On Invariant Subspaces of Linear Operators

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We prove that a linear operator $T : V \to V, V$ a finite dimensional vector space over $I\!R$, has an invariant subspace of dimension 1 or 2, by using complex equations. The spectral theorem for self-adjoint operators follows in the easiest way known to us. The result is useful in the study of orthogonal operators as well.

1. Introduction.

Let $T : V \to V$ be a linear operator, V a finite dimensional vector space over $I\!R$. We will prove the existence of a T-invariant subspace W of V of dimension one or two. Now, given an inner product $\langle , \rangle$ over V, then $W^\perp$ (the orthogonal complement) is T-invariant if T is self-adjoint or orthogonal with respect to $\langle , \rangle$ (the proofs in the literature are basically the same ([BW], [Li])). As self-adjoint or orthogonal operators are easily described in dimensions one or two ([BW], [Li]) the study of T reduces to the study of $T|_{W^\perp} : W^\perp \to W^\perp$. As $\dim W^\perp < \dim V$, iterating this procedure produces the spectral theorem for self-adjoint operators and a characterization of orthogonal operators (finite dimension).

2. Invariant Subspaces.

2.1 Lemma. Let $T : V \to V$ be a linear operator, V a vector space with n=dim $V < \infty$. There is a T-invariant subspace $W \subset V$ with dim $W = 1$ or 2.

Proof. Let $\alpha = (v_1, ..., v_n)$ be an ordered basis of V. Let A be the matrix of T with respect to α. Let $\lambda = \mu + i\nu \in \mathbb{C}$ be a root of the equation

$$\det(A - \lambda I) = 0,$$

* partially supported by FAPESP/BRAZIL grant 1996/6167-7

where I is the $n \times n$ identity matrix and det ( ) means determinant. The existence of $\lambda = \mu + i\nu$ is guaranteed by the fundamental theorem of Algebra. Now, solve the equation

$$(A - (\mu + i\nu)I)Z = 0,$$

Z a complex column vector with n coordinates, by performing elementary row operations ([AR], [BCRW]). This process will produce a matriz E in row-echelon form such that the above equation is equivalent to

$$EZ = 0.$$

As det $(A - (\mu + i\nu)I) = 0$, det $E = 0$ as well. Therefore, $E \neq I$ and so there is a solution $Z \neq 0$. Thus, there are real column vectors not both zero

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

such that $Z = X + iY$. It follows that

$$A(X + iY) = (\mu + i\nu)(X + iY).$$

Taking real and imaginary parts gives

$$AX, AY \in \text{Span}[X, Y],$$

where Span $[X, Y]$ is the subspace spanned by X, Y in the space of $n \times 1$ matrices. Next, let $u, v \in V$ be vectors such that its coordinates with respect to α are given by X, Y respectively. Set $W =$ Span $[u, v]$. Then W is T-invariant. Clearly, dim $W = 1$ or 2. ■

3. Concluding Remark.

In [Li], the above lemma is proved by first obtaining a polynomial $p(x)$ over $\mathbb{R}$ such that $p(T) = 0$, then factoring $p(x)$ over $\mathbb{R}$,then concluding that a linear or quadratic factor of $p(T)$ must be non-invertible. Now, use the kernel and image theorem.

We believe that our proof is less abstract than the proof outlined above.

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References

[AR] H. Anton & C. Rorres: Elementary Linear Algebra, 7^{th} ed., Wiley, 1994.

[BCRW] J. Boldrini, S. Costa, V. Ribeiro & H. Wetzler: Álgebra Linear, 3^{rd} ed, Harper & Row do Brasil, 1984.

[BW] T. Banchoff & J. Wermer: Linear Algebra Through Geometry, 2nd ed., Springer-Verlag, 1992.

[Li] E. Lima Álgebra Linear, 2nd ed., Coleção Matemática Universitária, IMPA, 1996.