \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{p^* S^{p^*/p}}\|u\|^{p^*} - \frac{\lambda}{r_1 S^{r_1/p}}\|c_1\|_{L^{\frac{p^*}{p^*-r_1}}} \|u\|^{r_1} - \\ &- \frac{\lambda}{r_2 S^{r_2/p}}\|c_2\|_{L^{\frac{p^*}{p^*-r_2}}} \|u\|^{r_2}. \end{aligned}$$

Case 1: $1 < r_1 \leq p$**.** We have

$$I_{\lambda,n}(u) \geq \|u\|^p\Big(\frac{1}{p} - \frac{1}{p^* S^{p^*/p}}\|u\|^{p^*-p}\Big) - \lambda\Big(\frac{\|c_1\|_{L^{\frac{p^*}{p^*-r_1}}}}{r_1 S^{r_1/p}}\|u\|^{r_1} + \frac{\|c_2\|_{L^{\frac{p^*}{p^*-r_2}}}}{r_2 S^{r_2/p}}\|u\|^{r_2}\Big).$$

Consider

$$Q(t) = \frac{1}{p^* S^{p^*/p}} t^{p^*-p} \quad \text{and} \quad R(t) = \frac{\|c_1\|_{L^{\frac{p^*}{p^*-r_1}}}}{r_1 S^{r_1/p}} t^{r_1} + \frac{\|c_2\|_{L^{\frac{p^*}{p^*-r_2}}}}{r_2 S^{r_2/p}} t^{r_2},$$

Since $Q(t) \to 0$, as $t \to 0$, there exists $\rho > 0$ such that

$$\frac{1}{p} - Q(\rho) > 0.$$

Now, we choose $\lambda^* > 0$ such that

$$\frac{1}{p} - Q(\rho) - \lambda^* R(\rho) > 0.$$

Consequently, there exist ρ and $\beta > 0$, with ρ and β independent of n, such that

$$I_{\lambda,n}(u) \geq \beta, \quad \|u\| = \rho.$$

Case 2: $p < r_1 < p^*$. We have

$$I_{\lambda,n}(u) \geq \|u\|^p \Big(\frac{1}{p} - \frac{1}{p^* S^{p^*/p}} \|u\|^{p^*-p} - \frac{\lambda}{r_1 S^{r_1/p}} \|c_1\|_{L^{\frac{p^*}{p^*-r_1}}} \|u\|^{r_1-p} - \\ - \frac{\lambda}{r_2 S^{r_2/p}} \|c_2\|_{L^{\frac{p^*}{p^*-r_2}}} \|u\|^{r_2-p} \Big).$$

Considering

$$Q(t) = \frac{1}{p^* S^{p^*/p}} t^{p^*-p} + \frac{\lambda}{r_1 S^{r_1/p}} \|c_1\|_{L^{\frac{p^*}{p^*-r_1}}} t^{r_1-p} + \frac{\lambda}{r_2 S^{r_2/p}} \|c_2\|_{L^{\frac{p^*}{p^*-r_2}}} t^{r_2-p},$$

we note that $Q(t) \to 0$, as $t \to 0$, since $p < r_1 \leq r_2$. Hence, there exists $\rho > 0$ such that

$$\frac{1}{p} - Q(\rho) > 0.$$

Consequently, we get ρ and $\beta > 0$, with ρ and $\beta > 0$ independent of n, such that

$$I_{\lambda,n}(u) \geq \beta, \quad \|u\| = \rho.$$

Lemma 4.1 is proved. ∎

Lemma 4.2 *Suppose f satisfies (f_2) and (f_3). Then, for every $\lambda > 0$ and $n \in I\!N$, $I_{\lambda,n}$ satisfies (Φ_2).*

Proof: Consider Ω_0 given by (f_3) and $\phi \in D(I\!R^N)$, a positive function with $supp(\phi) \subset \Omega_0$. For every $t > 0$, we have

$$I_{\lambda,n}(t\phi) \leq \frac{t^p}{p} \int_{I\!R^N} |\nabla \phi|^p \, dx - \frac{t^{p^*}}{p^*} \int_{I\!R^N} |\phi|^{p^*} \, dx - \lambda a t^q \int_{I\!R^N} |\phi|^q \, dx.$$

Since $p^* > p$, there exists $t > 0$ sufficiently large such that $I_{\lambda,n}(t\phi) < 0$ and $\|t\phi\| > \rho$, with ρ given by Lemma 4.1. This proves the lemma. ∎

5 Estimates

Considering Ω_0 given by (f_3), we take $x_0 \in \Omega_0$ and $r_0 > 0$ such that $B(x_0, 2r_0) \subset \Omega_0$. Now, let $n_0 \in I\!N$ be such that $B(x_0, 2r_0) \subset B(0, n_0)$. Choose $\phi \in D(I\!R^N)$ satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ on the ball $B(x_0, r_0)$, and $\phi \equiv 0$ on $I\!R^N \setminus B(x_0, 2r_0)$. Given $\varepsilon > 0$ and w_ε defined in Section 2, set

$$v_\varepsilon = \frac{\phi w_\varepsilon}{\|\phi w_\varepsilon\|_{L^{p^*}}}.$$

Then, v_ε satisfies (See, e.g., [4], [9])

$$X_\varepsilon \equiv \int_{I\!R^N} |\nabla v_\varepsilon|^p \, dx \le S + O(\varepsilon^{\frac{N-p}{p}}), \text{ as } \varepsilon \to 0. \tag{5.1}$$

Proposition 5.1 *Suppose f satisfies (f_2) and (f_3), with q satisfying condition (H). Then, for every $\lambda > 0$, there exist $\varepsilon > 0$, $n_0 \in I\!N$ and $d_\lambda > 0$ such that, for every $n \ge n_0$,*

$$\max\{I_{\lambda,n}(tv_\varepsilon) \mid t \ge 0\} \le d_\lambda < \frac{1}{N} S^{\frac{N}{p}}.$$

Proof: From (f_3) and the definitions of ϕ_n and v_ε, we have

$$\begin{aligned} I_{\lambda,n}(tv_\varepsilon) &= \frac{t^p}{p} X_\varepsilon - \frac{t^{p^*}}{p^*} - \lambda \int_{I\!R^N} F_n(x, tv_\varepsilon)\, dx \le \\ &\le \frac{t^p}{p} X_\varepsilon - \frac{t^{p^*}}{p^*} - \lambda t^q \int_{I\!R^N} a|v_\varepsilon|^q \, dx \equiv J_\lambda(tv_\varepsilon). \end{aligned}$$

Thus, to prove the proposition, it suffices to obtain $\varepsilon > 0$ and $d_\lambda > 0$ such that

$$\max\{J_\lambda(tv_\varepsilon) \mid t \ge 0\} \le d_\lambda < \frac{1}{N} S^{\frac{N}{p}}.$$

We argue as in the proof of Lemma 4.2. Given $\varepsilon > 0$, there exists some $t_\varepsilon > 0$ such that

$$\max_{t \ge 0} J_\lambda(tv_\varepsilon) = J_\lambda(t_\varepsilon v_\varepsilon) \quad \text{and} \quad \frac{d}{dt} J_\lambda(tv_\varepsilon) = 0 \text{ in } t = t_\varepsilon.$$

This implies

$$0 < t_\varepsilon \le X_\varepsilon^{\frac{1}{p^*-p}}.$$

On the other hand, from Lemma 4.1 and (5.1), we have

$$0 < \beta \le J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{t_\varepsilon^p}{p}(S + O(\varepsilon^{\frac{N-p}{p}})).$$

Hence, there exists $\alpha_0 > 0$ such that

$$\alpha_0 \le t_\varepsilon \le X_\varepsilon^{\frac{1}{p^*-p}}, \quad \forall\, \varepsilon > 0.$$

Since the function $h(s) = \frac{s^p}{p} X_\varepsilon - \frac{s^{p^*}}{p^*}$ is increasing on the interval $(0, X_\varepsilon^{\frac{1}{p^*-p}})$, we obtain

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N} X_\varepsilon^{\frac{N}{p}} - \lambda a \alpha_0{}^q \int_{I\!R^N} |v_\varepsilon|^q \, dx.$$

From (5.1) and using the inequality

$$(b+c)^\alpha \le b^\alpha + \alpha (b+c)^{\alpha-1} c \qquad \forall\, b,\ c \ge 0,\ \forall\, \alpha > 1,$$

with $b = S$, $c = O(\varepsilon^{\frac{N-p}{p}})$, and $\alpha = \frac{N}{p}$, we get

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N} S^{N/p} + O(\varepsilon^{\frac{N-p}{p}}) - \lambda \alpha_0^q \int_{I\!R^N} a |v_\varepsilon|^q \, dx.$$

Thus, there exists $M > 0$ such that

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N} S^{N/p} + \varepsilon^{\frac{N-p}{p}} \left(M - \frac{\lambda \alpha_0^q}{\varepsilon^{\frac{N-p}{p}}} \int_{I\!R^N} a |v_\varepsilon|^q \, dx \right) \le$$

$$\le \frac{1}{N} S^{N/p} + \varepsilon^{\frac{N-p}{p}} \left(M - \frac{\lambda a \alpha_0^q}{\varepsilon^{\frac{N-p}{p}}} \int_{B(0,1)} \frac{\varepsilon^{\frac{(N-p)q}{p^2}}}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{(N-p)q}{p}}} \, dx \right).$$

By changing variables, we obtain

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N} S^{N/p} + \varepsilon^{\frac{N-p}{p}} \Bigg(M -$$

$$- \lambda a w_{N-1} \alpha_0^q \varepsilon^{[(\frac{(N-p)}{p^2} - \frac{N-p}{p})q + \frac{(p-1)N}{p} + \frac{p-N}{p}]} \int_0^{\varepsilon^{\frac{1-p}{p}}} \frac{s^{N-1}}{(1 + s^{p/(p-1)})^{\frac{(N-p)q}{p}}} \, ds \Bigg).$$

Furthermore, for $\varepsilon > 0$ sufficiently small, we have

$$\int_0^{\varepsilon^{\frac{1-p}{p}}} \frac{s^{N-1}}{(1 + s^{p/(p-1)})^{\frac{(N-p)q}{p}}} \, ds \ge \int_0^1 \frac{s^{N-1}}{(1 + s^{p/(p-1)})^{\frac{(N-p)q}{p}}} \, ds \ge \frac{2^{\frac{(p-N)q}{p}}}{N},$$

because $g(s) = (1 + s^{p/(p-1)})^{-1} \ge g(1) = 2^{-1}$ for $s \in [0,1]$. Consequently, there exists a positive constant C, such that

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N} S^{N/p} + \varepsilon^{\frac{N-p}{p}} \left(M - \lambda C \varepsilon^{[(\frac{(N-p)}{p^2} - \frac{N-p}{p})q + \frac{(p-1)N}{p} + \frac{p-N}{p}]} \right).$$

Since $(\frac{(N-p)}{p^2}-\frac{(N-p)}{p})q+\frac{(p-1)N}{p}+\frac{p-N}{p}$ is negative, when q satisfies condition (H), we find $\varepsilon_0>0$ such that

$$\begin{aligned} J_\lambda(t_{\varepsilon_0}v_{\varepsilon_0}) &< d_\lambda \equiv \frac{1}{N}S^{N/p}+\varepsilon_0^{\frac{N-p}{p}}\left(M-\lambda C\varepsilon_0^{[(\frac{(N-p)}{p^2}-\frac{N-p}{p})q+\frac{(p-1)N}{p}+\frac{p-N}{p}]}\right) \\ &< \frac{1}{N}S^{N/p}. \end{aligned}$$

Proposition 5.1 is proved. ■

6 Theorem 1.1

In view of Lemmas 4.1 and 4.2, we may apply Theorem 2.2 to the sequence of functionals $I_{\lambda,n}$, obtaining a positive level $c_{\lambda,n}$, and a $(PS)_{c_{\lambda,n}}$ sequence $(u_j^{(n)})_j$ in $D^{1,p}$, i.e.,

$$I_{\lambda,n}(u_j^{(n)})\to c_{\lambda,n} \quad \text{and} \quad I'_{\lambda,n}(u_j^{(n)})\to 0 \quad \text{as} \quad j\to\infty.$$

Moreover, from Lemma 4.1 and Proposition 5.1 , we have

$$0<\beta\le c_{\lambda,n}=\inf_{\gamma\in\Gamma}\sup_{u\in\gamma} I_{\lambda,n}(u)\le d_\lambda<\frac{1}{N}S^{N/p}.$$

Taking a subsequence if necessary, we find $c_\lambda\in[\beta,d_\lambda]$ such that

$$c_\lambda=\lim_{n\to\infty}c_{\lambda,n}.$$

Thus, given $0<\epsilon<\min\{c_\lambda,\frac{1}{N}S^{N/p}\}$, there exists $n_0>0$ such that $c_{\lambda,n}\in(c_\lambda-\epsilon,\, c_\lambda+\epsilon)$ for every $n\ge n_0$. Now, for each $n\ge n_0$, there exists $u_n=u_{j_n}^{(n)}$ satisfying

$$c_\lambda-\epsilon<I_{\lambda,n}(u_n)<c_\lambda+\epsilon, \tag{6.1}$$

and

$$\|I'_{\lambda,n}(u_n)\|\le\frac{1}{n}. \tag{6.2}$$

Lemma 6.1 *The sequence (u_n) is bounded in $D^{1,p}$.*

<u>Proof:</u> From (f_4), there exist $\tau\in(p,p^*)$ and $\mu\in(1,p^*)$ such that

$$\begin{aligned} I_{\lambda,n}(u_n) &- \frac{1}{\tau}I'_{\lambda,n}(u_n)u_n=(\frac{1}{p}-\frac{1}{\tau})\|u_n\|^p+(\frac{1}{\tau}-\frac{1}{p^*})\|u_n^+\|_{L^{p^*}}^{p^*}+ \\ &+ \lambda\int_{I\!R^N}\left(\frac{1}{\tau}f_n(x,u_n)u_n-F_n(x,u_n)\right)dx\ge \\ &\ge (\frac{1}{p}-\frac{1}{\tau})\|u_n\|^p+(\frac{1}{\tau}-\frac{1}{p*})\|u_n^+\|_{L^{p^*}}^{p^*}-\lambda\int_{I\!R^N}c_3(x)|u_n^+|^\mu\,dx. \\ &\ge (\frac{1}{p}-\frac{1}{\tau})\|u_n\|^p+(\frac{1}{\tau}-\frac{1}{p^*})\|u_n^+\|_{L^{p^*}}^{p^*}-\lambda\|c_3\|_{L^{\frac{p^*}{p^*-\mu}}}\|u_n^+\|_{L^{p^*}}^{\mu}. \end{aligned} \tag{6.3}$$

On the other hand, from (6.1) and (6.2), we have

$$I_{n,\lambda}(u_n) - \frac{1}{\tau} I'_{n,\lambda}(u_n)u_n \leq C + \frac{1}{\tau}\|u_n\|. \tag{6.4}$$

Denoting $h(t) = (\frac{1}{\tau} - \frac{1}{p^*})t^{p^*} - \lambda\|c_3\|t^{\mu}$, for $t \geq 0$, from (6.3), (6.4), and using that $h(t)$ is bounded from below, we conclude that the sequence (u_n) is bounded in $D^{1,p}$. Lemma 6.1 is proved. ■

Applying Proposition 3.1 to the diagonal sequence (u_n), we obtain a weak solution u for problem (GP). The final step is the verification that u is nontrivial. First of all, we note that $u^- = 0$. Effectively

$$\begin{aligned} \int_{I\!R^N} |\nabla(u_n^-)|^p \, dx &\leq \int_{I\!R^N} \left(|\nabla(u_n^+)|^2 + |\nabla(u_n^-)|^2\right)^{\frac{p}{2}} dx = \\ &= \int_{I\!R^N} |\nabla u_n|^p \, dx. \end{aligned}$$

Thus, the sequence (u_n^-) is bounded in $D^{1,p}$. Consequently, $I'_{\lambda,n}(u_n)(u_n^-) \to 0$, as $n \to \infty$. Since $I'_{\lambda,n}(u_n)(u_n^-) = \frac{1}{p}\|u_n^-\|^p$, it follows that $u_n^- \to 0$ in $D^{1,p}$, as $n \to \infty$.

Now, we assume by contradiction that $u \equiv 0$ is the only possible solution of (GP). Let

$$l = \lim_{n\to\infty} \int_{I\!R^N} |u_n^+|^{p^*} \, dx.$$

From (f_3) and (6.2), we have

$$0 = \lim_{n\to\infty} I'_{\lambda,n}(u_n)u_n \geq \lim_{n\to\infty} \int_{I\!R^N} \left(|\nabla u_n|^p - |u_n^+|^{p^*} - \lambda c_1|u_n|^{r_1} - \lambda c_2|u_n|^{r_2}\right) dx.$$

Consequently, by Proposition 2.1, we have

$$\lim_{n\to\infty} \int_{I\!R^N} |\nabla u_n|^p \, dx \leq l. \tag{6.5}$$

We claim that $l > 0$. Effectively, arguing by contradiction, we suppose that $l = 0$. Under this assumption, from (6.5) and (f_3), $I_{\lambda,n}(u_n) \to 0$, as $n \to \infty$. But this is impossible in view of (6.1). The claim is proved.

Invoking (2.1), we have

$$\|\nabla u_n\|_p^p \geq \|\nabla(u_n^+)\|_p^p \geq S\Big(\int_{I\!R^N} |u_n^+|^{p^*}\Big)^{\frac{p}{p^*}}. \tag{6.6}$$

As a direct consequence of (6.6), and (6.5), we obtain

$$l \geq S^{\frac{N}{p}}. \tag{6.7}$$

By (f_4), (6.1) and (6.2), we get

$$\begin{aligned}\frac{1}{N}S^{N/p} &> d_\lambda \geq c_\lambda = \lim_{n\to\infty} I_{\lambda,n}(u_n) = \lim_{n\to\infty}\left[I_{\lambda,n}(u_n) - \frac{1}{\tau}I'_{\lambda,n}(u_n)u_n\right] = \\ &= \lim_{n\to\infty}\left[(\frac{1}{p}-\frac{1}{\tau})\|\nabla u_n\|^p + (\frac{1}{\tau}-\frac{1}{p^*})\|u_n^+\|^{p^*}_{L^{p^*}} - \int_{I\!R^N} c_3 u_n^\mu\, dx\right] \geq \\ &\geq \lim_{n\to\infty}\left[(\frac{1}{p}-\frac{1}{\tau})S\|u_n\|^p_{L^{p^*}} + (\frac{1}{\tau}-\frac{1}{p^*})\|u_n^+\|^{p^*}_{L^{p^*}} - \int_{I\!R^N} c_3 u_n^\mu\, dx\right].\end{aligned}$$

Consequently, from Proposition 2.1 and (6.7), we have

$$\begin{aligned}\frac{1}{N}S^{N/p} &> (\frac{1}{p}-\frac{1}{\tau})Sl^{p/p^*} + (\frac{1}{\tau}-\frac{1}{p^*})l \geq \\ &\geq (\frac{1}{p}-\frac{1}{\tau})S^{1+\frac{N}{p^*}} + (\frac{1}{\tau}-\frac{1}{p^*})S^{N/p} = \frac{1}{N}S^{N/p}.\end{aligned}$$

This concludes the proof of Theorem 1.1. ∎

7 Theorem 1.3

In this section we establish a proof of Theorem 1.3. The key ingredient is the verification of Proposition 5.1 under conditions (f_3) and (f_5). To obtain such result we exploit the positivity of the function $F(x,s)$. Considering the extremal functions w_ε defined by (2.2), we have

Proposition 7.1 *Suppose f satisfies $(f_2),(f_3)$, and (f_5). Then, for every $\lambda > 0$, there exist $\varepsilon > 0$, $n_0 \in I\!N$ and $d_\lambda > 0$ such that, for every $n \geq n_0$*

$$\max\{I_{\lambda,n}(tw_\varepsilon) \mid t \geq 0\} \leq d_\lambda < \frac{1}{N}S^{\frac{N}{p}}.$$

Proof: Let $n_0 \in I\!N$ such that $\hat{\Omega}_0 \equiv B(0,n_0)\cap\Omega_0 \neq \emptyset$. From $(f_2),(f_3),(f_5)$ and our definition of $I_{\lambda,n}$, for every $n \geq n_0$, we have

$$\begin{aligned}I_{\lambda,n}(tw_\varepsilon) &= (\frac{t^p}{p}-\frac{t^{p^*}}{p^*})S^{N/p} - \lambda\int_{I\!R^N}\phi_n(x)F(x,tw_\varepsilon)\,dx \leq \\ &\leq (\frac{t^p}{p}-\frac{t^{p^*}}{p^*})S^{N/p} - \lambda\int_{\hat{\Omega}_0}\phi_n(x)F(x,tw_\varepsilon)\,dx \leq \\ &\leq (\frac{t^p}{p}-\frac{t^{p^*}}{p^*})S^{N/p} - \lambda a t^q\int_{\hat{\Omega}_0}|w_\varepsilon|^q\,dx \equiv J_\lambda(tw_\varepsilon).\end{aligned}$$

Thus, it suffices to obtain $\varepsilon > 0$ and $d_\lambda > 0$ such that

$$\max\{J_\lambda(tw_\varepsilon) \mid t \geq 0\} \leq d_\lambda < \frac{1}{N} S^{N/p}.$$

To prove such result we follow the argument employed in [6]. By (2.2), the sequence w_ε is bounded in $L^{\frac{p^*}{q}}(I\!R^N)$ and $w_\varepsilon(x) \to 0$ a.e. in $I\!R^N$, as $\varepsilon \to 0$. Thus, $w_\varepsilon \rightharpoonup 0$ weakly in $L^{\frac{p^*}{q}}(I\!R^N)$, as $\varepsilon \to 0$. On the other hand, the restriction $w_\varepsilon |_{\hat{\Omega}_0}$ belongs to $W^{1,p}(\hat{\Omega}_0)$. Hence, by the Sobolev Embedding Theorem, $w_\varepsilon \to 0$ strongly in $L^r(B(0,2n))$, for every $1 \leq r < p^*$. Consequently,

$$\lim_{\varepsilon \to 0} \int_{\hat{\Omega}_0} |w_\varepsilon|^q \, dx = 0.$$

Therefore, there exists $\varepsilon_0 > 0$ such that

$$0 < J_\lambda(w_{\varepsilon_0}) < \frac{1}{N} S^{N/p}. \tag{7.1}$$

Now, arguing as in the proof of Lemma 4.2 , we take $t_{\varepsilon_0} > 0$ such that

$$J_\lambda(t_{\varepsilon_0} w_{\varepsilon_0}) = \max\{J_\lambda(tw_\varepsilon) \mid t \geq 0\}.$$

Since $\frac{d}{dt} I_{\lambda,n}(tw_{\varepsilon_0}) = 0$, for $t = t_{\varepsilon_0}$, we get

$$(t_{\varepsilon_0}^{p-1} - t_{\varepsilon_0}^{p^*-1}) S^{N/p} - \lambda a t_{\varepsilon_0}^{q-1} \int_{\hat{\Omega}_0} |w_{\varepsilon_0}|^q \, dx = 0. \tag{7.2}$$

From (7.1) and (7.2), we have

$$0 < t_{\varepsilon_0} < 1.$$

Observing that the function $h(t) = \frac{t^p}{p} - \frac{t^{p^*}}{p^*}$ achieves its maximum at $t = 1$, we get

$$J_\lambda(t_{\varepsilon_0} w_{\varepsilon_0}) \leq d_\lambda \equiv J_\lambda(w_{\varepsilon_0}) < \frac{1}{N} S^{N/p}.$$

Proposition 7.1 is proved. ∎

Finally, we observe that the proof of Theorem 1.3 follows by the same argument employed in the proof of Theorem 1.1, with Proposition 7.1 replacing Proposition 5.1.

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