THE ISOPERIMETRIC PROBLEM IN SPHERICAL CYLINDERS

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Abstract. The isoperimetric problem in $\mathbb{R}\times\mathbb{S}^n(1)$ is studied. Minimizers are shown to be invariant under the group $O(n)$ acting standardly on $\mathbb{S}^n$, via a symmetrization argument, and are of the same types of those of the simple case of the 2–dimensional cylinder $\mathbb{R}\times\mathbb{S}^1(1)$, i.e., balls (not round) and sections of the form $[a,b]\times\mathbb{S}^n$. It is shown that the minimizers may be of both types and, for $n=2$, that the transition occurs exactly once.

1. Introduction and statement of results

Let (M^n,g) be a complete Riemannian manifold. The isoperimetric problem for (M,g) is to classify the (compact) $\Omega\subset M^n$ of a given n–volume with boundary of minimal $(n-1)$–volume. The volume functions are given by the Hausdorff measures of the correct dimension, which will be denoted by $\mathcal{H}^k$ or vol_k. An $\mathcal{H}^n$–measurable (Lebesgue–measurable) set will be called simply measurable. The isoperimetric profile function $\mathcal{P}:[0,\mathrm{vol}_n(M))\to\mathbb{R}^+$ is

$$\mathcal{P}(v)=\inf\{\mathcal{H}^{n-1}(\partial\Omega):\ \Omega\subset M \text{ is compact, smooth, with } \mathcal{H}^n(\Omega)=v\}.$$

If $\mathrm{vol}_n(M)<\infty$, extend $\mathcal{P}$ by $\mathcal{P}(\mathrm{vol}_n(M))=0$. Ω may fail to be connected in this definition. Smooth and compact mean that it is the closure of a relatively compact open subset with smooth boundary $\partial\Omega$. A solution to the isoperimetric problem in (M,g) with volume v is a compact subset Ω, called either an isoperimetric region or domain, with $\mathcal{H}^n(\Omega)=v$ and $\mathcal{H}^{n-1}$–measurable boundary $\partial\Omega$ such that $\mathcal{H}^{n-1}(\partial\Omega)=\mathcal{P}(v)$. To avoid inessential subsets one usually requires a solution to be the closure of an open set.

Results on the isoperimetric profile and on general isoperimetric inequalities may be found in [Oss78, BP86, Gal88, Hsi92]. Very few cases of complete classification of isoperimetric domains are known, beyond the case of simply–connected space forms (see [BZ88]). Examples of recent work which include classification are [HH89], [RR92], [RR96], [BC96], [Pan98], [HHM], [PR], [Rit].

In this work we study the isoperimetric problem in the Riemannian product $\mathbb{R}\times\mathbb{S}^n$ and show that they reflect quite closely the elementary 2–dimensional case of the right cylinder $\mathbb{R}\times\mathbb{S}^1$. We describe next the results of the paper. Fix the radius of $\mathbb{S}^n$ to be 1.

We observe first that solutions exist for $\mathbb{R}\times\mathbb{S}^n$, by a result of F. Morgan [Mor94] providing existence of (compact) minimizers if M/G is compact, where G is the isometry group of (M,g). Also, the work of Gonzalez, Massari and Tamanini [GMT83] implies that the boundary of a minimizing domain is composed by a regular part, a hypersurface of constant mean curvature (CMC, for short), open in the boundary, and its complement, the singular part, a closed subset of Hausdorff codimension at least seven in the boundary.

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Let $y_0 \in \mathbb{S}^n$ be fixed and let $O(n)$ act on $\mathbb{R} \times \mathbb{S}^n$ by fixing the line $\mathbb{R} \times \{y_0\}$, with the standard action on the second factor. In Section 2 we show

Theorem 1. *An isoperimetric region Ω in $\mathbb{R} \times \mathbb{S}^n$ is smooth, connected and its intersection with the totally geodesic spheres $\{x\} \times \mathbb{S}^n$ may be empty, the whole totally geodesic sphere, or geodesic balls of $\{x\} \times \mathbb{S}^n$ (locally centered at a line of the form $\mathbb{R} \times \{y\} \subset \mathbb{R} \times \mathbb{S}^n$). Moreover, each connectd component of $\partial\Omega$ is isometric to one which is $O(n)$–invariant.*

The proof uses Schwartz–type symmetrization (Proposition 2.1) and is closely related to the one in Almgren's paper [Alm87], where he proves that boundary area minimizing drops inside a ball in $\mathbb{R}^n$ are pieces of round balls or half–balls. The symmetrization presented may be applied to other situations (like those in [HH89],[PR]).

In Section 3 we study the $O(n)$–invariant CMC hypersurfaces of $\mathbb{R} \times \mathbb{S}^n$ (Proposition 3.5) and prove the classification result for the isoperimetric regions in $\mathbb{R} \times \mathbb{S}^n$, described next. For the simple case $\mathbb{R} \times \mathbb{S}^1$, the isoperimetric domains are either (round) discs or sections of the cylinder, with a transition at the value of area 4π, for which value both regions are solutions. A similar situation occurs for $\mathbb{R} \times \mathbb{S}^n$: there are two competing families of candidates for solutions, a one–parameter family of regions Ω_h of ball type (not round), the parameter $h \in (0, \infty)$ being the mean curvature of $\partial\Omega_h$, and the cylindrical sections, and both types have regular boundaries. For the definition of Ω_h, see Proposition 3.5.(ii). The result is:

Theorem 2. *Let Ω be an isoperimetric region in $\mathbb{R} \times \mathbb{S}^n$. Then Ω is $O(n)$–invariant and, moreover, it is either a cylindrical section, i.e., $\Omega = [a, b] \times \mathbb{S}^n$, or congruent to a ball type region Ω_h.*

In Section 4 we show that the situation for $\mathbb{R} \times \mathbb{S}^2$ reproduces exactly the one for $\mathbb{R} \times \mathbb{S}^1$, using explicit integration.

Theorem 3. *Let Ω be a solution to the isoperimetric problem in $\mathbb{R} \times \mathbb{S}^2$ and $v = \mathcal{H}^3(\Omega)$. Then there exists $v_0 > 0$ such that:*

(i) *If $v < v_0$, there exists exactly one value of mean curvature $h > 0$ such that Ω is congruent to Ω_h;*
(ii) *If $v > v_0$, then $\Omega = [a, b] \times \mathbb{S}^2$, $b - a = v/4\pi$;*
(iii) *If $v = v_0$, one the above two possibilities occur.*

In the course of the proof we provide explict formulae describing the isoperimetric profile for $\mathbb{R} \times \mathbb{S}^2$. The proof of a similar result for the higher dimensional cases is much more involved (see comments at the end of the paper). Anyway, we also include two results that are valid for all $n \geq 3$, in the direction of proving Theorem 3 for general n. Theorem 4 in Section 4 shows that if the $(n+1)$–volume is sufficiently large, then the region is a section $[a, b] \times \mathbb{S}^n$, and, if the $(n+1)$–volume is small enough, Theorem 5 in Section 5 gives that the solution is unique and is the ball bounded by Ω_h given in Theorem 2.

The classification results in Theorems 2 and 3 are from the author's Ph.D. thesis, written under the supervision of Prof. Wu–Yi Hsiang. The original proofs used Hsiang's symmetry result [Hsi91] and symmetrization at the orbit level. We have chosen to present here a more direct approach, by means of symmetrization. The author thanks B. Kawohl, H. Lopes, M. Lopes, F. Mercuri, F. Morgan and M. Ritoré for comments and/or suggestions.

2. Proof of Theorem 1

Our approach to the symmetrization of a set in $\mathbb{R} \times \mathbb{S}^n$ follows closely the one in Almgren's paper on spherical symmetrization [Alm87]. We refer to that paper for the definitions and notations used below which are not standard in geometric measure theory.

We say that a set A is $\mathcal{H}^k$–almost (equal to) B ($A =_k B$) if $\mathcal{H}^k(A\Delta B) = 0$, where $A\Delta B = A \setminus B \cup B \setminus A$. The same may be applied to isometric images, and we say that A is $\mathcal{H}^k$–almost isometric to B. If $A \subset \mathbb{R} \times \mathbb{S}^n$ is measurable, there is a well–defined measure–theoretic boundary ∂A (consisting of those $p \in \mathbb{R} \times \mathbb{S}^n$ at which neither A nor its complement have density one) and A is said to have finite perimeter if $\mathcal{H}^n(\partial A) < \infty$. In this case, the reduced boundary of A, $\partial^* A$, is defined as the set of the points $p \in \mathbb{R} \times \mathbb{S}^n$ at which A has a measure–theoretic unit exterior normal vector $\nu(A, p)$ and we have $\partial^* A =_n \partial A$.

We define the symmetrization $\widetilde{\Omega}$ of a bounded measurable subset $\Omega \subset \mathbb{R} \times \mathbb{S}^n$ as follows. Let $\pi : \mathbb{R} \times \mathbb{S}^n \to \mathbb{R}$ be the projection onto the first factor, $\pi(x, y) = x$, denote the restriction of π to Ω by u. Choose a point $y_0 \in \mathbb{S}^n$, say the north pole, and consider the line $\mathbb{R} \times \{y_0\} \subset \mathbb{R} \times \mathbb{S}^n$. Let $D^n_r(y_0) \subset \mathbb{S}^n$ be the closed geodesic ball of radius r centered at y_0. Let $D_t = \Omega \cap \{t\} \times \mathbb{S}^n$, which is $\mathcal{H}^n$–measurable for almost all $t \in \mathbb{R}$. For each $t \in \mathbb{R}$, if $D_t \neq \emptyset$, let $\widetilde{D}_t = \{t\} \times D^n_r(y_0)$, where r is such that $\mathcal{H}^n(\widetilde{D}_t) = \mathcal{H}^n(D_t)$ ($r = 0$ if this is null). If $D_t = \emptyset$, $\widetilde{D}_t = \emptyset$. Define

$$\widetilde{\Omega} = \bigcup \{\widetilde{D}_t : \ t \in \mathbb{R}\}.$$

The fact that a slice is not $\mathcal{H}^n$–measurable does not introduce any difficulty, and one may avoid this by considering only Borel subsets (Ω closed, for example), in which case all slices would be $\mathcal{H}^n$–measurable, since the Hausdorff measures are Borel regular. In any case, the union is measurable, by the coarea formula ([Fed69], Theorem 3.2.22), and has finite perimeter if Ω has finite perimeter. This last property follows from an argument similar to the one given in the course of the proof that the Steiner symmetrized set of a finite perimeter set is of finite perimeter in [Tal93] (Lemma 4, p. 112).

Given a measurable set A, Almgren [Alm87] introduces the associated set A_* as the set of points $p \in \mathbb{R} \times \mathbb{S}^n$ at which the density $\theta^{n+1}(\mathcal{H}^{n+1} \lfloor A, p) = 1$. He defines the symmetrized domain as the $_*$ of the above union. This does not change the measure but eliminates points which are not in the "interior" of A, in some sense, and also adds points to fill in lower dimensional "holes" in A. Since assuming $A = A_*$, as in sections 5–7 of [Alm87], causes no loss of generality, we will do the same, and omit reference to it.

Suppose that Ω is a bounded measurable subset of $\mathbb{R} \times \mathbb{S}^n$ with finite perimeter. Since the Lebesgue measure of $\mathbb{R} \times \mathbb{S}^n$ is the product of the Lebesgue measures of $\mathbb{R}$ and $\mathbb{S}^n$, by Fubini's Theorem (or by the coarea formula) we have $\mathcal{H}^{n+1}(\widetilde{\Omega}) = \mathcal{H}^{n+1}(\Omega)$.

Proposition 2.1. $\mathcal{H}^n(\partial^* \widetilde{\Omega}) \leq \mathcal{H}^n(\partial^* \Omega)$.

Proof. Let $N = \partial \Omega$, $N^* = \partial^* \Omega$, $\widetilde{N} = \partial \widetilde{\Omega}$ and $\widetilde{N}^* = \partial^* \widetilde{\Omega}$. Also, let v (or v^*) be the restriction of π to N (or N^*) and $\widetilde{v}$ (or $\widetilde{v}^*$) be the restriction of π to $\widetilde{N}$ (or $\widetilde{N}^*$). Introduce the following sets:

(i) $\Omega_t = \Omega \cap (-\infty, t) \times \mathbb{S}^n = u^{-1}(-\infty, t)$;
(ii) $N^*_t = N^* \cap (-\infty, t) \times \mathbb{S}^n = v^{*-1}(-infty, t)$;
(iii) $\Omega_{t,\varepsilon} = \Omega \cap [t - \varepsilon, t + \varepsilon] \times \mathbb{S}^n = u^{-1}([t - \varepsilon, t + \varepsilon])$;

(iv) $N^*_{t,\varepsilon} = N^* \cap (t-\varepsilon, t+\varepsilon) \times \mathbb{S}^n = v^{*-1}((t-\varepsilon, t+\varepsilon))$,
(v) $B^*_{t,\varepsilon} = \partial^*\Omega_{t,\varepsilon} = N^*_{t,\varepsilon} \cup D^*_{t-\varepsilon} \cup D^*_{t+\varepsilon}$;
(vi) $C_t = N^* \cap \{t\} \times \mathbb{S}^n$,

and analogous subsets for $\widetilde{\Omega}$ (with tilda to denote the latter). The * in the slices $D^*_{t\pm\varepsilon}$ denote that we have removed from $D_{t\pm\varepsilon}$ the points not in $\partial^*\Omega_{t,\varepsilon}$. We will omit the * on v and $\widetilde{v}$, since they are just restrictions to subsets of same Hausdorff dimension, avoiding extra notation.

We also need some remarks on the assumptions on Ω and $\widetilde{\Omega}$. In order for the calculations performed below to work, we may, if necessary, use polihedral approximations of Ω (in this case they may be defined by embedding $\mathbb{R} \times \mathbb{S}^n$ into $\mathbb{R}^{n+2}$ and then performing the approximations there), by means of the Approximation Theorem 4.2.20 of [Fed69], as in Almgren's paper [Alm87], Section 7. Then, after the symmetrization is performed, we pass to the limit, using the flat norm convergence. Since $\mathcal{H}^n$ is lower semicontinuous with respect to flat convergence, the domination of $\mathcal{H}^n(\widetilde{N}^*)$ by $\mathcal{H}^n(N)$ is preserved.

Then, all conditions in Section 6 of [Alm87] also apply to the analogous objects in our situation, the most relevant being:

1. The generalized normal ν satisfies $\mathcal{H}^n(\{p \in N^*_{t,\varepsilon} : \nu(p) \text{ is parallel to } \nabla\pi\}) = 0$;
2. The function $\mathcal{H}^n(N^*_s)$ is absolutely continuous in s for $s \in [t-\varepsilon, t+\varepsilon]$;
3. The function $\mathcal{H}^n(D_s)$ is continuous in s for $s \in [t-\varepsilon, t+\varepsilon]$;
4. The same hold for the corresponding objects defined for $\widetilde{\Omega}$.

Once we have those properties, the use of the coarea formula in the many instances below is justified. We will omit further reference to such questions, performing all the computations assuming that the functions involved have the required regularity properties. For example, property 2. above implies that $\mathcal{H}^n(N^*_s)$ is differentiable $\mathcal{H}^1$-a.e. w.r.t. s in the interval $(t-\varepsilon, t+\varepsilon)$ and its derivative, using also property 1. and the coarea formula, is given by the first equation in (7) below. Similar comments apply to the rest of this section.

Consider the vector field $F(x,y) = x\partial/\partial x = \pi\nabla\pi$ in $\mathbb{R} \times \mathbb{S}^n$. Note that, giving $B^*_{t,\varepsilon}$ the outward orientation, the unit normal vector on $D^*_{t\pm\varepsilon}$ is $\pm\nabla\pi$. Let ν be the unit normal outward vector field on N^*. Using that $\mathrm{div}F = 1$, the Gauss-Green theorem ([Fed69], 4.5.6) implies

$$\begin{aligned}\mathcal{H}^{n+1}(\Omega_{t,\varepsilon}) &= \int_{N^*_{t,\varepsilon}} \langle F, \nu\rangle\, d\mathcal{H}^n + \int_{D^*_{t+\varepsilon}} \pi||\nabla\pi||^2\, d\mathcal{H}^n y - \int_{D^*_{t-\varepsilon}} \pi||\nabla\pi||^2\, d\mathcal{H}^n y \\ &= \int_{N^*_{t,\varepsilon}} \pi\langle\nabla\pi, \nu\rangle\, d\mathcal{H}^n + (t+\varepsilon)\mathcal{H}^n(D^*_{t+\varepsilon}) - (t-\varepsilon)\mathcal{H}^n(D^*_{t-\varepsilon}),\end{aligned}$$

since $||\nabla\pi|| = 1$ and π is constant, equal to t, on D^*_t. Here, $d\mathcal{H}^n y$ refers to the integral with respect to the second factor of the product $\mathbb{R} \times \mathbb{S}^n$. Now, on N^* we have

$$\nabla v = \nabla\pi - \langle\nabla\pi, \nu\rangle\nu, \tag{1}$$

so that $\langle\nabla\pi, \nu\rangle^2 = 1 - ||\nabla v||^2$. We use the usual gradiente applied to v (and to $\widetilde{v}$ below) but it must be understood in a generalized sense, since N^* is not a manifold in general. This is justified by the use of approximate derivatives ([Fed69], 3.2), but one may also use (1) as its definition. We obtain

$$\mathcal{H}^{n+1}(\Omega_{t,\varepsilon}) = \int_{N^*_{t,\varepsilon}} v\sqrt{1 - ||\nabla v||^2}\, d\mathcal{H}^n + (t+\varepsilon)\mathcal{H}^n(D^*_{t+\varepsilon}) - (t-\varepsilon)\mathcal{H}^n(D^*_{t-\varepsilon}),$$

and a similar equation for $\widetilde{\Omega}_t$:

$$\mathcal{H}^{n+1}(\widetilde{\Omega}_{t,\varepsilon}) = \int_{\widetilde{N}^*_{t,\varepsilon}} \widetilde{v}\sqrt{1-||\nabla\widetilde{v}||^2}\, d\mathcal{H}^n + (t+\varepsilon)\mathcal{H}^n(\widetilde{D}^*_{t+\varepsilon}) - (t-\varepsilon)\mathcal{H}^n(\widetilde{D}^*_{t-\varepsilon}).$$

These imply that

$$\int_{N^*_{t,\varepsilon}} v\sqrt{1-||\nabla v||^2}\, d\mathcal{H}^n = \int_{\widetilde{N}^*_{t,\varepsilon}} \widetilde{v}\sqrt{1-||\nabla\widetilde{v}||^2}\, d\mathcal{H}^n. \tag{2}$$

We now use the coarea formula ([Fed69], 3.2.22): if $g : M \to \mathbb{R}$ is Lipschitzian, M is $(\mathcal{H}^k, k)$–rectifiable and $\mathcal{H}^k$–measurable, $f : M \to \mathbb{R}$ is integrable, $M_t = g^{-1}((-\infty, t])$ and $S_s = g^{-1}(s)$,

$$\int_{M_t} f||\nabla g|| d\mathcal{H}^k = \int_{-\infty}^t \left[\int_{S_s} f d\mathcal{H}^{k-1}\right] ds.$$

Apply this with $f = h||\nabla v||^{-1}$ to the left–hand side and with $f = \widetilde{h}||\nabla\widetilde{v}||^{-1}$ to the right–hand side of (2), where h ($\widetilde{h}$, resp.) denotes the integrand of the left–hand (right–hand) side of (2). We obtain, after differentiating with respect to s,

$$\int_{C_s} \sqrt{1-||\nabla v||^2}\, ||\nabla v||^{-1}\, d\mathcal{H}^{n-1} = \int_{\widetilde{C}_s} \sqrt{1-||\nabla\widetilde{v}||^2}\, ||\nabla\widetilde{v}||^{-1}\, d\mathcal{H}^{n-1}, \tag{3}$$

where we have used that $v \equiv s$ ($\equiv \widetilde{v}$) is constant on C_s ($\widetilde{C}_s$).

Applying Schwarz's inequality to the left side of (3), which we denote by A, we obtain

$$A^2 \leq \int_{C_s} \left\{1-||\nabla v||^2\right\} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1} \int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1}, \tag{4}$$

which implies,

$$\int_{C_s} ||\nabla v||\, d\mathcal{H}^{n-1} \leq \left\{\left[\int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1}\right]^2 - A^2\right\}\left\{\int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1}\right\}^{-1}. \tag{5}$$

Observe that the first term on the right–hand side is positive, by (4). Now notice that, since on the right–hand side of (3) the whole integrand is constant on $\widetilde{C}_s$, by the $O(n)$–invariance of $\widetilde{\Omega}$, it is clear that the same type of expression holds, except that now it is an identity:

$$\int_{\widetilde{C}_s} ||\nabla\widetilde{v}||\, d\mathcal{H}^{n-1} = \left\{\left[\int_{\widetilde{C}_s} ||\nabla\widetilde{v}||^{-1}\, d\mathcal{H}^{n-1}\right]^2 - \widetilde{A}^2\right\}\left\{\int_{\widetilde{C}_s} ||\nabla\widetilde{v}||^{-1}\, d\mathcal{H}^{n-1}\right\}^{-1}, \tag{6}$$

where $\widetilde{A}$ is the right–hand side of (3).

Next, we consider the derivatives of $\mathcal{H}^n(N^*_s)$ and of $\mathcal{H}^n(\widetilde{N}^*_s)$, which are absolutely continuous by assumption. Using the coarea formula, in these cases with $f = ||\nabla v||^{-1}$ or $f = ||\nabla\widetilde{v}||^{-1}$, respectively, we obtain

$$\frac{d\mathcal{H}^n(N^*_s)}{ds} = \int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1} \quad\text{and}\quad \frac{d\mathcal{H}^n(\widetilde{N}^*_s)}{ds} = \int_{\widetilde{C}_s} ||\nabla\widetilde{v}||^{-1}\, d\mathcal{H}^{n-1}. \tag{7}$$

Applying Schwarz's inequality to the right–hand side of the first equation in (7), together with (5), we get

$$\int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1} \geq \frac{(\mathcal{H}^{n-1}(C_s))^2 \displaystyle\int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1}}{\left\{\displaystyle\int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1}\right\}^2 - A^2}, \tag{8}$$

which implies, since the denominator on the right–hand side is positive,

$$\int_{C_s} ||\nabla v||^{-1}\, d\mathcal{H}^{n-1} \geq \left\{A^2 + (\mathcal{H}^{n-1}(C_s))^2\right\}^{1/2}. \tag{9}$$

Now, using that the integrand of the second equation in (7) is constant on $\widetilde{C}_s$ and (6), we get a similar expression as (8), only that in this case with an equality sign, finally giving the identity

$$\int_{\widetilde{C}_s} ||\nabla \widetilde{v}||^{-1}\, d\mathcal{H}^{n-1} = \left\{\widetilde{A}^2 + (\mathcal{H}^{n-1}(\widetilde{C}_s))^2\right\}^{1/2}. \tag{10}$$

Hence, using that $A = \widetilde{A}$ and that $\mathcal{H}^{n-1}(C_s) \geq \mathcal{H}^{n-1}(\widetilde{C}_s)$, by the isoperimetric inequality for $\mathbb{S}^n$, (9) and (10) imply

$$\frac{d\mathcal{H}^n(N_s^*)}{ds} \geq \frac{d\mathcal{H}^n(\widetilde{N}_s^*)}{ds}$$

a.e. in $(t-\varepsilon, t+\varepsilon)$, which implies the proposition. ∎

The above proof has the following

Corollary 2.2. *Suppose that $\Omega_{t,\varepsilon}$ and $\widetilde{\Omega}_{t,\varepsilon}$ satisfy the conditions 1.–4. established in the beginning of the proof of Proposition 2.1, without requiring the polihedral approximation. Then, in the case of equality in the estimate given by Proposition 2.1, for almost all $s \in (t-\varepsilon, t+\varepsilon)$, D_s is $\mathcal{H}^n$–almost a geodesic ball in $\{s\} \times \mathbb{S}^n$ and the angle that the generalized normal ν makes with the $\partial/\partial x$–direction is $\mathcal{H}^{n-1}$–almost constant on C_s.*

Proof. The first assertion follows from the isoperimetric property of the geodesic balls in $\mathbb{S}^n$, in the last step of the above proof. The second from the equality in the steps of the proof that used Schwarz's inequality, which implies that $||\nabla \widetilde{v}|| = ||\nabla v||$ $\mathcal{H}^{n-1}$-a.e. along C_s, which means that $\langle \nabla \pi, \nu \rangle$ there. ∎

Remark . As noted by Almgren [Alm87], these results, in the case of equality, do not imply that Ω is $O(n)$–symmetric, even if it is smooth with connected boundary. For a simple example (which may be transported to our situation) think of the unit 2–disk in the (x, y)–plane in $\mathbb{R}^3$ from which a small 2–disk is removed. Then glue smoothly two vertical capped pieces of cylinders, one on the outer boundary and the other on the inner one. This may be done in a way that it results locally $O(2)$–invariant (except at $z = 0$), but not globally, unless the small disk is also centered at the origin.

Proof of Theorem 1. We start by showing that a compact solution $\Omega \subset \mathbb{R} \times \mathbb{S}^n$, which is the closure of its interior, must be connected (in fact, must have connected interior). It is enough to show that it cannot have two components. Suppose it does, and let them be Ω_1 and Ω_2. By moving Ω_2 using the translations on the first factor, we may bring it as close as possible to Ω_1. If they never touch, this means that they pass through each other and may have points with same first coordinates. Now, using the homogeneity of the spherical factor, we may move it until it touches Ω_1. But this would give a non–minimizing tangent cone at the touching point (since decomposable as a current), by Theorem 1 of [BG72], a contradiction.

Next, if $N = \partial\Omega$ (which may fail to be connected) has a piece of positive n-measure on a totally geodesic n-sphere $\{x\} \times \mathbb{S}^n$, then the whole of $\{x\} \times \mathbb{S}^n$ is a component of N, by the analiticity of the regular part of $\partial\Omega$. This vertical part is totally geodesic, and satisfies an elliptic equation, so it must be the whole n-sphere. So, the components of N which are not already "vertical" do not have any "vertical" parts of positive n-measure. Also, such a "vertical" component may only occur for extremal values of x in the first factor, i.e., for a minimum or maximum value of x on Ω, by the connectedness of Ω.

Now, let $[a, b] = \pi(\Omega)$ and apply Proposition 2.1 and its Corollary, obtaining that for almost all $t \in [a, b]$, D_t is $\mathcal{H}^n$–almost a geodesic ball in $\mathbb{S}^n$ and the angle that the unit normal vector to N^* makes with the $\mathbb{R}$–direction is constant there. In order to do that, we observe that the regularity of $N = \partial\Omega$ implies that the polihedral approximation mentioned in the proof of Proposition 2.1 may be performed directly for the piece of boundary $N \cap \Omega_{t,\varepsilon}$, so that not only the boundary measure is approximated but also the measures of the boundaries of the slices D_t are also approximated. This implies that if the inequalities that used the isoperimetric and Schwarz's inequalities are strict, they would pass to the limit.

Remark . Even though this is not explicitly mentioned by Almgren in [Alm87], he uses these facts in the last paragraph of the proof of Theorem 8, in p. 24, when he applies Cauchy-Kovalevsky to obtain that a solution is locally symmetric.

So, consider a connected component N_1 of $\partial\Omega$ which is not a totally geodesic slice, and let $\pi(N_1) = [a_1, b_1]$. Since the singularities on the boundary are of codimension at most seven there, the set C_t is $\mathcal{H}^{n-1}$–almost a round sphere $\mathbb{S}^{n-1}$ in $\mathbb{S}^n$, and may be seen as the (reduced) boundary of the slice D_t, for almost all $t \in [a_1, b_1]$. It has a well–defined center $c(t) = (t, \lambda(t))$, with $\lambda(t) \in \{t\} \times \mathbb{S}^n$, and radius $r(t)$, because again of the high codimension of the singular set. We show that the centers of slices must be on a line $\mathbb{R} \times \{y\}$, i.e., $\lambda(t) \equiv y$, for $t \in [a_1, b_1]$, and that both λ and r are smooth functions.

For a $t \in (a_1, b_1)$ where $r(t) > 0$, C_t must contain a regular point P of the boundary. But then there is an open neighborhood of regular points around P in $N = \partial\Omega$, and this determines the centers and radii of the $\mathcal{H}^{n-1}$–almost $(n-1)$–spheres C_s, s near t, and these are differentiable functions of s, so that D_s is indeed a smooth ball. If $r(t) = 0$, then t cannot be in the interior of $\pi(N_1) = [a_1, b_1]$, since it would imply that there is a singularity which has a non–minimizing tangent cone, as before. If $t = a_1$ or $t = b_1$ we have that the tangent cone there must be a hyperplane, implying that N_1 is smooth, again using the results of [BG72]. We only have to prove that the derivative of $\lambda(t)$ is zero to conclude the proof.

In order to do that, consider $\mathbb{R} \times \mathbb{S}^n$ embedded in $\mathbb{R}^{n+2}$ and let $\varphi_1, \varphi_2, \ldots, \varphi_n$ parametrize the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ as usual by

$$x_1 = \cos\varphi_1,\ x_2 = \sin\varphi_1 \cos\varphi_2,\ \ldots,\ x_{n+1} = \sin\varphi_1 \sin\varphi_2 \ldots \sin\varphi_n.$$

Now, we parametrize C_t, centered at $\lambda(t)$ with radius $r(t)$, using the variables $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ as follows: let v_i be the unit vector in the direction of $\partial/\partial\varphi_i$, then

$$X(t,\alpha) = a_1(t,\alpha)v_1(t) + a_2(t,\alpha)v_2(t) + \cdots + a_n(t,\alpha)v_n(t) + a_{n+1}(t)\lambda(t),$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and

$$\begin{aligned} a_1(t,\alpha) &= \sin r(t) \sin\alpha_1 \ldots \sin\alpha_{n-1}, \\ a_2(t,\alpha) &= \sin r(t) \sin\alpha_1 \ldots \sin\alpha_{n-2} \cos\alpha_{n-1}, \\ &\vdots \\ a_n(t,\alpha) &= \sin r(t) \cos\alpha_1, \\ a_{n+1}(t) &= \cos r \end{aligned}$$

gives the desired parametrization. Now, supposing that $\lambda(t) = (0, \ldots, 0, 1)$, i.e., $\varphi_i = \pi/2$, $i = 1, \ldots, n$, we get that $\langle \nu, \nabla\pi \rangle$ is constant on C_t if and only if

$$r'(t) + \langle\langle Q, (\varphi_1'(t), \ldots, \varphi_n'(t)) \rangle\rangle$$

is constant for all $Q \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, with $\langle\langle\ ,\ \rangle\rangle$ the euclidean inner product in $\mathbb{R}^n$. This can only happen if $\varphi_i'(t) = 0$, for all $i = 1, \ldots, n$, i.e., $\lambda'(t) = 0$.

■

Remarks .

1. For an example that would give a region such that all the intersections are disks, but it is not $O(1)$–symmetric, take a section $[a,b] \times \mathbb{S}^1 \subset \mathbb{R} \times \mathbb{S}^1$ and remove two round disks with centers not aligned on a meridian of the cylinder and sufficiently spaced. This cannot be a minimizer for various reasons, but it shows that one needs more explicit information on the possible boundaries to conclude on the $O(n)$–invariance. Anyway, one cannot have two components N_1 and N_2 of the boundary with $\pi(N_1) \cap \pi(N_2) \neq \emptyset$.
2. We could have adopted a similar approach to that of [Alm87], by showing that Ω must be locally $O(n)$–symmetric using Cauchy–Kovalevsky, and then using the results of the next section on the generating curves of $O(n)$–invariant CMC hypersurfaces.

3. Generating curves for invariant CMC hypersurfaces and the proof of Theorem 2

Let Ω be an isoperimetric domain in $\mathcal{C}^{n+1} = \mathbb{R} \times \mathbb{S}^n$. The boundary components are $O(n)$–invariant CMC hypersurfaces embbeded in $\mathcal{C}^{n+1}$, by Theorem 1. Let $\mathcal{B}$ denote the orbit space of the transformation group $(O(n), \mathcal{C}^{n+1})$, i.e. $\mathcal{B} = \mathcal{C}^{n+1}/O(n)$, equipped with the orbital distance metric. $\mathcal{B}$ is a strip of width π in $\mathbb{R}^2$, with the flat metric, which we parametrize as $\mathcal{B} = \{(x,y) : x \in \mathbb{R}, y \in [0,\pi]\}$. Now let Σ_h be an $O(n)$–invariant hypersurface in $\mathcal{C}^{n+1}$ with CMC h. Then the projection of Σ_h to the orbit space is a curve, denoted by γ_h, i.e. $\gamma_h = \Sigma_h/O(n) \subset \mathcal{B}$, the generating curve of Σ_h.

Remark . The structure of the orbit space already reduces drastically the possibilities for the connected components of the boundary of a minimizer. The generating curve may only be of three types: a closed curve in the interior of the orbit space, topologically an $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, a curve starting and ending at the same boundary of the orbit space, topologically an $\mathbb{S}^n$, or one starting and ending at each piece of the boundary of the orbit space, also an $\mathbb{S}^n$.

Using Theorem 1, the first possibility is not allowed, since it would violate the intersection property. The purpose of this section is to classify the other two.

Let $\gamma_h(s) = (x(s), y(s))$ be parametrized by arclength. Define $\sigma(s)$ by $\gamma_h' = (\cos\sigma, \sin\sigma)$, i.e., σ is the angle between γ and the positive x–axis. Straightforward computations give the following system of ordinary differential equations for γ_h. The result also includes the existence of a first integral for the system.

Proposition 3.1. *An $O(n)$–invariant hypersurface $\Sigma_h \subset \mathbb{R} \times \mathbb{S}^n$ is of constant mean curvature h if and only if its generating curve $\gamma_h = \Sigma_h/O(n)$ satisfies the following system of O.D.E.'s in $\mathcal{B}$:*

$$(\mathcal{S}) \begin{cases} dx/ds &= \cos\sigma \\ dy/ds &= \sin\sigma \\ d\sigma/ds &= h + (n-1)\cot y \cos\sigma \end{cases}.$$

Moreover, the functions $J^0_{n-1}(s) = (\sin y)^{n-1}\cos\sigma + hg_{n-1}(y)$ and $J^\pi_{n-1}(s) = (\sin y)^{n-1}\cos\sigma - hg_{n-1}(\pi - y)$, where $g_{n-1}(y) = \displaystyle\int_0^y (\sin t)^{n-1}\,dt$, are constant along solutions.

Figure 1 contains some computer–generated solutions of the system $(\mathcal{S})$ for $n = 2$; they are typical for all n.

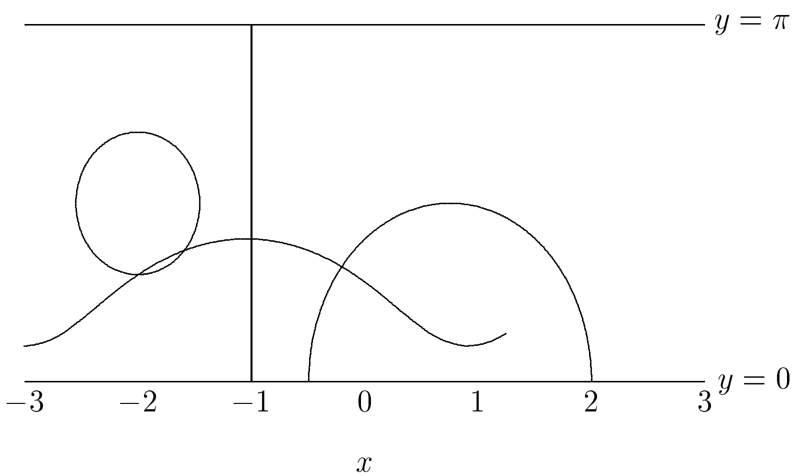

FIGURE 1. Typical solutions for the system $(\mathcal{S})$.

The equation for σ is singular at the boundary lines $y = 0$ and $y = \pi$ of $\mathcal{B}$ and, if a solution reaching those boundary lines exists, it must be perpendicular to them. Such a solution would pose no problems, since it would be the projection of a (locally defined) nonsingular hypersurface in $\mathbb{R} \times \mathbb{S}^n$. This kind of singular behavior has been studied previously, as in [HH89], and the conclusion is stated next.

Lemma 3.2. *There exists $\gamma(s) = (x(s), y(s), \sigma(s))$, which is the unique (local) analytical solution curve to the initial value problem given by the ODE system $(\mathcal{S})$, with initial conditions $x(0) = x_0$, $y(0) = 0$ (π, respectively) and $\sigma(0) = \pi/2$ ($-\pi/2$, respectively).*

This local solution may be extended indefinitely, using reflection along the boundaries. From now on, we will denote a common (complete) solution curve of Equations $(\mathcal{S})$ and

$J^0_{n-1}(s) = c$ by $\gamma_{h,c}$ (the case $J^\pi_{n-1}(s) = c$ is obtained by reflection through $y = \pi/2$ and does not need to be considered). By complete we mean infinitely extendable in both directions, including reflections at the boundary of $\mathcal{B}$. If a solution reaches the boundary, we get

$$J^0_{n-1}(s) = (\sin y)^{n-1} \cos\sigma + h \int_0^y (\sin t)^{n-1} dt = 0. \tag{11}$$

In that case, if $h = 0$, then it is clear that the only solution curves $\gamma_{0,0}$ are the vertical lines $x = a$, with both endpoints at the boundary lines, which correspond to the totally geodesic spheres $\{a\} \times \mathbb{S}^n$ embedded in $\mathbb{R} \times \mathbb{S}^n$. (In fact, the maximum principle implies that the only compact CMC hypersurfaces with $h = 0$ are the totally geodesic spheres $\{x\} \times \mathbb{S}^n$.) Otherwise, suppose that $h > 0$ (the case $h < 0$ is analogous, corresponding to a reversal of direction of the curve). We perform a change of variables. Define $u_{n-1} = -\cos\sigma$ along $\gamma_{h,0}$; also, define the function h_{n-1} on $(0, \pi]$ by

$$h_{n-1}(t) \;=\; \frac{(\sin t)^{n-1}}{g_{n-1}(t)}, \tag{12}$$

where g_{n-1} is given in Proposition 3.1. Let $\gamma_{h,0}$ be the solution curve with $y(0) = 0,\ x(0) = x_0$ and $\sigma(0) = \pi/2$. Then, as long as $\gamma_{h,0}$ is defined, we have that $u_{n-1}(h, y) = h/h_{n-1}(y)$. This equation makes sense for $y = 0$, since $u_{n-1}(0) = 0$ and $\lim_{y\searrow 0} h_{n-1}(y) = \infty$. Next, let h be fixed and consider u_{n-1} as a function of y only. This can be done locally whenever $dy/ds \neq 0$, i.e. $\sin\sigma \neq 0$. In particular, this holds for solutions starting from or terminating at the boundary of $\mathcal{B}$. The proof of the next lemma is straightforward.

Lemma 3.3. *A curve γ_h is a (local) solution of the Equations $(\mathcal{S})$ around $s = s_0$ with $dy/ds|_{s_0} \neq 0$ if and only if γ_h is a (local) solution of the system*

$$(\mathcal{S})^* \begin{cases} dx/dy(y) &= -\frac{u_{n-1}(y)}{\sqrt{1-(u_{n-1}(y))^2}} \\ du_{n-1}/dy(y) &= h - (n-1)u_{n-1}(y)\cot y \end{cases},$$

around $y(s_0)$.

Next, we state a collection of analytical facts about the functions u_{n-1} and h_{n-1}. The proof follows from elementary calculations using the differential equations and the first integral.

Lemma 3.4. *Let $h > 0$ and let $\gamma_{h,0}(y)$ be a (maximally defined) solution curve of the system $(\mathcal{S})^*$ on $[0, \eta) \subseteq [0, \pi]$, with the initial conditions $u_{n-1}(y)|_{y=0} = 0,\ \ x(y)|_{y=0} = x_0$; then:*

(i) $\lim_{y\searrow 0} u_{n-1}{}'(y) = h/n$ *and* $\lim_{y\searrow 0} u_{n-1}{}''(y) = 0$;
(ii) $u_{n-1}(y)$, $u_{n-1}{}'(y)$ *and* $u_{n-1}{}''(y)$ *are (strictly) monotonically increasing in* $(0, \eta)$;
(iii) $\lim_{y\nearrow\eta} u_{n-1}(y) = 1$ *and* $\lim_{y\nearrow\eta} u_{n-1}{}'(y) = h - (n-1)\cot\eta$.
(iv) $h_{n-1}(\eta) - (n-1)\cot\eta > 0$ *for any* $\eta \in (0, \pi)$;
(v) $h_{n-1}(\eta)$ *is (strictly) monotonically decreasing for* $\eta \in (0, \pi)$.

Using Lemma 3.4.(iii) we may extend the solution curve $\gamma_{h,0}(y)$ to include $y = \eta$. This means that we may omit the limit expressions and write directly $u_{n-1}(\eta) = 1$ and $u_{n-1}{}'(\eta) = h - (n-1)\cot\eta$. Even $x(\eta)$ makes sense, since the singularity in the first equation of $(\mathcal{S})^*$ is integrable.

Now, the restriction that a solution $\gamma_{h,0}$ of the system $(\mathcal{S})^*$ starting on the line $y = 0$ is defined for $y \leq \eta$ occurs only because we are parametrizing $\gamma_{h,0}$ by y. It is not difficult to check that a solution of $(\mathcal{S})$, $\gamma_{h,0}$, parametrized by arclength, can be extended further and the continuation is exactly the reflection of $\gamma_{h,0}$ through the line $x = x(\eta)$. This follows from Lemma 3.4.(v), which shows that η and $h_{n-1}(\eta)$ determine each other univocally, implying that the maximum value for y is the same for both curves. Also, this shows that the curve is convex. So, besides the vertical segments and the curves $\gamma_{h,0}$, the only other possibility that would generate a closed hypersurface is that of a simple closed curved completely contained in the interior of $\mathcal{B}$. We state these observations as

Proposition 3.5. *Let Σ_h be an $O(n)$–invariant closed hypersurface of constant mean curvature h embedded in $\mathbb{R} \times \mathbb{S}^n$. Then the generating curve $\gamma_h \subset \mathcal{B}$ of Σ_h is of one of the following three types:*

(i) *γ_h is a straight segment starting at one boundary line and terminating at the other, and Σ_h is a totally geodesic $\mathbb{S}^n$ embedded in $\mathbb{R} \times \mathbb{S}^n$.*

(ii) *γ_h starts and ends at the same boundary line, Σ_h is an embedded n–sphere in $\mathbb{R} \times \mathbb{S}^n$ with $h \neq 0$. Moreover, γ_h is convex, and if γ_h starts at the line $y = 0$, η is the y–maximum along γ_h, then $h = h_{n-1}(\eta)$ and γ_h can be characterized by*

$$x(y) = x_0 - \int_0^y \frac{u_{n-1}(\eta, t)}{\sqrt{1 - (u_{n-1}(\eta, t))^2}}\, dt,$$

for $y \in [0, \eta]$, where $u_{n-1}(\eta, t) = h/h_{n-1}(t) = h_{n-1}(\eta)/h_{n-1}(t)$, and γ_h is symmetric with respect to the line $x = x(\eta)$. The solutions that start and end at the line $y = \pi$ are obtained by reflecting those through $y = \pi/2$.

(iii) *γ_h is a simple closed curve in the interior of $\mathcal{B}$, and Σ_h is of the type $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.*

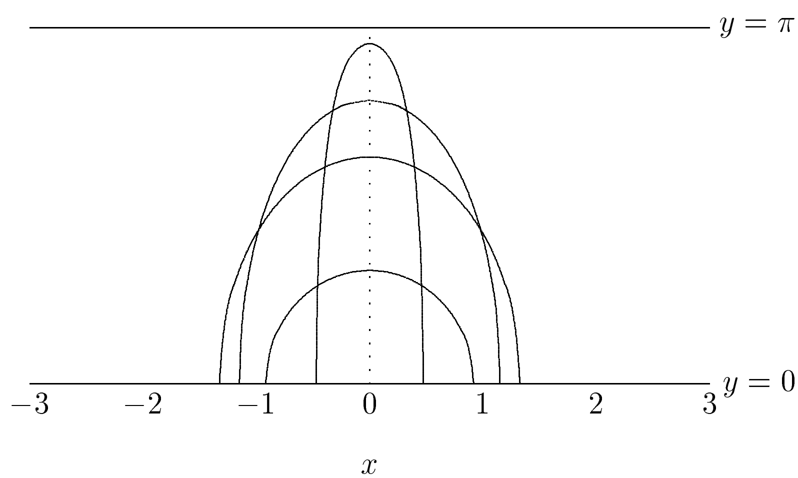

FIGURE 2. Generating curves γ_h for Ω_h ($n = 2$, $\eta = 1, 2, 2.5, 3$).

Figure 2 contains some computer–generated examples of solutions γ_h of type (ii). Denote the ball-type regions bounded by such curves by Ω_h.

Proof of Theorem 2. We prove the second assertion, which implies the first. As already observed at the beginning of this section, there cannot be components of $\partial\Omega$ generated by curves contained in the interior of the orbit space, because of the intersection property with the slices $\{x\} \times \mathbb{S}^n$, given by Theorem 1. For the same reason one cannot have balls removed from the interior of another ball (even if not aligned). One may also fill in the hole(s) and use a contraction in the $\mathbb{R}$–direction, correcting the $(n+1)$–volume and reducing the boundary volume. Also, one cannot have a section with balls removed, because the mean curvatures would not be the same (or use again the fill–in idea and contraction). Applying Proposition 3.5, the boundary of a solution Ω may have either one component, and the region is of the type Ω_h, or is composed of two totally geodesic spheres, and Ω is a section, completing the proof. ■

Even though CMC hypersurfaces generated by curves of type (iii) cannot occur as boundaries of isoperimetric domains, they do exist and we refer to [PR] for a discussion on these curves, where they are shown to generate unstable CMC hypersurfaces.

4. Volume formulae, the case $\mathbb{R}\times\mathbb{S}^2$ and large isoperimetric domains

Let $\Omega_h \subseteq \mathbb{R}\times\mathbb{S}^n$ be the regions given by Theorem 2 with $h>0$: then, $\partial\Omega_h$ is generated by a curve γ_h which starts and ends perpendicularly at the boundary line $y=0$ of $\mathcal{B}$, and is convex and symmetric with respect to the line $x=$ const. passing through the y–maximal point. By Proposition 3.5.(ii), if (ξ,η) is the y–maximal point along γ_h, then $h=h_{n-1}(\eta)$, where h_{n-1} is defined on $(0,\pi]$ by (12). We will denote by ω_n the n–volume of $\mathbb{S}^n$.

Proposition 4.1. *Let $\eta\in(0,\pi)$. Then the following hold:*

$$\operatorname{vol}_n(\partial\Omega_h) = 2\,\omega_{n-1}\int_0^\eta \frac{(\sin y)^{n-1}}{\sqrt{1-(u_{n-1}(\eta,y))^2}}\,dy; \tag{13}$$

$$\begin{aligned}\operatorname{vol}_{n+1}(\Omega_h) &= 2\,\omega_{n-1}\int_0^\eta \frac{(\sin y)^{n-1}\,(u_{n-1}(\eta,y))^2}{h\sqrt{1-(u_{n-1}(\eta,y))^2}}\,dy\\ &= 2\,\omega_{n-1}\int_0^\eta \frac{g_{n-1}(y)\,u_{n-1}(\eta,y)}{\sqrt{1-(u_{n-1}(\eta,y))^2}}\,dy,\end{aligned} \tag{14}$$

where $h=h_{n-1}(\eta)$ and $u_{n-1}(\eta,y)=h/h_{n-1}(y)$.

Proof. The $(n-1)$–volume of an orbit represented by $(x,y)\in\mathcal{B}=\mathcal{C}^{n+1}/O(n)$ is given by $\omega_{n-1}\,(\sin y)^{n-1}$, since it is isometric to $\mathbb{S}^{n-1}(\sin y)$. Since the generating curve γ_h is symmetric with respect to the line $x=\xi$ passing through (ξ,η), we consider, for the purpose of computing $\operatorname{vol}_n(\partial\Omega_h)$ and $\operatorname{vol}_{n+1}(\Omega_h)$, only the part of γ_h where y varies from 0 to η, which we denote by ${\gamma_h}^+$. By simple geometric analysis we have

$$\operatorname{vol}_n(\partial\Omega_h) = 2\,\omega_{n-1}\int_{{\gamma_h}^+}(\sin y)^{n-1}\,ds.$$

Since $dy/ds=\sin\sigma=\sqrt{1-(u_{n-1}(\eta,y))^2}$, (13) follows. Notice that the singularity appearing in the integrand of (13) is of the type $(\eta-y)^{-1/2}$, thus integrable. Next, (14) follows from an application of Green's Theorem to R_h, the region under ${\gamma_h}^+$. ■

Notation: let $A_n(\eta)$ denote $\operatorname{vol}_n(\partial\Omega_h)$ and $V_n(\eta)$ denote $\operatorname{vol}_{n+1}(\Omega_h)$, for $h=h_{n-1}(\eta)$. Both are well–defined functions on $(0,\pi)$.

The integrals appearing in the formulae above are of a general elliptic type, and therefore not explicitly computable in terms of elementary functions, with the exception of the case

$n = 2$, treated next. For $n = 2$, Equation (11) takes the form $u(\eta, y)\, \sin y \;=\; h\,(1 - \cos y)$, so that we get

$$\sin y \;=\; \frac{2\,h\,u}{h^2 + u^2}\,, \quad \cos y \;=\; \frac{h^2 - u^2}{h^2 + u^2}.$$

Then, the formulae in the next lemma follow from elementary calculus.

Lemma 4.2. *Let $\eta \in (0, \pi)$ and $h = h_1(\eta)$. Then*

$$x(y) = x_0 \;-\; \frac{h}{\sqrt{1+h^2}} \left[\log \frac{\sqrt{1+h^2}+1}{\sqrt{1+h^2}-1} \;-\; \log \frac{\sqrt{1+h^2}+\sqrt{1-u^2}}{\sqrt{1+h^2}-\sqrt{1-u^2}} \right] ;$$

$$A_2(\eta) =\; 4\,\pi \left[\frac{2}{1+h^2} \;+\; \frac{h^2}{(1+h^2)^{3/2}}\; \log \frac{\sqrt{1+h^2}+1}{\sqrt{1+h^2}-1} \right] ;$$

$$V_2(\eta) = 4\,\pi\,h \left[\frac{2+h^2}{(1+h^2)^{3/2}}\; \log \frac{\sqrt{1+h^2}+1}{\sqrt{1+h^2}-1} \;-\; \frac{2}{1+h^2} \right] .$$

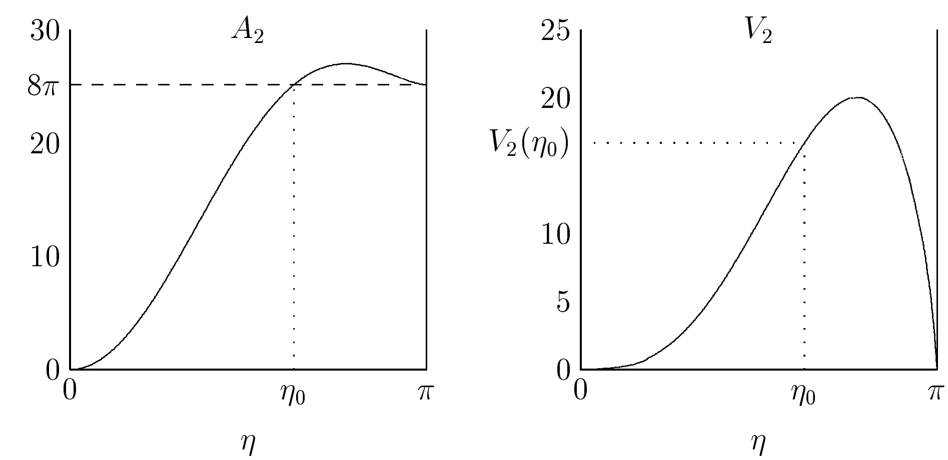

FIGURE 3. The area and volume functions for $n = 2$

Figure 3 contains the plottings of A_2 and V_2 as functions of η. The derivatives of A_2 and V_2 are easy to compute, implying the following estimates.

Corollary 4.3. *Let $\alpha = 2.20$, $\beta = 2.19$.*

(i) $\lim_{\eta \nearrow \pi} A_2(\eta) = 8\,\pi$.
(ii) *There exists $\eta \in (0, \alpha)$ such that $A_2(\eta) > 8\,\pi$.*
(iii) $A_2{}'(\eta) > 0$ *if $\eta \in (0, \alpha)$.*
(iv) $V_2{}''(\eta) < 0$ *if $\eta \in (\beta, \pi)$.*

Remark . It is easy to check that the first derivatives satisfy $A_2{}'(\eta) = hV_2{}'(\eta)$. This also follows from a general result (see Lemma 5.2 below).

Proof of Theorem 3. By Corollary 4.3.(ii) and (iii), there exists $\eta_0 \in (0, \alpha)$ such that $A_2(\eta_0) = 8\pi$, since $\lim_{\eta \searrow 0} A_2(\eta) = 0$. By (iii) and the remark above, the relationships $\eta \longleftrightarrow A_2(\eta)$ and $\eta \longleftrightarrow V_2(\eta)$ are bijective, in that range. Again using that $A_2{}'(\eta) = h\, V_2{}'(\eta)$,

(iv) implies that ${A_2}'(\eta)$ can change sign at most once on (β, π). But (i), (ii) and (iii) imply that ${A_2}'(\eta)$ must change sign exactly once on (β, π), hence $A_2(\eta) > 8\pi$ for $\eta \in (\eta_0, \pi)$, completing the proof. ∎

Figure 4 contains the isoperimetric profile of $\mathbb{R} \times \mathbb{S}^2$, where the exponents are such that the profile for $\mathbb{R}^3$ is a line (the dotted line). We complete the discussion by giving the estimates for v_0, η_0, $h_1(\eta_0)$ and $k_2(\eta_0)$:

$$\eta_0 \approx 1.97, \quad v_0 = V_2(\eta_0) \approx 16.66, \quad h_1(\eta_0) \approx 0.66, \quad k_2(\eta_0) \approx 57.22\,.$$

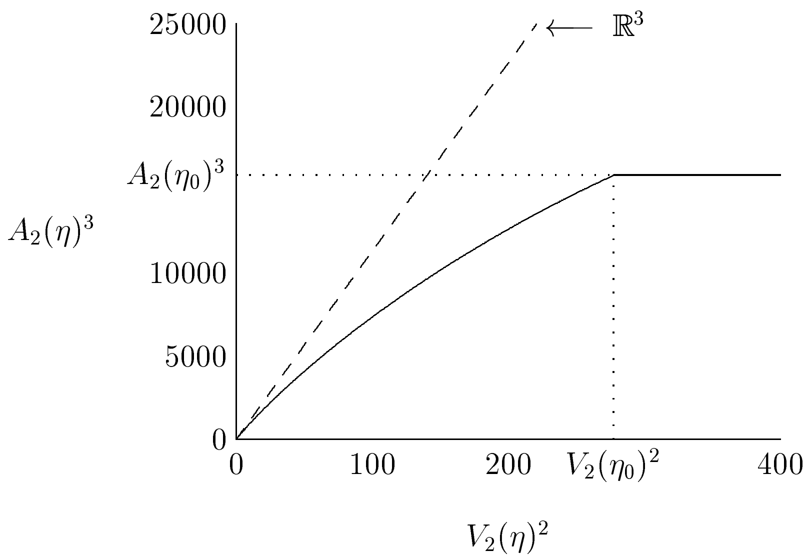

FIGURE 4. The isoperimetric profile for $\mathbb{R} \times \mathbb{S}^2$.

Theorem 4. *Let Ω be an isoperimetric domain in $\mathbb{R} \times \mathbb{S}^n$ such that*

$$\mathrm{vol}_{\mathrm{n}+1}(\Omega) \geq 2\,\pi\,\omega_{n-1} \int_0^{\pi} \left[\int_0^{y} (\sin t)^{\mathrm{n}-1}\,dt\right] dy\,;$$

then Ω is a cylindrical section of $\mathbb{R} \times \mathbb{S}^n$ with height $\mathrm{vol}_{n+1}(\Omega)/\omega_{\mathrm{n}}$.

Proof. We consider a region Ω_h, with boundary is generated by γ_h (Proposition 3.5.(ii)). Let $\eta \in (0, \pi)$ be fixed but arbitrary. Since ${u_{n-1}}''(y) > 0$ for $y \neq 0$, by Lemma 3.4.(ii), then $u_{n-1}(y) < y/\eta$ for $y \in (0, \eta)$. Letting $w = y/\eta$ in (14) we get

$$\mathrm{vol}_{n+1}(\Omega_h) \;<\; 2\,\pi\,\omega_{n-1} \int_0^{\pi} g_{n-1}(y)\,dy \int_0^1 \frac{w}{\sqrt{1-w^2}}\,dw.$$

The result follows from the volume formula for a section $\Omega_{[a,b]} = [a, b] \times \mathbb{S}^n$, $\mathrm{vol}_{n+1}(\Omega_{[a,b]}) = (b-a)\omega_n = (b-a)\omega_{n-1}g_{n-1}(\pi)$, recalling that $g_{n-1}(y) = \int_0^y (\sin t)^{n-1} dt$. ∎

This result is far from being sharp, but provides a lower bound which depends on the dimension only. The lower bound obtained from Theorem 4 for $\mathbb{R} \times \mathbb{S}^2$ is about 124, which is much larger than the actual transition value (≈ 16.7).

Remarks . 1. Gonzalo [Gon90] has proved a general result of this type for any product of a compact manifold and an Euclidean space, without specific estimates.

2. In a personal communication, M. Ritoré has pointed out that, once we can prove that the volume functions of Ω_h are decreasing, the boundary is unstable as a CMC hypersurface. It seems feasible to improve on the above result by means of this observation (but see the next section for the technical difficulties involved in dealing with the monotonicity of the volume functions).

5. Some comments on the volume formulae and small regions

Even though we know from the volume formulae $A_n(\eta)$ and $V_n(\eta)$ for Ω_h, $h = h(\eta)$, that isoperimetric regions of small volume must be of ball type, it is not clear if, in that range, there is uniqueness, since the behavior of the volume functions $A_n(\eta)$ and $V_n(\eta)$ have not been completely established. In this section we prove the following result.

Theorem 5. *Let $n \geq 3$. There is $\eta_0(n) \in (0, \pi]$ such that both A_n and V_n are strictly increasing on $(0, \eta_0(n))$ and, moreover, in that range of volume, the regions Ω_h are the (unique) isoperimetric domains in $\mathbb{R} \times \mathbb{S}^n$.*

Remark . F. Morgan [Mor] has recently announced a proof that in any Riemannian manifold (M, g) such that M/G is compact, G the isometry group of (M, g), small isoperimetric regions must be very close to a round ball (very close means smoothly close after rescaling).

First note that A_n and V_n must be differentiable functions of η, but it is not immediately clear how to obtain formulae for the derivatives, since we are differentiating with respect to the parameter for which the integrand becomes singular. We do this next, with both ${A_n}'(\eta)$ and ${V_n}'(\eta)$ obtained in a similar way. Let $I_n(\eta, y)$ denote the integrand in the formula for either $A_n(\eta)$ or $V_n(\eta)$, including constants. Let F_n denote either A_n or V_n.

Lemma 5.1. *$A_n(\eta)$ (or $V_n(\eta)$) is a C^1 function of $\eta \in (0, \pi)$ and*

$$ {F_n}'(\eta) = \int_0^\eta \left[\frac{\partial I_n}{\partial \eta}(\eta, y) + \frac{\partial I_n}{\partial y}(\eta, y) \right] dy \, . $$

Proof. . Fix $\eta_0 \in (0, \pi)$. Let $\varepsilon > 0$ be such that $[\eta_0 - \varepsilon, \eta_0 + \varepsilon] \subset (0, \pi)$. Now let $\eta \in (\eta_0 - \varepsilon/2, \eta_0 + \varepsilon/2)$ and consider the following sequence of functions

$$ (F_n)_k(\eta) = \int_0^{\eta - \varepsilon/2^{k+1}} I_n(\eta, y)\, dy \, , \quad k \geq 1 \, . $$

$\{(F_n)_k\}$, $k \geq 1$, is a sequence of well–defined functions on $[\eta_0 - \varepsilon/2, \eta_0 + \varepsilon/2]$. Since there are no singularities involved here, $(F_n)_k$ is smooth and its derivative satisfies a formula like the one in the statement of the lemma. The proof is completed by computing the integrands for these derivatives explicitly in both cases and showing that there is uniform convergence in small compact subintervals. ∎

Since formula for ${V_n}'(\eta)$ is the more manageable of the two, the following standard relationship between ${A_n}'(\eta)$ and ${V_n}'(\eta)$ will be useful. Let M be an (connected) orientable Riemannian manifold and $\Omega \subset M$ a compact region with smooth boundary $N = \partial\Omega$. Let $f_t : N \to M$, $t \in (-\varepsilon, \varepsilon)$, be a variation of N through smooth (embedded) hypersurfaces N_t, with $f = f_0$ being the inclusion, all of them bounding regions Ω_t. Suppose further that the mean curvature of $N = N_0$, $h = h(0)$, is constant.

Lemma 5.2. [BdCE88] *Let $V(t)$ and $A(t)$ be the volumes of Ω_t and N_t, respectively. Then they are differentiable functions of t and $A'(0) = hV'(0)$. In particular, considering the regions Ω_h as a family parametrized by η, ${A_n}'(\eta) = h{V_n}'(\eta)$.*

From now on we will use this result and consider only of ${V_n}'(\eta)$, given by Lemma 5.1:

$$(15) \qquad {V_n}'(\eta) = 2\omega_{n-1} \int_0^{\eta} \frac{u\,(\sin y)^{n-1}}{(1+u)\sqrt{1-u^2}}\, \Psi_{n-1}(\eta, y)\, dy,$$

where

$$(16) \qquad \Psi_{n-1}(\eta, y) = 2 + u - \frac{(n-1)\, u\, \sin(\eta - y)}{h_{n-1}(\eta)\, \sin\eta\, \sin y\,(1-u)}$$

and u is shorthand for $u_{n-1}(\eta, y) = h_{n-1}(\eta)/h_{n-1}(y)$ (see (12) for the definition of h_{n-1}). Next, we define some functions related to ${V_n}'$ and state a lemma with their relevant properties, which is proved by routine application of Lemma 3.4.

Definition 5.3. Let $n \geq 3$, $\eta \in (0, \pi)$ and $y \in [0, \eta)$. We define:

(i) $\theta_{n-1}(\eta) \;=\; \sin\eta\; h_{n-1}(\eta)\,;$
(ii) $\rho_{n-1}(\eta) \;=\; \sin\eta\,[h_{n-1}(\eta) - (n-1)\,\cot\eta]\,;$
(iii) $\zeta_{n-1}(\eta) \;=\; h_{n-1}(\eta) - n\,\cot\eta\,;$
(iv) $\varphi_{n-1}(\eta, y) \;=\; \sin(\eta - y)\,/\,[1 - u_{n-1}(\eta, y)]\,.$

Lemma 5.4. *Let $n \geq 3$. The following hold:*

(i) $\lim_{\eta \searrow 0} \zeta_{n-1}(\eta) = 0$, *and ζ_{n-1} is (strictly) monotonically increasing on $(0, \pi)$;*
(ii) $\lim_{\eta \searrow 0} \theta_{n-1}(\eta) = n$, $\theta_{n-1}(\pi) = 0$ *and θ_{n-1} is (strictly) monotonically decreasing on $(0, \pi)$;*
(iii) $\rho_{n-1}(\eta) \geq 0$, $\lim_{\eta \searrow 0} \rho_{n-1}(\eta) = 1$, $\rho_{n-1}(\pi) = n - 1$ *and ρ_{n-1} is (strictly) monotonically increasing on $(0, \pi)$;*
(iv) $\varphi_{n-1}(\eta, 0) = \sin\eta$, $\lim_{y \nearrow \eta} \varphi_{n-1}(\eta, y) = (h_{n-1}(\eta) - (n-1)\,\cot\eta)^{-1}$, *$\varphi_{n-1}(\eta, y)$ has exactly one extremum on $(0, \eta)$, a maximum, and its minimum is assumed when $y \nearrow \eta$. As a consequence, on any interval $[y_1, y_2] \subseteq [0, \eta]$, the minimum of $\varphi_{n-1}(\eta, y)$ is assumed at one of its endpoints.*

Proof of Theorem 5. We start by showing that for any $\eta \in (0, \pi)$, $y \in [0, \eta]$,

$$(17) \qquad \Psi_{n-1}(\eta, y) \geq 2 - \frac{n\,(n-1)}{[\sin\eta\, h_{n-1}(\eta)]^2}\,.$$

Observe that, using (16), that $u_{n-1}(\eta, y) \in [0, 1]$ and that $u_{n-1}(\eta, y) = h_{n-1}(\eta)/h_{n-1}(y)$, it follows that

$$\Psi_{n-1}(\eta, y) \geq 2 - \frac{n\, \sin(\eta - y)}{\sin\eta\, h_{n-1}(y)\, \sin y\,[1 - u_{n-1}(\eta, y)]}.$$

Now, from Lemma 5.4.(iv), we know that $\varphi_{n-1}(\eta, y) = \sin(\eta - y)/[1 - u_{n-1}(\eta, y)]$ assumes its maximum on $y_0 \in (0, \eta)$, and that is the only extremum of φ_{n-1} on $(0, \eta)$. Differentiating φ_{n-1} with respect to y, we have that $\partial\varphi_{n-1}/\partial y(\eta, y_0) = 0$ implies

$$\varphi_{n-1}(\eta, y_0) \;=\; \frac{\cos(\eta - y_0)}{\partial u_{n-1}/\partial y(\eta, y_0)}.$$

We maximize this on $[0,\eta]$, obtaining

$$\frac{\cos(\eta-y)}{\partial u_{n-1}/\partial y(\eta,y)} \leq \frac{1}{\lim_{y\searrow 0}\partial u_{n-1}/\partial y(\eta,y)} = \frac{n}{h_{n-1}(\eta)},$$

where we used Lemma 3.4. Thus,

$$\Psi_{n-1}(\eta,y) \geq 2 - \frac{n\,(n-1)}{\sin\eta\, h_{n-1}(\eta)\ \sin y\, h_{n-1}(y)},$$

and (17) follows from Lemma 5.4.(ii), which implies that $\sin y\, h_{n-1}(y)$ is decreasing.

Now, the result just proved and Lemma 5.4.(ii) imply that, if η is small enough, then both A_n and V_n are initially strictly increasing. Let $\eta_1(n)$ be the supremum of $\eta\in(0,\pi)$ such that if $0<\eta<\eta_1$, then ${A_n}'(\eta)>0$. Now, if $\eta\geq\eta_1(n)$, then $A_n(\eta)\ \geq\ 2\,\omega_{n-1}\int_0^{\eta_1}(\sin y)^{n-1}dy$. The proof is completed by letting $\eta_0(n)$ be the maximum of $0<\eta\leq\eta_1$ such that

$$A_n(\eta)\ \leq\ 2\omega_{n-1}\int_0^{\eta_1}(\sin y)^{n-1}dy\ \leq\ 2\omega_n.$$

■

6. Comments

The formulae for the derivatives of the volume functions given by Lemma 5.1 are quite difficult to deal with for $n\geq 3$. In [Ped] we present a few more analytical results about these formulae and show how these may be applied to obtain a proof of Theorem 3 for such n. The main problem is that the proof depends on very critical estimates on the values of both the volume formulae and their derivatives, which seem to depend on the particular case, i.e., the value of n, considered. The behavior of the functions A_n and V_n for the case $n=2$, presented here, seems to be typical for all n.

The generalized Delaunay curves of the unduloid type are solutions for the system $(\mathcal{S})$, but were not relevant for our study, since the generated hypersurfaces are not compact. In a joint work with M. Ritoré [PR], the isoperimetric problem in spaces of the form $\mathbb{S}^1(r)\times\mathbb{Q}^n(c)$ is solved ($\mathbb{Q}^n(c)$ is the simply connected space form of constant curvature $c=-1,0,1$). In those cases we need to study these hypersurfaces, which we show are unstable as CMC hypersurfaces (for any section equal to or longer than half a period). These questions are also related to the problem of describing the stable liquid drops between two parallel hyperplanes in $\mathbb{R}^n$, which was worked out for $\mathbb{R}^3$ by Athanassenas [Ath86] and Vogel [Vog87], independently. We reprove their result and extend it for higher dimensions.

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