Hermitian structures on flag manifolds

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Abstract

In this note using the tournament technique, we study invariant almost complex structures and invariant metrics on complex flag manifolds and produce some new harmonic maps into them.

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§0 Introduction

Let (M, g, J_M) be a cosymplectic almost Hermitian manifold and (N, h, J_N) a (1,2)-symplectic almost Hermitian manifold. A traditional approach for constructing harmonic maps between M and N is to find holomorphic maps among them because in this case, holomorphic maps are rather special harmonic maps ([8] and [15]).

When the target manifolds are complex flag manifolds, an alternative method is to study horizontal almost complex structures more generally, horizontal f-structures, since the horizontal holomorphic maps are harmonic for all invariant metrics [2]. For instance, Eells and Wood showed that linearly full harmonic maps of S^2 into $\mathbb{C}P^n$ arises by homogeneous projection from a horizontally holomorphic map into the manifold of full flag in $\mathbb{C}^{n+1}$. One of the authors established that these maps (which he calls Eells-Wood maps) are harmonic for all the invariants metrics [17].

Let $F = F(n_1, ..., n_k; N)$ denote the complex flag manifold $\dfrac{U(N)}{U(n_1) \times ... \times U(n_k)}$ where $n_1 + n_2 + ... + n_k = N$. The results of Borel and Hirzebruch [4] show that there are $2^{\binom{k}{2}}$ $U(N)$-invariant almost complex structures on F. Among of them, the natural and remarkable ones are the canonical complex structure J_1 and corresponding parabolic almost complex structure J_2 [10] [13]. Denote by F the set $\{0 = V_0 \subset V_1 \subset ... \subset V_k = \mathbb{C}^N; \dim V_i = \sum_{j=1} n_j, n_j \geq 1\}$. We say that such F has height $k-1$ (see [9] for more details).

The canonical complex structure J_1 on F is the one induced by the inclusion of F into a product of Grassmannians is given by: $(V_0 \subset V_1 \subset ... \subset V_k) \mapsto (V_1, V_2, ..., V_k)$.

The corresponding parabolic almost complex structure is obtained by flipping the orientation of J_1 on $\ker d\pi$, where π is the homogeneous projection from F to a complex Grassamannian given by

$$(E_1, ..., E_k) \mapsto \bigoplus_i E_{2i}$$

and $(E_1, ..., E_k)$ are the legs of the flag $0 = V_0 \subset V_1 \subset ... \subset V_k = \mathbb{C}^N$ [4]. There are many maps into F which are holomorphic with respect to these almost complex structures. For example, we can prove that every harmonic map $S^2 \to G_r(\mathbb{C}^N)$

is covered by a J_2-holomorphic map into a suitable flag manifold [11], [6]. The Gauss maps for totally geodesic immersions of Kähler manifolds to complex projective spaces are J_1-holomorphic maps into certain flag manifolds [13]. However, an arbitrary almost complex structure J on F in general, is not horizontal. (See §8 for more details).

Notice that the height two flag manifolds admit a homogeneous metric which is (1,2)-symplectic for J_2 (i.e., if Ω is the corresponding Kähler form, then $d\Omega^{(1,2)} = 0$) [9], [7] and [14]. Hence in this special case we are able to obtain harmonic maps into $F(n_1, n_2, n_3; N)$ from J_2-holomorphic maps via the use of the well known Theorem of Lichnerowicz [15]. Hence our problem of producing harmonic maps into flag manifolds is reduced to the question of finding a J_2-(1,2)-symplectic metric for an arbitrary height flag manifolds

All these left invariant metrics on F have dimension $\binom{k}{2}$. Among of then, a natural one is the normal metric [13]. It is induced from the natural bi-invariant metric on $U(N)$ and it has the largest isometry group. But such a metric is not well behaved from the point of view of complex geometry. We remark, except in the case where the Lie algebra is equal to $u(2)$, the normal metric is not Kähler [17] [14]. For the description of invariant Kähler-Einstein metrics for see [1]. From [8], we obtain that the normal metric on $F(1,1,1;3)$ is (1,2)-symplectic. The general case is unknown to us.

According to Burstall and Salamon [10] $U(N)$-invariant almost complex structures on F of height $k-1$ are 1-1 correspondence with directed graphs with k nodes (or equivalently k-tournaments). In a similar way we can see that each left invariant metric on F corresponds to associate a strictly positive scalar to each edge in the tournament (see [17] and [2]).

Using the k-tournament technique we, in this note, study Hermitian structures on flag manifolds and produce some new examples of harmonic maps into flag manifolds. Our main results are:

1) Every symplectic or Hermitian (1,2)-symplectic complex flag manifold must

be a Kähler one. (See Theorem 3.4 and Theorem 3.5).

2) Every integrable (resp. parabolic) invariant almost complex structure on a flag manifold admits families of dimension n-1 (resp. n) of left invariant Kähler (resp. (1,2)-symplectic) metrics.

3) The normal metric is (1,2)-symplectic with respect to some invariant almost complex structure into the flag manifold F if and only if the height of F is less than 3.

4) In constrast to 1) and 2) when the height of F is at least 3, there are two large classes of invariant almost complex structures on F which don't admit any left invariant (1,2)-symplectic metric. In particular, any invariant almost complex structure in a height three flag manifold doesn't admit a left invariant (1,2)-symplectic if the almost complex structure is not integrable and parabolic.

§1 introduces invariant Hermitian metrics and invariant almost complex structures on complex flag manifolds and in §2 we investigate the Kähler forms and their exterior differentiations.

§3 establishes integrability conditions for almost complex structures, and gives explicit formulas for the Kähler metrics.

§4 is devoted to describing parabolic invariant almost complex structures. We construct n-dimensional (1,2)-symplectic metric for each of such structure.

In §5 we discuss almost complex structures that don't admit any left-invariant metric. A explicit description was given for Kähler metrics and (1.2)-sympletic metrics in the case of F(3) and F(4) in §6. In §7 we apply Gale's inequality in [12] to the geometry of the invariant metric arising from the Killing form. Several necessary remarks will be presented in §8. Finally, as application, in §9, we construct some new harmonic maps from surfaces or flag manifolds into flag manifolds.

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§1 Invariant almost complex structures and Hermitian metrics on $F(N)$

Without loss of generality, we consider full complex flag manifolds $F = F(1,1,...1;N) := F(N)$ (see §8).

We indicate with ω the Maurer-Cartan form of $U(N)$, that is $dZ = \omega.Z$, for a unitary frame $Z = (Z_1, ..., Z_n)$. At the identity of $U(N)$, we have

$$dZ = \omega = (\omega_{i\bar{j}}) \tag{1.1}$$

and at the Lie algebra level, we write

$$\begin{aligned} u^*(n)^{\mathbb{C}} &= \bigoplus_{i,j}(\text{span}\{\omega_{i\bar{j}}\} \oplus \text{span}\{\omega_{\bar{i}j}\}) \\ &= \bigoplus_{i}(\text{span}\{\omega_{i\bar{i}}\} \oplus \text{span}\{\omega_{\bar{i}i}\})[\bigoplus_{i\neq j}(\text{span}\{\omega_{i\bar{j}}\} \oplus \text{span}\{\omega_{\bar{i}j}\})] \\ &= (u(1) + ... + u(1))^{*\mathbb{C}} \oplus (\bigoplus_{i\neq j} D_{ij}^{\mathbb{C}}) \end{aligned} \tag{1.2}$$

where

$$D_{ij} = \text{span}\{Re\omega_{i\bar{j}}, Im\omega_{\bar{i}j}\} \tag{1.3}$$

Each real vector space D_{ij} has two invariant almost complex structures, with its (1,0)-type forms generated by $\omega_{i\bar{j}}$ and $\omega_{\bar{i}j}$ respectively. The results of Borel and Hirzebruch [4] show that there are $2^{\binom{N}{2}}$ $U(N)$-invariant almost complex structure J on $F(N)$ determined by the choice of one of these two structures in each D_{ij}. We see that such a choice defines a tournament $\mathcal{J}(J)$ with players $T = \{1, 2, ..., N\}$. Indeed the space of $(1,0)$-cotangent vector at the identity coset, can be identified with

$$m_{1,0} = \underset{i \to j}{\text{Span}} \quad \{\omega_{i\bar{j}}\} \tag{1.4}$$

where

$$\mathcal{J}(J) = \{i \to j; i, j = 1, ..., n \text{ with } i \neq j\} \tag{1.5}$$

The basic references are Moon [16] and Reid and Beineke [18].

Now we define all of left-invariant metrics on $F(N)$ (see [2]), namely

$$ds^2_\Lambda = \sum_{i,j} \lambda_{ij} \omega_{i\bar{j}} \otimes \omega_{\bar{i}j} \tag{1.6}$$

where

$$\Lambda = (\lambda_{ij}) \tag{1.7}$$

is a real symmetric matrix anel satisfies that

$$\lambda_{ij} \begin{cases} > 0 & \text{if} \quad i \neq j \\ = 0 & \text{if} \quad i = j \end{cases} \tag{1.8}$$

For an alternative description, see for example [17]

(1.6)-(1.8) defines an Hermitian metric on $F(N)$ for each invariant almost complex structure J, because:

$$\begin{aligned} ds^2_\Lambda(JX, JY) &= \sum_{i,j} \lambda_{ij} \omega_{i\bar{j}}(JX) \omega_{\bar{i}j}(JY) \\ &= \sum_{i,j} \lambda_{ij} J\omega_{i\bar{j}}(X) J\omega_{\bar{i}j}(Y) \\ &= \sum_{i,j} \lambda_{ij} \varepsilon_{ij} \sqrt{-1} \omega_{i\bar{j}}(X) \varepsilon_{ij}(-\sqrt{-1}) \omega_{\bar{i}j}(Y) \\ &= \sum_{i,j} \lambda_{ij} \varepsilon^2_{ij} \omega_{i\bar{j}} \otimes \omega_{\bar{i}j}(X, Y) = ds^2_\Lambda(X, Y) \end{aligned} \tag{1.9}$$

for any vector fields X and Y where

$$\varepsilon_{ij} = \begin{cases} 1 & i \to j \\ -1 & j \to i \\ 0 & i = j \end{cases} \tag{1.10}$$

It is clear that $\varepsilon := (\varepsilon_{ij})$ is anti-symmetric.

§2 Kähler forms and their exterior differentiations

Let $\sum_N$ be the permutation group of N elements with identity e. For each $\tau \in \sum_N$, the Kähler form Ω, with respect to the $U(N)$-invariant almost Hermitian structure corresponding tournament $\mathcal{J}(J)$, (see (1.5)) and left-invariant Hermitian metric ds^2_Λ (see (1.6)), is defined by

$$\begin{aligned}
&\Omega(X,Y) := ds^2_\Lambda(X, JY) \\
&= \sum_{i,j} \lambda_{\tau(i)\tau(j)} \omega_{\tau(i)\overline{\tau(j)}}(X) J \omega_{\overline{\tau(i)}\tau(j)}(Y) \\
&= \sum \lambda_{\tau(i)\tau(j)} \varepsilon_{\tau(i)\tau(j)} (-\sqrt{-1}) \omega_{\tau(i)\overline{\tau(j)}}(X) \omega_{\overline{\tau(i)}\tau(j)}(Y) \\
&= -\sqrt{-1} (\sum_{i<j} + \sum_{i>j}) \varepsilon_{\tau(i)\tau(j)} \lambda_{\tau(i)\tau(j)} [\omega_{\tau(i)\overline{\tau(j)}}(X) \omega_{\overline{\tau(i)}\tau(j)}(Y)] \\
&= -\sqrt{-1} \sum_{i<j} \varepsilon_{\tau(i)\tau(j)} \lambda_{\tau(i)\tau(j)} [\omega_{\tau(i)\overline{\tau(j)}}(X) \omega_{\overline{\tau(i)}\tau(j)}(Y) - \omega_{\tau(j)\overline{\tau(i)}}(X) \omega_{\overline{\tau(j)}\tau(i)}(Y)] \\
&= -\sqrt{-1} \sum_{i<j} \varepsilon_{\tau(i)\tau(j)} \lambda_{\tau(i)\tau(j)} [\omega_{\tau(i)\overline{\tau(j)}}(X) \omega_{\overline{\tau(i)}\tau(j)}(Y) - \omega_{\tau(i)\overline{\tau(j)}}(Y) \omega_{\overline{\tau(i)}\tau(j)}(X)] \\
&= -2\sqrt{-1} \sum_{i<j} \varepsilon_{\tau(i)\tau(j)} \lambda_{\tau(i)\tau(j)} \omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)}(X,Y) \qquad (2.1)
\end{aligned}$$

where ε_{ij} is defined in (1.10). the Kähler form Ω is given by: It follows that

$$\Omega = -2\sqrt{-1} \sum_{i<j} \mu_{\tau(i)\tau(j)} \omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)} \qquad (2.2)$$

for arbitrary $\tau \in \sum_N$ and

$$\mu_{ij} := \varepsilon_{ij} \lambda_{ij} \qquad (2.3)$$

satisfies that

$$\mu_{ij} + \mu_{ji} = 0 \qquad (2.4)$$

By differentiating (2.2) and using the Maurer-Cartan equations for $U(N)$ one deduces the following

$$\begin{aligned}
\frac{\sqrt{-1}}{2} d\Omega &= \sum_{i<j} \mu_{\tau(i)\tau(j)} [(d\omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)} - \omega_{\tau(i)\overline{\tau(j)}} \wedge (d\omega_{\overline{\tau(i)}\tau(j)})] \\
&= \sum_{i<j} \mu_{\tau(i)\tau(j)} [\sum_k \omega_{\tau(i)\overline{\tau(k)}} \wedge \omega_{\tau(k)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)} - \sum_k \omega_{\tau(i)\overline{\tau(j)}} \wedge \overline{\omega_{\tau(i)\overline{\tau(k)}}} \wedge \overline{\omega_{\tau(k)\overline{\tau(j)}}}] \\
&= \sum_{i<j} \mu_{\tau(i)\tau(j)} [(\omega_{\tau(i)\overline{\tau(i)}} + \omega_{\overline{\tau(i)}\tau(i)}) \wedge \omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)} + \omega_{\tau(i)\overline{\tau(j)}} \wedge (\omega_{\tau(j)\overline{\tau(j)}} + \omega_{\overline{\tau(j)}\tau(j)}) \wedge \omega
\end{aligned}$$

$$+ \sum_{k\neq i,j} \omega_{\tau(i)\overline{\tau(k)}} \wedge \omega_{\tau(k)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)} - \sum_{k\neq i,j} \overline{\omega_{\tau(i)\overline{\tau(k)}} \wedge \omega_{\tau(k)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)}}\ ]$$
$$= 2\sqrt{-1} \sum_{i<j} \mu_{\tau(i)\tau(j)} \sum_{k\neq i,j} Im(\omega_{\tau(i)\overline{\tau(k)}} \wedge \omega_{\tau(k)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)})$$

Hence we get

$$\frac{1}{4}d\Omega = (\sum_{k<i<j} + \sum_{i<k<j} + \sum_{i<j<k})\mu_{\tau(i)\tau(j)} \cdot Im(\omega_{\tau(i)\overline{\tau(k)}} \wedge \omega_{\tau(k)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)})$$
$$= \sum_{i<j<k} [\mu_{\tau(j)\tau(k)} Im(\omega_{\tau(j)\overline{\tau(i)}} \wedge \omega_{\tau(i)\overline{\tau(k)}} \wedge \omega_{\overline{\tau(j)}\tau(k)}) + \mu_{\tau(i)\tau(k)} Im(\omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\tau(j)\overline{\tau(k)}} \wedge \omega_{\overline{\tau(i)}\tau(k)})$$
$$+\mu_{\tau(i)\tau(j)} Im(\omega_{\tau(i)\overline{\tau(k)}} \wedge \omega_{\tau(k)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)})]$$
$$= \sum_{i<j<k} C_{\tau(i)\tau(j)\tau(k)} \Psi_{\tau(i)\tau(j)\tau(k)} \tag{2.6}$$

where

$$C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk} \tag{2.7}$$

and

$$\Psi_{ijk} = Im(\omega_{i\bar{j}} \wedge \omega_{\bar{i}k} \wedge \omega_{j\bar{k}}) \tag{2.8}$$

We denote by $\mathbb{C}^{p,q}$ the space of complex forms with degree (p,q) in $F(N)$. Then for any i,j,k we either have

$$\Psi_{ijk} \in \mathbb{C}^{0,3} \oplus \mathbb{C}^{3,0} \tag{2.9}$$

or

$$\Psi_{ijk} \in \mathbb{C}^{1,2} \oplus \mathbb{C}^{2,1} \tag{2.10}$$

§3 Integrability of invariant almost complex structures

An almost complex structure is said to be integrable if it has no torsion. i.e

$$[JX, JY] = [X, Y] + J[X, JY] + J[JX, Y]$$

for arbitrary vector fields X and Y. Burstall and Salamon have shown:

Theorem 3.1. ([10]) The invariant almost complex strucutre J is integrable if and only if $\mathcal{J}(J)$ is isomorphic to the canonical tournament.

Notice that the canonical N-tournament $\mathcal{J}_N$ is defined by setting:

$$i \to j \text{ if and only if } i < j \tag{3.1}$$

The following theorem was proved by Moon [16, Theorem 9] and two tournaments J and J are isomorphic if there is a map ϕ: $\{1, ..., N\} \to \{1, ..., N\}$ such that: $s \to t \Rightarrow f(s) \to f(t)$ for an arbitrary $s \to t$ in $\mathcal{J}$

Theorem 3.2. An N-tournament is isomorphic to the canonical tournament if and only if there are no circuits, i.e. closed paths $i_1 \to i_2 \to ... \to i_k \to i_1$

We can associate to each tournament a directed digraph in which every node (or vertice) represents a player, and each dominance relation is represented by an oriented edge $i \longleftarrow j$. We improve theorem 3.2 as follows:

Lemma 3.3. An N-tournament is isomorphic to the canonical tournament if and only if it has no 3 cycles, i.e its associated digraph contains no configuration of the type

Proof: It is sufficient to show that any N-tournament $\mathcal{J}$ with a circuit,

$$i_1 \to i_2 \to ... \to i_\ell \to i_1. \tag{3.2}$$

has a 3-cycle.

For $\ell = 3$, (3.2) is clearly true. Suppose for $\ell = r(> 3)$ our conclusion is true. We consider now the following $(r+1)$-cycle

$$i_1 \to i_2 \to \ldots \to i_r \to i_{r+1} \to i_1 \tag{3.3}$$

Then either

$$i_1 \to i_r \tag{3.4}$$

or

$$i_r \to i_1$$

When (3.3) holds we have a 3-cycle

$$i_1 \to i_r \to i_{r+1} \to i_1$$

otherwise we have a r-cycle

$$i_1 \to i_2 \to \ldots \to i_r \to i_1$$

By using induction it includes a 3-cycle. So does (3.3). (See [16] for more comments).

We are now in position of describing the equivalence condition for invariant almost complex structures on complex flag manifolds $F(N)$ to be integrable.

Theorem 3.4. Let J be an invariant complex structure on $F(N)$, and Ω the Kähler form with respect to any left-invariant metric. Then the (0,3) part of $d\Omega$ is zero. Therefore any Hermitian (1,2)-symplectic complex flag manifold is a Kähler one.

Remark Recall that an almost Hermitian manifold is (1,2)-symplectic if the (1,2) part of $d\Omega$ is zero.

Proof of Theorem 3.4. Suppose that, there exists a left-invariant metric ds^2_Λ, such that

$$(d\Omega)^{(0,3)} \neq 0 \tag{3.5}$$

According to (2.6), one gets $\Psi_{ijk} \in \mathbb{C}^{1,3}\Theta\mathbb{C}^{3,0}$ for some (i, j, k).

From (1.4) and (2.8) we have either

$$\omega_{i\bar{j}}, \omega_{\bar{i}k}, \omega_{j\bar{k}} \in m_{1,0} \tag{3.6}$$

or

$$\omega_{\bar{i}j}, \omega_{i\bar{k}}, \omega_{\bar{j}k} \in m_{1,0} \tag{3.7}$$

When (3.6) is true we have

$$i \to j \to k \to i \tag{3.8}$$

otherwise

$$i \to k \to j \to i \tag{3.9}$$

Combine with theorem 3.1 and lemma 3.3 we get that J is non-integrable. QED

Conversely, we have the following:

Theorem 3.5. Let Ω be the Kähler form related to to the invariant almost complex structure J and the left-invariant metric ds^2_Λ. If the (0,3) part of $d\Omega$ is zero then J is integrable.

Proof. Suppose that J is non-integrable. We know from theorem 3.1 and lemma 3.3, there exists Ψ_{ijk} such that (2.9) is true. Furthermore, either (3.8) or (3.9) holds. Together with (1.10) we have

$$\varepsilon_{ij} = -\varepsilon_{ik} = \varepsilon_{jk} \tag{3.10}$$

Combining (2.3) and (2.7) we get

$$C_{ijk} \neq 0 \tag{3.11}$$

So we obtain that $(d\Omega)^{(0,3)} \neq 0$ since $\{\omega_{i\bar{j}} \wedge \omega_{\bar{i}k} \wedge \omega_{j\bar{k}}\}$ are linearly independent.

QED

Summarizing we have the following:

Theorem 3.6. Let J be an invariant almost complex structure on $F(N)$. The following conditions are equivalent:

(i) J is integrable;

(ii)

$$d\Omega \equiv 0 \mod (d\Omega)^{1,2} \tag{3.12}$$

for some left-invariant metric dS^2_Λ. (iii) (3.12) is true related to any dS^2_Λ.

Now we write out the explicit formulas for Kähler metrics on $F(N)$. Let J be an invariant complex structure in $F(n)$. By theorem 3.1 there exists a permutation $\tau \in \sum_N$ such that $\mathcal{J}(J)$ is given by:

$$i < j \Leftrightarrow \tau(i) \to \tau(j) \tag{3.13}$$

Combining with (1.4) and (2.8) we have

$$\Psi_{\tau(i)\tau(j)\tau(k)} \in \mathbb{C}^{1,2} \oplus \mathbb{C}^{2,1}$$

for any $i < j < k$.

So for any $i < j < k$ we have:

$$\Psi_{\tau(i)\tau(j)\tau(k)} := Im(\omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(k)} \wedge \omega_{\tau(j)\overline{\tau(k)}}) \in \mathbb{C}^{1,2} + \mathbb{C}^{2,1}$$

It follows that ds^2_Λ is a Kähler metric with respect to J if and only if $C_{\tau(i)\tau(j)\tau(k)} = 0$ for any $i < j < k$ from (2.6). In fact, if we use (1.10), (2.3), (2.7), and (3.12) for any $i < j < k$ we have

$$\lambda_{\tau(i)\tau(j)} - \lambda_{\tau(i)\tau(k)} + \lambda_{\tau(j)\tau(k)} \tag{3.14}$$

Put $\lambda_{\tau(j)\tau(j+1)} = a_j$. If ds^2_Λ is Kähler, from (3.13), we have:

$$\begin{aligned}
&\lambda_{\tau(i)\tau(k)} = \lambda_{\tau(i)\tau(i+1)} + \lambda_{\tau(i+1)\tau(k)} = \\
&= \lambda_{\tau(i)\tau(i+1)} + \lambda_{\tau(i+1)\tau(i+1)} + \lambda_{\tau(i+2)\tau(k)} = \ldots = \sum_{j=1}^{k-1} \lambda_{\tau(j)\tau(j+1)} \\
&= a_i + a_{i+1} + \ldots + a_{k-1}
\end{aligned} \tag{3.15}$$

Conversely, (3.15) is the solution of $C_{\tau(i)\tau(j)\tau(k)} = 0$ for any $i < j < k$ because

$$\lambda_{\tau(i)\tau(j)} + \lambda_{\tau(j)\tau(k)} = a_i + \ldots + a_{j-1} + a_j + \ldots + a_{k-1} = \lambda_{\tau(i)\tau(k)} \tag{3.16}$$

Thus we get the following:

Theorem 3.7. Let J be an invariant complex structure with associated tournament $\tau(i) \to \tau(j) \Leftrightarrow i < j$. Then:

(i) If ds^2_Λ is a Kähler metric with respect to J, then $\Lambda = (\lambda_{ij})$ satisfies (3.15);

(ii) Every ds^2_Λ for which $\Lambda = (\lambda_{ij})$ satisfies (3.15) is a Kähler metric with respect to J.

§4 Parabolic invariant almost complex structures

Now define parabolic almost complex structures corresponding to integrable ones, (See §0).

Definition 4.1. An invariant almost complex J on $F(N)$ is called parabolic if there exists a permutation τ such that $\mathcal{J}(J)$ is given by

$$\tau(i) \to \tau(j) \quad \Leftrightarrow \quad i - j \in 2\mathbb{N} \quad \text{or} \quad j - i \in 2\mathbb{N} - 1 \tag{4.1}$$

Remark. For an equivalent description, see for example [6]. These almost complex structures are examples of a large class of invariant almost complex structures defined on generalized flag manifolds (i.e. homogeneous spaces $G^{\mathbb{C}}/P$ where P is a parabolic subgroups of a complex semi-simple Lie group $G^{\mathbb{C}}$) which have been studied by F.E. Burstall and J.H. Rawnsley (see [5], [7] and [10].).

The main goal of this section is to show that each parabolic almost complex structure on $F(N)$, there exists a family of dimension N of almost Hermitian (1,2)-symplectic metrics and to write out that explicit formulas. More precisely we have the following.

Theorem 4.2. Suppose that J is a parabolic invariant almost complex structure on $F(N)$ with corresponding tournament given by (4.1). Then an almost left-invariant metric ds^2_Λ is (1,2)-symplectic related to J is and only if Λ satisfies that

$$\lambda_{\tau(i)\tau(k)} = \begin{cases} a_i + a_{i+2} + ... + a_{k-2} \quad \text{if} \quad k - i \in 2\mathbb{N} \\ a_k + a_{k+2} + ... + a_{N-1} + a_1 + a_3 + ... + a_{i-2} \text{ if } i, N \in 2\mathbb{N} - 1, k \in 2\mathbb{N}; \\ a_k + a_{k+2} + ... + a_N + a_2 + a_4 + ... + a_{i-2} \text{ if } N, k \in 2\mathbb{N} - 1, i \in 2\mathbb{N} \\ a_k + a_{k+2} + ... + a_{N-2} + a_{N-1} + a_1 + a_3 + ... + a_{i-2} \text{ if } N, \\ \qquad k \in 2\mathbb{N}, i \in 2\mathbb{N} - 1; \\ a_k + a_{k+2} + ... + a_{N-3} + a_N + a_2 + a_4 + ... + a_{i-2} \text{ if } i, N \in 2\mathbb{N}, k \in 2\mathbb{N} - 1 \end{cases} \tag{4.2}$$

where $a_0 = a_N, a_{-1} = a_{N-1}$.

Proof. For any $i < j < k$

$$\Psi_{\tau(i)\tau(j)\tau(k)} \in \mathbb{C}^{1,2} + \mathbb{C}^{2,1} \tag{4.3}$$

if and only if one of the following is true:

$1^{\underline{o}}$ $j-i \in 2\mathbb{N}, k-j \in 2\mathbb{N}-1$;

$2^{\underline{o}}$ $k-j \in 2\mathbb{N}, j-i \in 2\mathbb{N}-1$;

$3^{\underline{o}}$ $j-i, k-j \in 2\mathbb{N}$.

for any $i<j<k$. The corresponding $C_{\tau(i)\tau(j)\tau(k)}$ vanishes if and only if:

(I) $\lambda_{\tau(j)\tau(k)} = \lambda_{\tau(i)\tau(j)} + \lambda_{\tau(i)\tau(k)}$ (4.4)

(II) $\lambda_{\tau(i)\tau(j)} = \lambda_{\tau(i)\tau(k} + \lambda_{\tau(j)\tau(k)}$ (4.5)

(III) $\lambda_{\tau(i)\tau(k)} = \lambda_{\tau(i)\tau(j)} + \lambda_{\tau(j)\tau(k)}$ (4.6)

If follows that respectively ds^2_Λ is an almost Hermitian (1,2)-symplectic with respect to J if and only if (4.4)-(4.6) hold, where $i<j<k$ satisfy $1^{\underline{o}}$ - $3^{\underline{o}}$ respectively. Put

$$a_j = \begin{cases} \lambda_{\tau(j)\tau(j+2)} & \text{if} \quad j=1,2,...,N-2 \\ \lambda_{\tau(1)\tau(N-1)} & \text{if} \quad j=N-1 \in 2\mathbb{N} \\ \lambda_{\tau(1)\tau(N)} & \text{if} \quad j=N-1 \in 2\mathbb{N}-1 \\ \lambda_{\tau(2)\tau(N-1)} & \text{if} \quad j=N \in 2\mathbb{N} \\ \lambda_{\tau(2)\tau(N)} & \text{if} \quad j=N \in 2\mathbb{N}-1 \end{cases} \tag{4.7}$$

Assume that ds^2_Λ is a (1,2)-symplectic metric with respect to J. Then:

a) If $k-i \in 2\mathbb{N}$, we have from (4.6)

$$\begin{aligned} \lambda_{\tau(i)\tau(k)} &= \lambda_{\tau(i)\tau(i+2)} + \lambda_{\rho(i+2)\rho(k)} \\ &= \lambda_{\tau(i)\tau(i+2)} + \lambda_{\tau(i+2)\tau(i+4)} + \lambda_{\tau(i+4)\tau(k)} \\ &= \lambda_{\tau(i)\tau(i+2)} + \lambda_{\tau(i+2)\tau(i+4)} + ... + \lambda_{\tau(k-2)\tau(k)} \\ &= a_i + a_{i+2} + ... + a_{k-2}. \end{aligned} \tag{4.8}$$

b) If $i, N \in 2\mathbb{N}-1, k \in 2\mathbb{N}$, then

$$\begin{aligned} \lambda_{\tau(i)\tau(k)} &\overset{(4.4)}{=\!=} \lambda_{\tau(1)\tau(i)} + \lambda_{\tau(1)\tau(k)} \\ &\overset{(4.5)}{=\!=} \lambda_{\tau(1)\tau(i)} + \lambda_{\tau(1)\tau(N-1)} + \lambda_{\tau(k)\tau(N-1)} \\ &\underset{(4.8)}{\overset{(4.7)}{=\!=}} a_1 + a_3 + ... + a_{i-2} + a_{N-1} + a_k + a_{k+2} + ... + a_{N-3} \end{aligned} \tag{4.9}$$

c) If $N, k \in 2\mathbb{N} - 1$ and $i \in 2\mathbb{N}$

$$\begin{aligned}
\lambda_{\tau(i)\tau(k)} &\overset{(5.6)}{=\!=} \lambda_{\tau(i)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\
&\overset{(5.5)}{=\!=} \lambda_{\tau(2)\tau(i)} + \lambda_{\tau(2)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\
&\underset{(4.9)}{\overset{(4.8)}{=\!=}} a_2 + a_4 + \ldots + a_{i-2} + a_N + a_k + a_{k+2} + \ldots + a_{N-2}
\end{aligned} \tag{4.10}$$

d) When $N, k \in 2\mathbb{N}$ and $i \in 2\mathbb{N} - 1$ we have:

$$\begin{aligned}
\lambda_{\tau(i)\tau(k)} &\overset{(4.6)}{=\!=} \lambda_{\tau(i)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\
&\overset{(4.5)}{=\!=} \lambda_{\tau(1)\tau(i)} + \lambda_{\tau(1)\tau(N)} + \lambda_{\tau(k)\tau(N)} \\
&\underset{(4.9)}{\overset{(4.8)}{=\!=}} a_1 + a_3 + \ldots + a_{i-2} + a_{N-1} + a_k + a_{k+2} + \ldots + a_{N-2}
\end{aligned} \tag{4.11}$$

e) If $i, N \in 2\mathbb{N}$ and $k \in 2\mathbb{N} - 1$

$$\begin{aligned}
\lambda_{\tau(i)\tau(k)} &\overset{(4.6)}{=\!=} \lambda_{\tau(2)\tau(i)} + \lambda_{\tau(2)\tau(k)} \\
&\overset{(4.5)}{=\!=} \lambda_{\tau(2)\tau(i)} + \lambda_{\tau(2)\tau(N-1)} + \lambda_{\tau(k)\tau(N-1)} \\
&\underset{(4.9)}{\overset{(4.8)}{=\!=}} a_2 + a_4 + \ldots + a_{i-2} + a_N + a_k + a_{k+2} + \ldots + a_{N-3}
\end{aligned} \tag{4.12}$$

So (4.2) holds. Conversely, assume $\Lambda = (\lambda_{ij})$ satisfies (4.2). Then

$1^{\underline{o}}$) If $j - i \in 2\mathbb{N}, k - j \in 2\mathbb{N} - 1$

$$\begin{aligned}
& \lambda_{\tau(i)\tau(j)} + \lambda_{\tau(i)\tau(k)} = a_i + a_{i+2} + \ldots + a_{j-2} \\
+ & \begin{cases}
a_k + a_{k+2} + \ldots + a_{N-1} + a_1 + a_3 + \ldots + a_{i-2} \ (i, N \in 2\mathbb{N} - 1) \\
a_k + a_{k+2} + \ldots + a_{N-2} + a_{N-1} + a_1 + a_3 + \ldots + a_{i-2} \ (i \in 2\mathbb{N} - 1, N \in 2\mathbb{N}) \\
a_k + a_{k+2} + \ldots + a_N + a_2 + a_4 + \ldots + a_{i-2} \ (i \in 2\mathbb{N}, N \in 2\mathbb{N} - 1) \\
a_k + a_{k+2} + \ldots + a_{N-3} + a_N + a_2 + a_4 + \ldots + a_{i-2} \ (i, N \in 2\mathbb{N})
\end{cases} \\
= & \begin{cases}
a_k + a_{k+2} + \ldots + a_{N-1} + a_1 + a_3 + \ldots + a_{i-2} \\
a_k + a_{k+2} + \ldots + a_{N-2} + a_{N-1} + a_1 + a_3 + \ldots + a_{i-2} \\
a_k + a_{k+2} + \ldots + a_N + a_2 + a_4 + \ldots + a_{i-2} \\
a_k + a_{k+2} + \ldots + a_{N-3} + a_N + a_2 + a_4 + \ldots + a_{i-2}
\end{cases} \\
= & \lambda_{\tau(j)\tau(k)}
\end{aligned}$$

$2^{\underline{o}}$) If $j-i \in 2\mathbb{N}-1, k-j \in 2\mathbb{N}$, then

$$\begin{aligned}
\lambda_{\tau(i)\tau(k)} + \lambda_{\tau(j)\tau(k)} &= \begin{cases} a_k + a_{k+2} + ... + a_{N-2} + a_{N-1} + a_1 + a_3 + ... + a_{i-2} \\ \qquad\qquad (i, N \in 2\mathbb{N}-1) \\ a_k + a_{k+2} + ... + a_{N-2} + a_{N-1} + a_1 + a_3 + ... + a_{i-2} \\ \qquad\qquad (i \in 2\mathbb{N}-1, N \in 2\mathbb{N}) \\ a_k + a_{k+2} + ... + a_N + a_2 + a_4 + ... + a_{i-2} \ (i \in 2\mathbb{N}, N \in 2\mathbb{N}-1) \\ a_k + a_{k+2} + ... + a_{N-3} + a_N + a_2 + a_4 + ... + a_{i-2} \ (i, N \in 2\mathbb{N}) \end{cases} \\
&+ \ a_j + a_{j+2} + ... + a_{k+2} \\
&= \begin{cases} a_j + a_{j+2} + ... + a_{N-1} + a_1 + a_3 + ... + a_{i-2} \\ a_j + a_{j+2} + ... + a_{N-2} + a_{N-1} + a_1 + a_3 + ... + a_{i-2} \\ a_j + a_{j+2} + ... + a_N + a_2 + a_4 + ... + a_{i-2} \\ a_j + a_{j+2} + ... + a_{N-3} + a_N + a_2 + a_4 + ... + a_{i-2} \end{cases} \\
&= \lambda_{\tau(i)\tau(j)}
\end{aligned}$$

$3^{\underline{o}}$) If $j-i, k-j \in 2\mathbb{N}$

$$\begin{aligned}
\lambda_{\tau(i)\tau(j)} + \lambda_{\tau(j)\tau(k)} &= (a_i + a_{i+2} + ... + a_{j-2}) + (a_j + a_{j+2} + ... + a_{k-2}) \\
&= a_i + a_{i+2} + ... + a_{k-2} = \lambda_{\tau(i)\tau(k)}
\end{aligned}$$

Hence ds^2_Λ is an almost Hermitian (1,2)-symplectic metric with respect to J.

§5 Almost complex structures without left-invariant metrics

In section §3 and §4 we saw that each integrable (respectively parabolic) almost complex structure admits a family of dimension $N-1$ (respectively N) of Kähler (respectively (1,2)-symplectic) metrics. Since the Kähler condiction implies the (1,2)-symplectic one, a natural question is the following one: "Is there a (1,2)-simplectic metric for any $U(N)$-invariant almost complex structure on $F(N)$?". The answer is no!. In fact, we have:

Theorem 5.1. If J is a $U(N)$-invariant almost complex structure whose associated digraph contains configurations of the following type:

(i) (ii)

Then J does not admit any left-invariant (1,2)-symplectic metric.

Proof: If the tournament $\mathcal{J}(J)$ contains (i) then we can mark this 4-subtournament by

for some permutation $\tau \in \sum_n$. Suppose that ds^2_Λ is (1,2)-simplectic related to J. Because

$$\omega_{\tau(1)\overline{\tau(2)}}, \omega_{\overline{\tau(1)}\tau(3)}, \omega_{\overline{\tau(1)}\tau(4)}, \omega_{\overline{\tau(2)}\tau(3)}, \omega_{\tau(2)\overline{\tau(4)}}, \omega_{\tau(3)\overline{\tau(4)}}$$

are (1,0)-forms, so

$$\left\{ \begin{array}{l} C_{\tau(1)\tau(2)\tau(3)} = 0 \\ C_{\tau(1)\tau(3)\tau(4)} = 0 \\ C_{\tau(2)\tau(3)\tau(4)} = 0 \end{array} \right. \qquad (5.1)$$

From (1.10), (5.1) is equivalent to

$$\begin{cases} \lambda_{\tau(1)\tau(2)} + \lambda_{\tau(1)\tau(3)} - \lambda_{\tau(2)\tau(3)} = 0 \\ -\lambda_{\tau(1)\tau(3)} + \lambda_{\tau(1)\tau(4)} + \lambda_{\tau(3)\tau(4)} = 0 \\ -\lambda_{\tau(2)\tau(3)} - \lambda_{\tau(2)\tau(4)} + \lambda_{\tau(3)\tau(4)} = 0 \end{cases} \tag{5.2}$$

Hence

$$\lambda_{\tau(2)\tau(3)} = \lambda_{\tau(1)\tau(2)}+\lambda_{\tau(1)\tau(3)} = \lambda_{\tau(1)\tau(2)}+\lambda_{\tau(1)\tau(4)}+\lambda_{\tau(3)\tau(4)} = \lambda_{\tau(1)\tau(2)}+\lambda_{\tau(1)\tau(4)}+\lambda_{\tau(2)\tau(3)}+\lambda_{\tau(2)\tau(4)}$$

which implies that

$$\lambda_{\tau(1)\tau(2)} + \lambda_{\tau(1)\tau(4)} + \lambda_{\tau(2)\tau(3)} = 0.$$

So by using (1.8) we derive a contradiction. In a similar manner we can prove the theorem for the type (ii). Q.E.D.

§6 Invariant almost complex structures on $F(3)$ and $F(4)$

6.1. The full flag manifold $F(3)$

$F(3)$ carries 8 invariant almost complex structures [11]. From (2.6) we have

$$\frac{1}{4}d\Omega = C_{\tau(1)\tau(2)\tau(3)}\Psi_{\tau(1)\tau(2)\tau(3)} \tag{6.1}$$

Hence either

$$d\Omega \in \mathbb{C}^{0,3} + \mathbb{C}^{3,0} \tag{6.2}$$

or

$$d\Omega \in \mathbb{C}^{1,2} + \mathbb{C}^{2,1} \tag{6.3}$$

The equation (6.2) (resp. (6.3)) means that ds^2_Λ is non-integrable (1,2)-symplectic (resp. integrable) from theorems 3.4 and 3.5. It follows that:

Theorem 6.1. Among all almost complex structures of $F(3)$ 6 are integrable and 2 are parabolic. Each left invariant metric, in particular, the normal metric, is (1,2)-symplectic but not sympletic for parabolic structures.

The almost complex structures up to a sign and their left invariant metrics are listed below:

Almost complex structure ε	Kähler metric Λ	(1,2)-symplectic non-symplectic metric Λ
$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1 & \lambda_1+\lambda_2 \\ \lambda_1 & 0 & \lambda_2 \\ \lambda_1+\lambda_2 & \lambda_2 & 0 \end{pmatrix}$	no
$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_1 & \lambda_1 \\ \lambda_1+\lambda_2 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & 0 \end{pmatrix}$	no
$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1 & \lambda_2 \\ \lambda_1 & 0 & \lambda_1+\lambda_2 \\ \lambda_2 & \lambda_1+\lambda_2 & 0 \end{pmatrix}$	no
$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	no	$\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$

6.2 The full flag manifold $F(4)$

If we use Figure 1 below (which is taken from [18] we see all the isomorphism classes of a 4-tournament (see [18, pg.87] for more details).

(i) (ii) (iii) (iv)

Figure 1

Clearly, (i) is canonical, (ii) and (iii) are listed in theorem 5.1 and (iv) is parabolic. Together with theorem 3.4, 3.5, 4.2 and 5.1 we have

Theorem 6.2

An almost complex structure on $F(4)$ is integrable (resp. parabolic if and only if it admits a symplectic (resp. non-symplectic (1,2)-symplectic) left invariant metric.

The integrable (resp parabolic) almost complex structures up to sign and the corresponding left-invariant Kähler (resp. non-symplectic (1,2)-symplectic) metrics are listed below:

Case 1: Integrable structures

ε	Λ
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1 & \lambda_1+\lambda_2 & \lambda_1+\lambda_2+\lambda_3 \\ \lambda_1 & 0 & \lambda_2 & \lambda_2+\lambda_3 \\ \lambda_1+\lambda_2 & \lambda_2 & 0 & \lambda_3 \\ \lambda_1+\lambda_2+\lambda_3 & \lambda_2+\lambda_3 & \lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1 & \lambda_1+\lambda_2+\lambda_3 & \lambda_1+\lambda_2 \\ \lambda_1 & 0 & \lambda_2+\lambda_3 & \lambda_2 \\ \lambda_1+\lambda_2+\lambda_3 & \lambda_2+\lambda_3 & 0 & \lambda_3 \\ \lambda_1+\lambda_2 & \lambda_2 & \lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_2 & \lambda_1+\lambda_2+\lambda_3 & \lambda_1 \\ \lambda_1+\lambda_2 & 0 & \lambda_3 & \lambda_2 \\ \lambda_1+\lambda_2+\lambda_3 & \lambda_3 & 0 & \lambda_2+\lambda_3 \\ \lambda_1 & \lambda_2 & \lambda_2+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_2 & \lambda_1 & \lambda_1+\lambda_2+\lambda_3 \\ \lambda_1+\lambda_2 & 0 & \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_2 & 0 & \lambda_2+\lambda_3 \\ \lambda_1+\lambda_2+\lambda_3 & \lambda_3 & \lambda_2+\lambda_3 & 0 \end{pmatrix}$

$\varepsilon = (\varepsilon_{ij})$	$\Lambda = (\lambda_i)$
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_2+\lambda_3 & \lambda_1 & \lambda_1+\lambda_2 \\ \lambda_1+\lambda_2+\lambda_3 & 0 & \lambda_2+\lambda_3 & \lambda_3 \\ \lambda_1 & \lambda_2+\lambda_3 & 0 & \lambda_2 \\ \lambda_1+\lambda_2 & \lambda_3 & \lambda_2 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_2+\lambda_3 & \lambda_1+\lambda_2 & \lambda_2 \\ \lambda_1+\lambda_2+\lambda_3 & 0 & \lambda_3 & \lambda_2+\lambda_3 \\ \lambda_1+\lambda_2 & \lambda_3 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2+\lambda_3 & \lambda_2 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_2 & \lambda_2+\lambda_3 & \lambda_1 \\ \lambda_2 & 0 & \lambda_3 & \lambda_1+\lambda_2 \\ \lambda_2+\lambda_3 & \lambda_3 & 0 & \lambda_1+\lambda_2+\lambda_3 \\ \lambda_1 & \lambda_1+\lambda_2 & \lambda_1+\lambda_2+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_2 & \lambda_1 & \lambda_1+\lambda_2+\lambda_3 \\ \lambda_1+\lambda_2 & 0 & \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_2 & 0 & \lambda_2+\lambda_3 \\ \lambda_1+\lambda_2+\lambda_3 & \lambda_3 & \lambda_2+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_2+\lambda_3 & \lambda_2 & \lambda_1 \\ \lambda_2+\lambda_3 & 0 & \lambda_3 & \lambda_1+\lambda_2+\lambda_3 \\ \lambda_2 & \lambda_3 & 0 & \lambda_1+\lambda_2 \\ \lambda_1 & \lambda_1+\lambda_2+\lambda_3 & \lambda_1+\lambda_2 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_2 & \lambda_1 & \lambda_2+\lambda_3 \\ \lambda_2 & 0 & \lambda_1+\lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_1+\lambda_2 & 0 & \lambda_1+\lambda_2+\lambda_3 \\ \lambda_2+\lambda_3 & \lambda_3 & \lambda_1+\lambda_2+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_3 & \lambda_1+\lambda_2 & \lambda_2 \\ \lambda_3 & 0 & \lambda_1+\lambda_2+\lambda_3 & \lambda_2+\lambda_3 \\ \lambda_1+\lambda_2 & \lambda_1+\lambda_2+\lambda_3 & 0 & \lambda_1 \\ \lambda_2 & \lambda_2+\lambda_3 & \lambda_1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_3 & \lambda_2 & \lambda_1+\lambda_2 \\ \lambda_3 & 0 & \lambda_2+\lambda_3 & \lambda_1+\lambda_2+\lambda_3 \\ \lambda_2 & \lambda_2+\lambda_3 & 0 & \lambda_1 \\ \lambda_1+\lambda_2 & \lambda_1+\lambda_2+\lambda_3 & \lambda_1 & 0 \end{pmatrix}$

Case 2: Parabolic almost complex structures

$\varepsilon = (\varepsilon_{ij})$	$\Lambda = (\lambda_i)$
$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_2+\lambda_3 & \lambda_1 & \lambda_3 \\ \lambda_2+\lambda_3 & 0 & \lambda_4 & \lambda_2 \\ \lambda_1 & \lambda_4 & 0 & \lambda_1+\lambda_3 \\ \lambda_3 & \lambda_2 & \lambda_1+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1 & \lambda_1+\lambda_3 & \lambda_4 \\ \lambda_1 & 0 & \lambda_3 & \lambda_2+\lambda_3 \\ \lambda_1+\lambda_3 & \lambda_3 & 0 & \lambda_2 \\ \lambda_4 & \lambda_2+\lambda_3 & \lambda_2 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_3 & \lambda_2+\lambda_3 & \lambda_1 \\ \lambda_3 & 0 & \lambda_2 & \lambda_1+\lambda_3 \\ \lambda_2+\lambda_3 & \lambda_2 & 0 & \lambda_4 \\ \lambda_1 & \lambda_1+\lambda_3 & \lambda_4 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_3 & \lambda_1 & \lambda_4 \\ \lambda_1+\lambda_3 & 0 & \lambda_3 & \lambda_2 \\ \lambda_1 & \lambda_3 & 0 & \lambda_2+\lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_2+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_2+\lambda_3 & \lambda_3 & \lambda_1 \\ \lambda_2+\lambda_3 & 0 & \lambda_2 & \lambda_4 \\ \lambda_3 & \lambda_2 & 0 & \lambda_1+\lambda_3 \\ \lambda_1 & \lambda_4 & \lambda_1+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1 & \lambda_4 & \lambda_1+\lambda_3 \\ \lambda_1 & 0 & \lambda_2+\lambda_3 & \lambda_3 \\ \lambda_4 & \lambda_2+\lambda_3 & 0 & \lambda_2 \\ \lambda_1+\lambda_3 & \lambda_3 & \lambda_2 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_1+\lambda_3 & \lambda_4 & \lambda_1 \\ \lambda_1+\lambda_3 & 0 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_2 & 0 & \lambda_2+\lambda_3 \\ \lambda_1 & \lambda_3 & \lambda_2+\lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_3 & \lambda_1 & \lambda_2+\lambda_3 \\ \lambda_3 & 0 & \lambda_1+\lambda_3 & \lambda_2 \\ \lambda_1 & \lambda_1+\lambda_3 & 0 & \lambda_4 \\ \lambda_2+\lambda_3 & \lambda_2 & \lambda_4 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_4 & \lambda_2+\lambda_3 & \lambda_2 \\ \lambda_4 & 0 & \lambda_1 & \lambda_1+\lambda_3 \\ \lambda_2+\lambda_3 & \lambda_1 & 0 & \lambda_3 \\ \lambda_2 & \lambda_1+\lambda_3 & \lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_4 & \lambda_2 & \lambda_2+\lambda_3 \\ \lambda_4 & 0 & \lambda_1+\lambda_3 & \lambda_1 \\ \lambda_2 & \lambda_1+\lambda_3 & 0 & \lambda_3 \\ \lambda_2+\lambda_3 & \lambda_1 & \lambda_3 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_2 & \lambda_1+\lambda_3 & \lambda_3 \\ \lambda_2 & 0 & \lambda_4 & \lambda_2+\lambda_3 \\ \lambda_1+\lambda_3 & \lambda_4 & 0 & \lambda_1 \\ \lambda_3 & \lambda_2+\lambda_3 & \lambda_1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda_2 & \lambda_3 & \lambda_1+\lambda_3 \\ \lambda_2 & 0 & \lambda_2+\lambda_3 & \lambda_4 \\ \lambda_3 & \lambda_2+\lambda_3 & 0 & \lambda_1 \\ \lambda_1+\lambda_3 & \lambda_4 & \lambda_1 & 0 \end{pmatrix}$

§7 Normal metric

Among all Hermitian metrics ds^2_Λ (see §1 (1.6)) a very natural one is namely:

$$\lambda_{ij} = 1 - \delta_{ij} \tag{7.1}$$

in (1.6), induced from the natural bi-invariant metric on $U(N)$. We call it the normal metric [13], [17], [2]. It is well-known that the normal metric is not Kähler with respect to any complex structure on $F(N)$.

On the other hand, when $N = 3$ the normal metric is (1,2)-symplectic with respect to a parabolic almost complex structure, see §6. In fact, we have

Theorem 7.1. The normal metric on $F(N)$ is not (1,2)-symplectic with respect to any almost complex structure on $F(N)$ if $N > 3$.

Proof: Put τ equal to the identity in (2.6) then

$$\frac{1}{4}d\Omega = \sum_{i<j<k} C_{ijk}\Psi_{ijk} \tag{7.2}$$

Hence we have

$$\begin{aligned} ds^2_\Lambda \text{ is } (1,2)-\text{symplectic } &\Leftrightarrow (d\Omega)^{1,2} = 0 \\ \underset{(2.9)\ (2.10)}{\overset{(7.2)}{\Longleftrightarrow}} \quad & C_{ijk} = 0 \text{ if } \Psi_{ijk} \in \mathbb{C}^{1,2} + \mathbb{C}^{2,1} \end{aligned} \tag{7.3}$$

However from, (7.1), (2.7), (2.3) and (1.10) one gets

$$C_{ijk} = \varepsilon_{ij} - \varepsilon_{ik} + \varepsilon_{jk} \neq 0 \tag{7.4}$$

So (7.3) is equivalent $\Psi_{ijk} \in \mathbb{C}^{0,3} + \mathbb{C}^{3,0}$ for any $i < j < k$. From the proof of theorem 3.4 it follows that the number of 3-cycles in the tournament $\mathcal{J}(J) = \binom{N}{3}$. However, this is impossible because if $N > 3$ from [12, pg 7] and also from [14, pg.16] we know that the number of 3-cycles in $\mathcal{J}(J)$ is less then or equal to $\frac{1}{24}(N^2 - N)$ if N is odd or $\frac{1}{24}(N^3 - 4N)$ if N is even. Q.E.D

§8 Remarks

In §5, we saw that a sufficient condition for an almost complex structure on $F(N)$ not to admit a (1,2)-symplectic metric is: its corresponding digraph contains 4-subtournaments in theorem 5.1.

Combine with Theorem 3.7 and Theorem 4.2 we have

Proposition 8.1. Tournaments arising from integrable or parabolic almost complex structures contains no configurations of type (i) and (ii) in theorem 5.1.

From Fig.1 in §6.2 the converse of Proposition 8.1 is true if $N = 4$. Nevertheless, the following result shows that the converse is false in general.

Proposition 8.2. There is an almost complex structure J in F(5) such that:

(a) J is neither integrable nor parabolic;

(b) $\mathcal{J}(J)$ contains no configuration as in Theorem 5.1;

(c) J has a 5-dimensional family of (1,2)-symplectic metrics.

Proof: Consider the almost complex structure J on $F(5)$ such that $\mathcal{J}(J)$ is defined by

It is easy to see that the score vector (i.e., the number of games that each player won) of $\mathcal{J}(J)$ is (1,1,2,3,3). On the other hand, integrable (resp. parabolic) almost complex structures have score vector (0,1,2,3,4) (resp. (2,2,2,2,2)). Furthemore isomorphic neither tournaments have the same score vector. So J is neither parabolic non parabolic.

There are five 4-subtournaments in $\mathcal{J}(J)$. The number of 3-cycles in them is 0 or 2. However, the diagrams in theorem 5.1 have only one 3-cycle. So we have (b) of Proposition 8.2. According to the definition of J and (7.2) we have

$$\frac{1}{4}[(d\Omega)^{1,2} + (d\Omega)^{2,1}] = C_{124}\Psi_{124} + C_{125}\Psi_{125} +$$
$$+C_{134}\Psi_{134} + C_{135}\Psi_{135} + C_{145}\Psi_{145} + C_{245}\Psi_{245} + C_{345}\Psi_{345}$$

Together with (1.10), (2.3) and (2.7) we see that ds^2_Λ is (1,2)-symplectic if and only if $(\lambda_{ij}) = \Lambda$ satisfy:

$$\begin{cases} \lambda_{24} = \lambda_{12} + \lambda_{14} = \lambda_{25} + \lambda_{45} \\ \lambda_{25} = \lambda_{12} + \lambda_{15} \\ \lambda_{13} = \lambda_{14} + \lambda_{34} = \lambda_{15} + \lambda_{35} \\ \lambda_{14} = \lambda_{15} + \lambda_{45} \\ \lambda_{35} = \lambda_{34} + \lambda_{45} \end{cases}$$

It has the following solution:

$$\begin{pmatrix} 0 & \lambda_1 & \lambda_2+\lambda_4+\lambda_5 & \lambda_2+\lambda_5 & \lambda_2 \\ \lambda_1 & 0 & \lambda_3 & \lambda_1+\lambda_4+\lambda_5 & \lambda_1+\lambda_2 \\ \lambda_2+\lambda_4+\lambda_5 & \lambda_3 & 0 & \lambda_4 & \lambda_4+\lambda_5 \\ \lambda_2+\lambda_5 & \lambda_1+\lambda_4+\lambda_5 & \lambda_4 & 0 & \lambda_5 \\ \lambda_2 & \lambda_1+\lambda_2 & \lambda_4+\lambda_5 & \lambda_5 & 0 \end{pmatrix}$$

We would like to mention that an arbitrary non-necessary - full complex flag manifold $F = F(n_1, ..., n_k; N)$ similar Hermitian structures that $F(k)$. For example, we can consider $F(1,1,2;4)$. The family of left-invariant metrics on it can be described in the following way:

$$\begin{aligned} & ds^2_\Lambda = \lambda_1(\omega_{1\bar{2}}\omega_{\bar{1}2} + \omega_{2\bar{1}}\omega_{\bar{2}1}) \\ & +\lambda_2(\omega_{1\bar{3}}\omega_{\bar{1}3} + \omega_{3\bar{1}}\omega_{\bar{3}1} + \omega_{1\bar{4}}\omega_{\bar{1}4} + \omega_{4\bar{1}}\omega_{\bar{4}1}) \\ & +\lambda_3(\omega_{2\bar{3}}\omega_{\bar{2}3} + \omega_{3\bar{2}}\omega_{\bar{3}2} + \omega_{2\bar{4}}\omega_{\bar{2}4} + \omega_{4\bar{2}}\omega_{\bar{4}2}) \end{aligned}$$

Now we consider an invariant almost complex structure on $F(1,1,2;4)$. We define ε_i $(i = 1,2,3)$ by:

$$\varepsilon_1 = \begin{cases} 1 \text{ if } \omega_{1\bar{2}} \text{ is a } (1,0) - \text{form} \\ \\ -1 \text{ if } \omega_{1\bar{2}} \text{ is a } (0,1) - \text{form} \end{cases}$$

$$\varepsilon_2 = \begin{cases} 1 \text{ if } \omega_{1\bar{3}} \text{ and } \omega_{1\bar{4}} \text{ are } (1,0) - \text{forms} \\ \\ -1 \text{ if } \omega_{1\bar{3}} \text{ and } \omega_{1\bar{4}} \text{ are } (0,1) - \text{forms} \end{cases}$$

$$\varepsilon_3 = \begin{cases} 1 \text{ if } \omega_{2\bar{3}} \text{ and } \omega_{2\bar{4}} \text{ are } (1,0) - \text{forms} \\ \\ -1 \text{ if } \omega_{2\bar{3}} \text{ and } \omega_{2\bar{4}} \text{ are } (0,1) - \text{forms} \end{cases}$$

Then each choice $\varepsilon = \varepsilon_1, \varepsilon_2, \varepsilon_3)$ determines an invariant almost complex structure so the associated Kähler form is given by:

$$\Omega = -2\sqrt{-1}[\mu_1(\omega_{1\overline{2}} \wedge \omega_{\overline{1}2}) + \mu_2(\omega_{1\overline{3}} \wedge \omega_{\overline{1}3} + \omega_{1\overline{4}} \wedge \omega_{\overline{1}4}) + \mu_3(\omega_{2\overline{3}} \wedge \omega_{\overline{2}3} + \omega_{2\overline{4}} \wedge \omega_{\overline{2}4})]$$

where

$$\mu_j = \varepsilon_j \lambda_j$$

Hence it is easy to show that

$$\frac{1}{4} d\Omega = (\mu_1 - \mu_2 + \mu_3) Im[(\omega_{1\overline{3}} \wedge \omega_{3\overline{2}} + \omega_{1\overline{4}} \wedge \omega_{4\overline{2}}) \wedge \omega_{\overline{1}2}] \tag{8.1}$$

Notice that (8.1) is very similar to (6.1) where τ is the identity permutation. Our final remarks is:

1. The canonical complex structure J_1 on $F(n_1, ..., n_k; N)$ (see §0) is not horizontal if $k \geq 3$
2. The parabolic almost complex structure J_2 is not horizontal if $k \geq 4$.

Proof. When $k \geq 3$, with respect to J_1 we have

$$\overline{E}_1 E_3 \subset m^{1,0}, \quad \overline{E}_3 E_2 \subset m^{0,1}$$

where $(E_1, ..., E_k)$ denote the legs of the flag $0 = V_0 \subset V_1 \subset ... \subset V_k = \mathbb{C}^N$, and $m^{1,0}$ (resp. $m^{0,1}$) the space of (1.0) (resp. (0.1)) tangent vectors at the identity coset. From elementary representation theory we have

$$[\overline{E}_1 E_3, \overline{E}_3 E_2] = \overline{E}_1 E_2 \not\subset u(n_1) + u(n_2) + u(n_3)$$

Hence J_1 is not horizontal (see [2]). When $k \geq 4$, with respect to J_2, we have

$$[\overline{E}_1 E_4, \overline{E}_4 E_3] = \overline{E}_1 E_3 \not\subset u(n_1) + u(n_2) + u(n_3) + u(n_4).$$

It follows that J_2 is not horizontal if $k \geq 4$.

§9 Harmonic maps into flag manifolds

In this section, we construct new examples of harmonic maps into flag manifolds by using the following.

Theorem 9.1. [15] Let $\phi : (M, g) \to (N, h)$ be a $\pm$-holomorphic map between almost Hermitian manifolds where M is cosymplectic and N is (1,2)-symplectic. Then ϕ is harmonic.
Recall that M is cosymplectic if its Kähler form is co-closed, and a (1,2)-symplectic metric is cosymplectic.

The combination of theorem 9.1 with those in §3 and §4 will enable us to produce new harmonic maps into flag manifolds.

Theorem 9.2. Let $\phi : S^2 \to G_r(\mathbb{C}^N)$ be a harmonic map. Then there exists a flag manifold $F = F(n_1, ..., n_k; N)$ and a harmonic map $\Psi : S^2 \to (F, ds^2_\Lambda)$ such that either ϕ or $\phi^\perp$ is given by $\pi_e \circ \Psi$ where $\Lambda = (\lambda_{ij})$ is given in (4.2) (take τ=identity) and $k \leq 2r + 1$ is odd.

Proof: From Proposition (2.5), (2.6), Theorem (2.9) and Corollary (3.3) in [6], there exists $k(\in 2\mathbb{N} - 1) \leq 2r + 1, F = F(n_1, ..., n_k; N)$ and a holomorphic map $\Psi : S^2 \to (F, J_2)$ such that either ϕ or $\phi^\perp$ is given by $\pi_e \circ \Psi$, where J_2 the canonical parabolic almost complex structure and $\pi_e : F \to G_r(\mathbb{C}^N)$ is a homogeneous Riemannian fibration with respect to the identity permutation e. Now our conclusion can be obtained as an immediate consequence of Theorem 4.2 and 9.1.

Remark. It is clear that we can extend Theorem 9.2 to anynilconformal harmonic map of order k from a connected Riemann surface (see [6] for details).

Now we are in the position of constructing new harmonic maps between flag manifolds.

Define a homogeneous fibration $\pi : F(n_1, ..., n_k, N) \rightarrow F(n_1 + n_k, n_2, ..., n_{k-1}, N)$ by:

$$\pi(E_1, ..., E_k) = (E_1 \oplus E_k, E_2, ..., E_{k-1})$$

Then π is harmonic with respect to a (1,2)-symplectic metric of J_2.

Proof: Combine and Proposition 4.2. in [6] Theorem 9.1 with one gets that
Proposition 9.3. Let $k \in 2\mathbb{N}$. Then π is harmonic with respect to all (1,2)-symplectic metrics of J_2 given in §4.

Remark. Using Kähler metrics on $F(n_1, ..., n_k; N)$ we can construct new harmonic maps into certain flag manifolds. As an example of this fact, we consider a totally geodesic holomorphic immersion from a Kähler manifold M^m to $\mathbb{C}P^N$, then its Grauss map from M^m to $F(m, N - m, 1; N + 1)$ is harmonic with respect to a 2-dimensional family of canonical Kähler metrics (See, proposition in [13]).

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