Hopf-Zero Bifurcations of Reversible Vector Fields

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Abstract

In the present paper we study the dynamics of a class of reversible vector fields having eigenvalues $(0, \alpha i, -\alpha i)$ around their symmetric equilibria. We give a complete list of all normal forms for such vector fields, their versal unfoldings, and the corresponding bifurcation diagrams of codimensional one case. We also obtain some important conclusions on the existence of homoclinic and heteroclinic orbits, invariant tori and symmetric periodic orbits.

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1 Introduction

In this paper we study a class of time-reversible vector fields having the form

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where $F(x)$ is a smooth function, $F(0) = 0$. The vector field (1) is called time-reversible if there is a germ of a smooth involution $\phi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ ($\phi \circ \phi = id.$) satisfying the relation

$$F(\phi(x)) = -\phi'(x) \cdot F(x), \quad x \in \mathbb{R}^n, 0. \tag{2}$$

In particular, if the dimension of the fixed point set of ϕ, $S = Fix\{\phi\}$, is equal to k, then (1) is said to be of (n, k)-type reversibility. It is clear that $0 \leq k \leq n$.

The paper is primarily devoted to studying topological classification of $(3, 1)$-type time-reversible vector fields around their symmetric singular points. A singular point of a reversible vector field is called symmetric if it lies on the fixed point set S of the involution. In the present case S can be put into the form $S = \{x = 0,\ y = 0\}$, this is because any involution of a (3,1)-type reversible vector field is smoothly conjugated to $\phi_0(x, y, z) = (-x, -y, z)$, according to [11].

Since the 1-jet of a ϕ_0-reversible vector field takes the form $(\mu_1 + az)\partial/\partial x + (\mu_2 + bz)\partial/\partial y + (cx + dy)\partial/\partial z$ where $\mu_1, \mu_2,\ a, b, c$ and d are parameters, it follows that generically (3,1)-type reversible vector fields do not have symmetric singularities. Indeed one can see that the occurence of a symmetric singularity is in itself a codimension one phenomenon. In this paper, we shall restrict to those vector fields which have a symmetric singularity at the origin, i.e., $\mu_1 = \mu_2 = 0$, and consider the corresponding unfolded systems (for a study of families of vector fields depending on certain parameters, see [7, 8]). It is easy to see that the eigenvalues of these vector fields around the origin are either $(0, \pm\alpha)$ or $(0, \pm\alpha i)$ where α is a nonzero real parameter. We here treat the latter case only, justifying the term *Hopf-zero* shown in the title of the paper .

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We notice that some study on bifurcation and classification for general (not necessarily reversible) vector fields having eigenvalues $(0, \pm\alpha i)$ was given in [7, 13] among others. Systems having such eigenvalues but in other settings, say, divergent free systems, also attract attentions (see [2]). In particular, one sees easily that divergent free systems in $\mathbb{R}^3$ in generic case are topologically orbitally equivalent to one special type of reversible system (see X_4 in equation (11)). In [3] there is a brief exposition on the relations between various categories.

Even within the world of reversible vector fields, various types of reversible systems have been investigated, too. For example, in [15] all $(2, 1)$-type reversible systems are classified, in [4] $(2, 0)$-type, and in [10] $(3, 2)$-type. In [6], there is an exploration on (3,1)-type reversible vector fields having nilpotent linear part.

The basic motivation of the present paper comes from the study of the local dynamics of reversible vector fields around a critical point with a pair of purely imaginary eigenvalues and a zero one. Such kind of vector fields are very often encountered in applications. For detailed models from physics and hydrodynamics, please refer to [12].

The methods used in the paper come from [7, 5, 1, 9, 15, 10]. More precisely, the methods used in [15, 10] are to perform special changes of coordinates around the singularity. Thus the analysis of the full system can be transfered to a study of the contact between a general system and a codimension one submanifold in $\mathbb{R}^n$. These techniques can be generalized to the $(n, n-1)$-type. To make use of these techniques to the (3,1)-type reversible vector fields, we perform the S^1 reduction to change the original vector field to a vector field in the cylindrical polar coordinates in which the azimuthal coordinate takes the form $\dot{\theta} = \pm 1$. It follows that the topological classification of the original vector field can be reduced to the corresponding classification of the restricted planar systems to which the blow-up techniques can be applied (because the polar coordinates and the azimuthal one are decoupled). Moreover, since the reduced vector field is of the (2,1)-type, this makes possible to apply the techniques developed in [15].

On the other hand, since the planar system satisfies an inequality of the Lojasiewicz type, therefore by [5] the above 2-dimensional classification holds not only in C^0 equivalence but also in C^0 conjugacy category. We shall take this observation into account when considering the 3-dimensional *pull-back* classification.

To obtain the topological classification of the reduced system and consequently to recover the dynamical properties of the original system, however, certain difficulties and obstacles arise, either due to the reversibility or due to the reduction procedure. Firstly, the reversibility assumption generally imposes some constraints on the original system, that in turn results in a degeneracy of the reduced planar vector field. For example, the reduced system typically excludes all the cubic terms, that causes certain difficulties to obtain the reduced normal forms and to perform "blowing-ups". Another difficulty is more mathematical. In studying the reduced planar vector field, we have to take the reversibility of the original system into consideration in order to determine the dynamics patterns. For example, we have to face the center-focus problem in the unfolded planar system (see for instance case $X^+_{3,-,\frac{1}{4}}$ in Figure 2), and in such cases, we proceed as follows. Note that a pair of singularities in the unfolded system in fact exactly give rise to a symmetric periodic solution of the original system therefore the orbit cannot be an attractor or a repellor. It follows that the singularities must be of center type. This implies that the original system has a family of invariant tori. In a similar way, when the unfolded planar systems admit symmetric singularities of saddle type, we can draw conclusion that the original systems have homoclinic orbits (see, for instance, the case $X^-_{3,+,\frac{1}{4}}$ in Figure 2) or heteroclinic orbits (see, for instance, the case $X^-_{5,+}$ in

Figure 3). We emphasize that the phenomena described above are due to the reversibility of the original system. In other words, a general vector field (not necessarily reversible) with the same linear approximation may not enjoy such properties. Therefore the approach employed in the paper is not a simple application of [7, 2].

We remark that there is still another obstacle to go back to the original system from the planar one, i.e., the possible existence of flat terms. Indeed, flat terms generally may break the formal S^1 normal form symmetry. In this regard, in the present paper we shall depart from the normalized 3-dimensional vector fields, assuming that the objects under considerations take the normal forms as specified in the context.

The main results of this paper include a classification and a qualitative description of symmetric singularities occurring generically for 1-parameter families of (3,1)-type vector fields (the study on 2-parameter families of (3,1)-type vector fields are essentially related to the present case, due to the special unfolded forms, see Appendix). From the classification one draws some interesting conclusions on the existence of invariant tori, periodic orbits, homoclinic and heteroclinic orbits for vector fields considered. We also give a complete list of bifurcation diagrams.

The rest of the paper is arranged as follows: In section 2 we recall some preliminaries of reversible vector fields. We also give the normal forms on which our discussion relies. Section 3 includes the results of the generic case and the codimensional one case in the space of all (3,1)-type reversible vector fields having $(0, \pm\alpha i)$ eigenvalues at the symmetric singular point. The main part of this section lies in the characterization of the sets Σ_0 and Σ_1 (see Section 3) and in showing how useful a former result of Dumortier is (its application was not considered anywhere until now, as far as we know). We point out that the study of the bifurcation diagram of Σ_1 is hard and is essential, since the 2-parameter unfolding actually is nothing more than an addition of a constant in the first component of the vector fields. In section 4 we shall give a brief discussion on the genericity conditions as well as the proof of the results. Section 5 contains an analysis and a discussion on the existence of invariant tori and homoclinic orbits, whereas the bifurcation diagrams are illustrated in the last part of the paper. We remark that in these diagrams the arrows indicating the direction of the flow of the vector fields do not necessarily reflect the direction of that of the original system. This is because in deriving the orbital C^0 normal form we may multiply by a nonvanishing negative function. We also include an appendix commenting on the unfoldings where the singular point disappears.

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2 Preliminaries

In this section we recall some basic properties of reversible vector fields and introduce certain notations and definitions. In this part we shall also normalize the objects under considerations to some convenient forms from which our further study departs.

Let X be a ϕ-reversible system on $\mathbb{R}^n$. Then the following statements hold.

- The phase portrait of X is symmetric with respect to S, the fixed point set of the involution.
- Any periodic orbit γ is symmetric if and only if $S \bigcap \gamma \neq 0$. If $\gamma(t)$ is a solution of X then so is $\phi\gamma(-t)$.

- Any symmetric singular point or symmetric periodic orbit cannot be an attractor or a repellor.

The topological classification of the objects is based on the following definition.

Definition 2.1 *Two vector fields X and Y at their singularities are said to be C^0 conjugated in neighborhoods of the singular points if there is a homeomorphism carrying one singular point into the other and conjugating the local phase flows of the systems at these singular points. They are called orbitally C^0 equivalent if there is a homeomorphism mapping the local phase curves of X into that of Y.*

Definition 2.2 *We say that vector field X defined on $\mathbb{R}^n, 0$ satisfies an inequality of Lojasiewicz type if there exists an integer $k > 0$ and $c > 0$ such that $\|X(x)\| \geq c\|x\|^k$ for all x in a neighborhood of the singular point.*

The following statements can be deduced from [5].

Theorem 1 *The singularity of any smooth vector field defined on $\mathbb{R}^2, 0$ satisfying an inequality of Lojasiewicz type and having a characteristic orbit is finitely determined in C^0 conjugacy. Moreover, if two finitely determined singularities on the plane are C^0-equivalent, then they are also C^0 conjugated.*

In what follows we shall consider the normalization of the (3,1)-type vector fields. Let X be a (3,1)-type reversible vector field having the eigenvalues $(0, \pm\alpha i)$. Assume that it takes the form.

$$X: \begin{cases} \dot{x} = a_0 z + a_1 x^2 + a_2 y^2 + a_3 z^2 + a_4 xy + \cdots \\ \dot{y} = b_0 z + b_1 x^2 + b_2 y^2 + b_3 z^2 + b_4 xy + \cdots \\ \dot{z} = c_0 x + d_0 y + c_1 xz + c_2 yz + \cdots, \end{cases} \tag{3}$$

where a_0, b_0, c_0 and d_0 are parameters satisfying $a_0 c_0 + b_0 d_0 < 0$. It is easy to see that under a linear change of coordinates (together with a multiplication of a constant, if necessary), one can always reduce the linear part of the vector field (3) to the form

$$j^1 X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

therefore throughout the paper, we assume that the linear part of (3) has been normalized to the above form.

System (3) can be put into resonant normal form (see [16]). To realize so, we rewrite the involution ϕ_0 in the form $\phi_0(x, \xi) = (-x, -\bar{\xi})$, where $\xi = y + iz$. The resonant normal form means that X takes the form

$$\dot{x} = f(x, r^2), \quad \dot{\xi} = \xi g(x, r^2), \tag{4}$$

where $r^2 = \xi\bar{\xi}$, f is a real function and g a function having complex coefficients. It is easy to see that the reversibility of X leads to the following equalities:

$$f(x, r^2) = f(-x, r^2), \quad \overline{\xi g(x, r^2)} = -\bar{\xi} g(-x, r^2). \tag{5}$$

It follows from these relations that f is even in x and g can be decomposed into the form $g(x, r^2) = xg_1(x^2, r^2) + i(\alpha + g_2(x^2, r^2))$, where $\alpha \neq 0$. In other words, X has the following form.

$$\begin{cases} \dot{x} = f(x^2, r^2) \\ \dot{y} = z(\alpha + g_2(x^2, r^2)) + yxg_1(x^2, r^2) \\ \dot{z} = -y(\alpha + g_2(x^2, r^2)) + zxg_1(x^2, r^2). \end{cases} \tag{6}$$

In the following we shall use the symbol $\mathscr{X}$ to denote the space the germs of C^r vector fields having the form (6) and endowed with the C^r-topology, $r > 3$.

To further simplify (6), we first multiply it by a function $h = 1/(\alpha + g_2(x^2, r^2))$ and then put the multiplied system into cylindrical polar system $\dot{x} = \tilde{f}(x^2, r^2)$, $\dot{r} = xrg(x^2, r^2)$, $\dot{\theta} = \pm 1$. Namely, we have the following expansion.

$$X: \quad \begin{cases} \dot{x} = \delta_1 x^2 + \delta_2 r^2 + \delta_3 x^4 + \delta_4 r^4 + \alpha x^2 r^2 + \cdots \\ \dot{r} = cxr + \beta x^3 r + \gamma x r^3 + \cdots \\ \dot{\theta} = \pm 1, \end{cases} \tag{7}$$

where $\delta_1 = \{0, 1\}, \delta_2 = \{0, \pm 1\}$, $\delta_3, \delta_4, \alpha, \beta, \gamma$ and c are parameters depending on a_i, b_i, c_i, d_i of (3). Note that this normal form contains no θ-dependent terms. Thus by treating the azimuthal coordinate as time, we arrive at the reduced form

$$\tilde{X}: \quad \begin{cases} \dot{x} = \delta_1 x^2 + \delta_2 r^2 + \delta_3 x^4 + \delta_4 r^4 + \alpha x^2 r^2 + \cdots \\ \dot{r} = cxr + \beta x^3 r + \gamma x r^3 + \cdots. \end{cases} \tag{8}$$

Convention: *In the paper by saying a 3-dimensional $X \in \mathscr{X}$ is C^0 orbitally equivalent to $\tilde{X}$ of the form (8), we mean that X has been processed in the above way to (7) where the azimuthal coordinate is omitted. Namely, given a vector field $X \in \mathscr{X}$ we denote by $\tilde{X}$ the corresponding planar normal form of X. We denote by $\tilde{\mathscr{X}}$ the set of $\tilde{X}$ with the associated $X \in \mathscr{X}$. Therefore there is a 1-1 correspondence between $\mathscr{X}$ and $\tilde{\mathscr{X}}$ via (7) and (8).*

3 Statement of the Results

In terms of (8), we introduce the following notation.

$$\begin{aligned} \Sigma_0^1 = & \ \{X \in \mathscr{X} : \delta_1 \neq 0,\ \delta_2 = 1, c \neq 1\}, \\ \Sigma_0^2 = & \ \{X \in \mathscr{X} : \delta_1 \neq 0,\ \delta_2 = -1, c \neq 0, 1\}, \\ \Sigma_0 = & \ \Sigma_0^1 \bigcup \Sigma_0^2. \end{aligned} \tag{9}$$

Then the algebraic relations in Σ_0 characterize the genericity conditions whose derivation will be given in the coming section. We have the following statement.

Theorem 2 *Vector fields orbitally equivalent to those of Σ_0 form an open and dense set in $\mathscr{X}$.*

In generic case, the classification and the normal forms of such systems are known, see, for example, [14, 7]. For the sake of completeness, here we recall these results.

Theorem 3 *The set Σ_0 can be divided into the following five subsets $\Sigma_0(i)$, $(i = 1,2,3,4,5)$, such that any two vector fields belonging to the same subset are C^0 orbitally equivalent:*

$$\begin{array}{rl} \Sigma_0(1) = & \{X \in \mathcal{X} : \ \delta_1 = 1,\ \delta_2 = -1,\ c > 1\}, \\ \Sigma_0(2) = & \{X \in \mathcal{X} : \ \delta_1 = 1,\ \delta_2 = 1,\ c < 1\}, \\ \Sigma_0(3) = & \{X \in \mathcal{X} : \ \delta_1 = 1,\ \delta_2 = -1,\ 0 < c < 1\}, \\ \Sigma_0(4) = & \{X \in \mathcal{X} : \ \delta_1 = 1,\ \delta_2 = -1,\ c < 0\}, \\ \Sigma_0(5) = & \{X \in \mathcal{X} : \ \delta_1 = 1,\ \delta_2 = 1,\ c > 1\}. \end{array} \tag{10}$$

Moreover, in each set, a representative vector field $Y \in \mathcal{X}$ is C^0 orbitally equivalent to one of the normal forms

$$\begin{array}{ll} \tilde{X}_1: & \tilde{X} = (x^2 - r^2, 2xr), \\ \tilde{X}_2: & \tilde{X} = (x^2 + r^2, \frac{1}{2}xr), \\ \tilde{X}_3: & \tilde{X} = (x^2 - r^2, \frac{1}{2}xr), \\ \tilde{X}_4: & \tilde{X} = (x^2 - r^2, -xr), \\ \tilde{X}_5: & \tilde{X} = (x^2 + r^2, 2xr). \end{array} \tag{11}$$

The corresponding 3-dimensional normal forms can be given by $X_i = (\tilde{X}_i, \pm 1)$. On the other hand, due to Theorem 1, the C^0 orbital equivalence in the above theorem can be improved to C^0 conjugacy when only planar singularities are involved.

When the genericity conditions are violated we obtain degenerated vector fields. In particular, if one (and only one) of the following conditions

$$\delta_1 = 0, \quad \delta_2 = 0, \quad c = 0, \quad \delta_1 = c = 1, \tag{12}$$

is satisfied and at the same time the higher order terms are generic, then we have the codimensional 1 singularity of X whose classification, as a rule, depends on the higher order terms. In fact, we shall show that its classification is related to the parameters $\delta_3, \delta_4, \alpha, \beta, \gamma$ and c.

We introduce the following sets to describe the topological types of the singularities of the 3-dimensional systems. The topological invariance of the algebraic conditions characterizing these sets will be explained in the coming section.

$$\begin{array}{l} \Sigma_1(1.1) = \{X \in \mathcal{X} : \ \delta_1 = 0,\ \delta_2 = 1,\ \delta_3 = 1,\ c < 0\}, \\ \Sigma_1(1.2) = \{X \in \mathcal{X} : \ \delta_1 = 0,\ \delta_2 = 1,\ \delta_3 = -1,\ c < 0\}, \\ \Sigma_1(2.1) = \{X \in \mathcal{X} : \ \delta_1 = 0,\ \delta_2 = 1,\ \delta_3 = 1,\ c > 0\}, \\ \Sigma_1(2.2) = \{X \in \mathcal{X} : \ \delta_1 = 0,\ \delta_2 = 1,\ \delta_3 = -1,\ c > 0\}, \\ \Sigma_1(3.1) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = 1,\ c < 0\}, \\ \Sigma_1(3.2) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = -1,\ c < 0\}, \\ \Sigma_1(3.3) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = 1,\ 0 < c < \frac{1}{2}\}, \\ \Sigma_1(3.4) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = -1,\ 0 < c < \frac{1}{2}\}, \\ \Sigma_1(3.5) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = 1,\ \frac{1}{2} < c < 1\}, \\ \Sigma_1(3.6) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = -1,\ \frac{1}{2} < c < 1\}, \\ \Sigma_1(3.7) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = 1,\ c > 1\}, \\ \Sigma_1(3.8) = \{X \in \mathcal{X} : \ \delta_2 = 0,\ \delta_1 = 1,\ \delta_4 = -1,\ c > 1\}, \\ \Sigma_1(4.1) = \{X \in \mathcal{X} : \ \delta_1 = c = 1,\ \delta_2 = 1\}, \\ \Sigma_1(4.2) = \{X \in \mathcal{X} : \ \delta_1 = c = 1,\ \delta_2 = -1\}, \\ \Sigma_1(5.1) = \{X \in \mathcal{X} : \ c = 0,\ \delta_1 = 1,\ \delta_2 = -1,\ \beta + \gamma > 0\}, \\ \Sigma_1(5.2) = \{X \in \mathcal{X} : \ c = 0,\ \delta_1 = 1,\ \delta_2 = -1,\ \beta + \gamma < 0\}. \end{array} \tag{13}$$

We let $\mathscr{X}_1 = \mathscr{X} - \Sigma_0$ and denote by Σ_1 the union of the above 16 sets, i.e.,

$$\Sigma_1 := \bigcup \Sigma_1(i,j). \tag{14}$$

In this paper we prove the following

Theorem 4 *Vector fields which are orbitally equivalent to those of Σ_1 form an open and dense set in $\mathscr{X}_1$. Σ_1 is a codimensional 1 embedded submanifold of $\mathscr{X}$.*

As to the singularities of generic 1-parameter families of reversible vector fields of $\mathscr{X}$ we have the following results.

Theorem 5 *1) Two vector fields X and Y in $\mathscr{X}_1$ are C^0 orbitally equivalent if and only if they belong to the same subset $\Sigma_1(i,j)$ of (13).*

2) Any one-parameter family $\tilde{X}^\lambda$, with $X^0 \in \Sigma_1$, in generic case (transversal to Σ_1) is C^0 orbitally equivalent to one of the following 16 normal forms.

$$\begin{array}{rl} \tilde{X}^\lambda_{1,\pm}: & (\lambda x^2 + r^2 \pm x^4, -xr), \\ \tilde{X}^\lambda_{2,\pm}: & (\lambda x^2 + r^2 \pm x^4, xr), \\ \tilde{X}^\lambda_{3,\pm,c}: & (x^2 + \lambda r^2 \pm r^4, cxr), \\ \tilde{X}^\lambda_{4,\pm}: & ((1+\lambda)x^2 \pm r^2, xr), \\ \tilde{X}^\lambda_{5,\pm}: & (x^2 - r^2, \lambda xr \pm xr^3), \end{array} \tag{15}$$

where c takes one of the following values $\{-1, \frac{1}{4}, \frac{3}{4}, 2\}$, and λ is the unfolding parameter.

The above theorem together with Theorem 1 lead to the following

Corollary 3.1 *Any planar singularity $\tilde{X} \in \tilde{\mathscr{X}}$, where $X \in \Sigma_1$, in generic case is C^0 conjugated to one of the normal form (15) where $\lambda = 0$.*

Note that the normal form $\tilde{X}^\lambda$ is not necessarily continuous with respect to the parameter. Note also that the second statement of the theorem does not imply that the corresponding 3-dimensional unfoldings $X^\lambda = (\tilde{X}^\lambda, 1)$ are C^0 stable in the space of all one-parameter families of vector fields of $\mathscr{X}$. In other words, the normal forms $X^0 = (\tilde{X}^0, \pm 1)$, where $\tilde{X}^0$ is from (15) with $\lambda = 0$, only give a topological classification of singularities of 3-dimensional systems, not the classification of the unfolded systems.

4 Proof of The Main Theorem

4.1 Comments on the generic case

It is known from [14] that the planar vector field (8) in generic case is jet-2 determined with respect to C^0 conjugacy. In what follows we shall verify that the algebraic conditions in (9) coincide with the genericity conditions of this system in our terminology. In other words, if one of the conditions in (12) is violated then (8) is not generic. To this regard, consider the 2-jet of

$\tilde{X}$: $\dot{x} = \delta_1 x^2 + \delta_2 r^2$, $\dot{r} = cxr$, where $\delta_1 = 0, 1$, and $\delta_2 = 0, \pm 1$. This system, after once blowing-up under $x = \rho\cos\theta, r = \rho\sin\theta$, can be put into the following form

$$\begin{cases} \dot{\rho} = \rho(\delta_1 \cos^3\theta + (\delta_2 + c)\cos\theta\sin^2\theta) \\ \dot{\theta} = -\delta_2 \sin^3\theta + (c - \delta_1)\sin\theta\cos^2\theta. \end{cases} \tag{16}$$

Note that generically $\delta_2(c - \delta_1) \neq 0$. If $\delta_2(c - \delta_1) < 0$ then system (16) has only one singular point $(0, 0)$. In this case it is easy to see that the blown-up system is hyperbolic if and only if $\delta_1 \neq 0$. If $\delta_2(c - \delta_1) > 0$ then system (16) has three singular points $(0, 0)$, $(0, \pm arctan\sqrt{\frac{c-\delta_1}{\delta_2}})$. A little calculation shows that in this case system (16) is hyperbolic at all these singular points if and only if $c \neq 0$. Thus we verified the genericity conditions specified in (9).

4.2 Codimension 1 case

From the previous discussion we know that if one of the conditions (12) is broken then the vector field is degenerated. In this part we first precise the conditions under which the system is of codimension 1. This means that certain genericity conditions should be imposed on the coefficients of other terms. Obviously, we need only to consider the following four cases.

(1). $\delta_1 = 0,\ \delta_2 \neq 0,\ c \neq 0$;

(2). $\delta_2 = 0,\ \delta_1 \neq 0,\ c \neq 0, 1$;

(3). $\delta_1 = c = 1,\ \delta_2 \neq 0$;

(4). $c = 0,\ \delta_1 \neq 0,\ \delta_2 \neq 0$;

In what follows we only treat case (1) in more details. In the remaining cases we shall only point out the main differences.

Taking case (1), i.e., $\delta_1 = 0$, $\delta_2 = \pm 1, c \neq 0$, we are interested in the following points: the genericity conditions imposed on the higher order terms, the unfolding of the system, and the bifurcation of the unfolded system. Note that in this case we can put $\delta_2 = 1$ and $\delta_3 \in \{0, 1, -1\}$. This can be done by scaling x, and thus time is preserved (recall that in all the 2-dimensional cases, the discussion should be in the C^0 conjugacy setting, not the orbital equivalence setting, because the azimuthal coordinate has been taken as time). We put (8) into the form

$$\begin{cases} \dot{x} = r^2 + \delta_3 x^4 + \delta_4 r^4 + \alpha x^2 r^2 + \cdots \\ \dot{r} = cxr + \beta x^3 r + \gamma x r^3 + \cdots, \end{cases} \tag{17}$$

where $c \neq 0$ and the dots denote the terms of degrees higher than 4.

It is straightforward to check that (17) is of codimension 1 in $\tilde{\mathfrak{X}}$ if and only if $\delta_3 \neq 0$. If $\delta_3 \neq 0$ then according to signs of δ_3 and c we can divide (17) into four different cases: $c > 0, \delta_3 = \pm 1$; $c < 0, \delta_3 = \pm 1$. By performing blowing-ups, one can show that these four cases are topologically different from each other.

Below we prove that we can choose $\tilde{X}^{\lambda}_{1,\pm}$ and $\tilde{X}^{\lambda}_{2,\pm}$ (see (15)) as the corresponding unfoldings.

First we clarify the equivalence of two 1-parameter families of vector fields. We say $X^{\lambda} \sim Y^{\mu}$, if there exists a function h: $(-\epsilon, \epsilon) \to (-\epsilon, \epsilon)$, where ϵ is small and $h(\lambda) = \mu$, such that X^{λ} is conjugated to Y^{μ}.

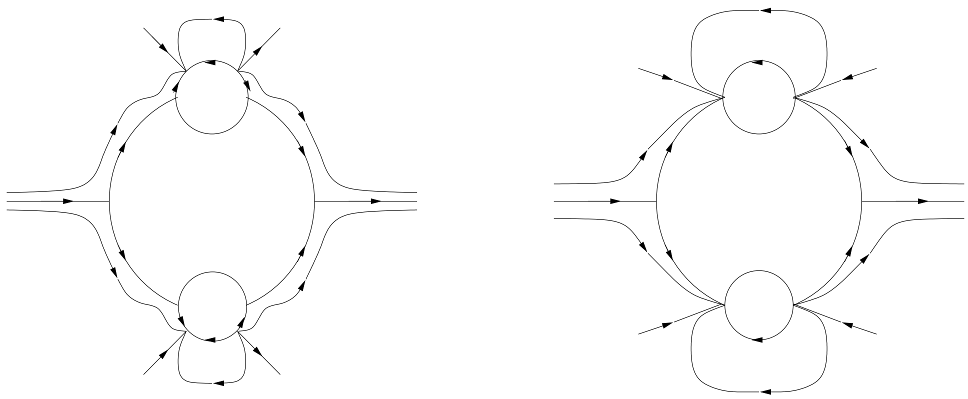

Figure 1: Blow-ups of cases $\tilde{X}^0_{3,-,\frac{1}{4}}$ and $\tilde{X}^0_{3,-,\frac{3}{4}}$

Let $\tilde{X}_\mu$ be an unfolding such that $\tilde{X}_0$ is in one of the first four sets of (13). Assume that, say, $\tilde{X}_{\mu_0} \in \Sigma_1(1,1)$. Then $\delta_1(\mu_0) = 0$, and for any μ_1 and μ_2 such that $(\mu_1 - \mu_0)(mu_2 - \mu_0) < 0$, one has $\tilde{X}_{\mu_1}, \tilde{X}_{\mu_2} \in \Sigma_0$, and $\tilde{X}_{\mu_1}$ is topologically different from $\tilde{X}_{\mu_2}$. This fact implies that for $\mu_1 > \mu_0$ (resp. $\mu_1 < \mu_0$) there are only two possibilities: $\delta_1 > 0$ (resp. $\delta_1 < 0$), or $\delta_1 < 0$ (resp. $\delta_1 > 0$). Take the first case (the other possibility can be treated exactly in the same way). Then for any $\mu > \mu_0$ one has $\tilde{X}_\mu \in \Sigma_0(i)$ and $\mu < \mu_0$ one has $\tilde{X}_\mu \in \Sigma_0(j)$, where $i \neq j$. Considering $\mu(\lambda) = \lambda - \mu_0$ we get the unfolding $\tilde{X}^\lambda_{1,+}$.

As to the validity of Theorem 4 it is sufficient to notice that Σ_1 given by (14) is a submanifold of codimensional 1 of $\mathscr{X}$. The proof of this fact is omitted here.

Now let us briefly discuss the remaining cases. In case (2), i.e., $\delta_2 = 0,\ \delta_1 = 1,\ c \neq 0, 1$, we can rescale x and r such that δ_4 takes one of the values $0, \pm 1$. In this case, when blowing-up the corresponding system, one can see that if $c = \frac{1}{2}$ then the vector field considered shall have higher codimension. Correspondingly, we have eight subcases due to all the possible combinations between $\delta_4 = \pm 1$ and c lies in $(-\infty, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$ and $(1, \infty)$. Since c takes values from these four sets, consequently, there is no modality in the classification. The unfolding is given by $\tilde{X}^\lambda_{3,\pm,c}$ (see (15)).

Remark 4.1 *The phase portraits of the cases $\tilde{X}^\lambda_{3,-,\frac{1}{4}}$ and $\tilde{X}^\lambda_{3,-,\frac{3}{4}}$ seem to be identical in the bifurcation figures at the end of the paper. Their blowing-ups at $\lambda = 0$, however, show that these two cases are topologically different. The two blow-ups are shown in figure 1.*

Case (3), i.e., $\delta_1 = c = 1, \delta_2 \neq 0$, can be treated in a similar way. The unfoldings are given by $\tilde{X}^\lambda_{4,\pm}$ (see (15)).

In case (4), i.e., $c = 0,\ \delta_1 \neq 0,\ \delta_2 \neq 0$, we can only consider the higher order terms of $\dot{r}$. The essential difference from the previous cases lies only in showing the invariance of $(\beta + \gamma)$, and this can be proved by blowing-up the system. Consequently, there are two subcases, according to the signs of $(\beta + \gamma)$ The corresponding unfoldings are given by $\tilde{X}^\lambda_{5,\pm}$ (see (15)).

5 Symmetric periodic orbits, invariant tori, homoclinic and heteroclinic orbits

With the help of the classification of the planar systems obtained in the previous sections, we can give an analysis of the 3-dimensional systems. In this part of the paper, we shall discuss the related dynamical properties of the 3-dimensional vector fields, with the emphasis on the existence of symmetric periodic orbits, homoclinic orbits, invariant tori and other important properties.

5.1 Symmetric periodic orbits and invariant tori

From the bifurcation diagrams we see that in the unfolded systems $\tilde{X}^-_{3,+,-1}$, $\tilde{X}^+_{3,-,\frac{1}{4}}$, $\tilde{X}^+_{3,-,\frac{3}{4}}$, and $\tilde{X}^+_{3,-,\frac{3}{2}}$ the corresponding blown-up systems are of center-focus type. In each case, there are a pair of singularities in the unfolded system, and we face the center-focus problem. Since these two symmetric singularities of the planar vector field in fact correspond to a symmetric periodic orbit of the original system. Consequently, the orbit cannot be an attractor or a repellor (see Section 2.1). This implies that the type of the planar unfolded system is a center. In other words, we have two families of circles centered around the singularities. Therefore the corresponding 3-dimensional system has a family of invariant tori which link these two families of tori. It is clear that the 3-dimensional unfolded system has no singularities on these invariant tori, and the *pull-back* system $X = (\tilde{X}_\cdot, \pm 1)$, where $\tilde{X}_\cdot$ takes the 2-dimensional normal form with $\lambda = 0$, only gives a classification of 3-dimensional singularities of the systems, not the 3-dimensional unfoldings.

The following cases also admit symmetric periodic orbits: $\tilde{X}^-_{3,+,\frac{1}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{2}}$ $\tilde{X}^+_{3,-,-1}$, $\tilde{X}^-_{5,+}$ and $\tilde{X}^+_{5,-}$.

5.2 Homoclinic and Heteroclinic orbits

From the bifurcation diagram we see that in the cases $\tilde{X}^-_{3,+,\frac{1}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{4}}$, $\tilde{X}^-_{3,+,\frac{3}{2}}$ and $\tilde{X}^+_{3,-,-1}$ the unfolded systems also have a pair of symmetric singularities. These singularities, however, are of saddle type. Moreover, one can see that except the case $\tilde{X}^+_{3,-,-1}$ in all the other cases in a neighborhood of γ there exist a family of homoclinic orbits tending to 0.

The other cases where homoclinic orbits exist are $\tilde{X}^+_{1,+}$, $\tilde{X}^0_{1,-}$, $\tilde{X}^+_{1,-}$, $\tilde{X}^-_{3,-,\frac{1}{4}}$, $\tilde{X}^0_{3,-,\frac{1}{4}}$, $\tilde{X}^-_{3,-,\frac{3}{4}}$, $\tilde{X}^0_{3,-,\frac{3}{4}}$, $\tilde{X}^\lambda_{3,-,\frac{3}{2}}$, $\tilde{X}^\lambda_{4,-}$, $\tilde{X}^-_{5,+}$ and $\tilde{X}^0_{5,+}$

In a similar way, we know that in the cases $\tilde{X}^-_{5,+}$ and $\tilde{X}^+_{5,-}$ the unfolded planar problem has two pairs of symmetric singularities. For example, from the case $\tilde{X}^+_{5,-}$ we deduce that for positive λ, the unfolding parameter, $\tilde{X}^\lambda_{5,-} \in \tilde{\mathfrak{X}}$ has a loop connecting the two periodic orbits and the symmetric equilibrium point P_λ. Moreover, there is at P_λ a 1-parameter family of heteroclinic orbits.

In the case $\tilde{X}^-_{1,-}$ there are also a family of heteroclinic orbits.

The 3-dimensional systems corresponding to $\tilde{X}^+_{5,-}$ possess homoclinic as well as heteroclinic orbits.

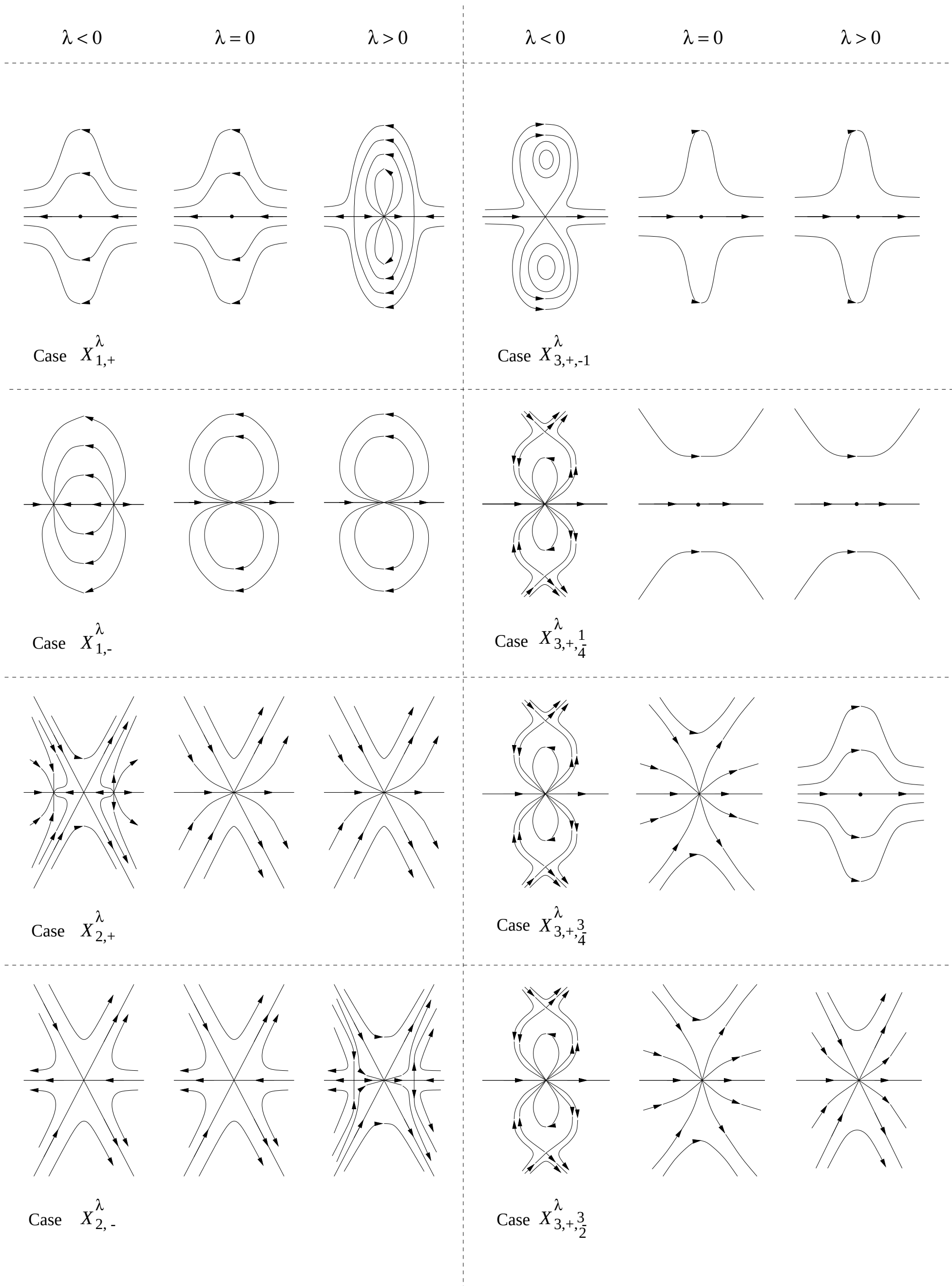

Figure 2: Bifurcation Diagrams I

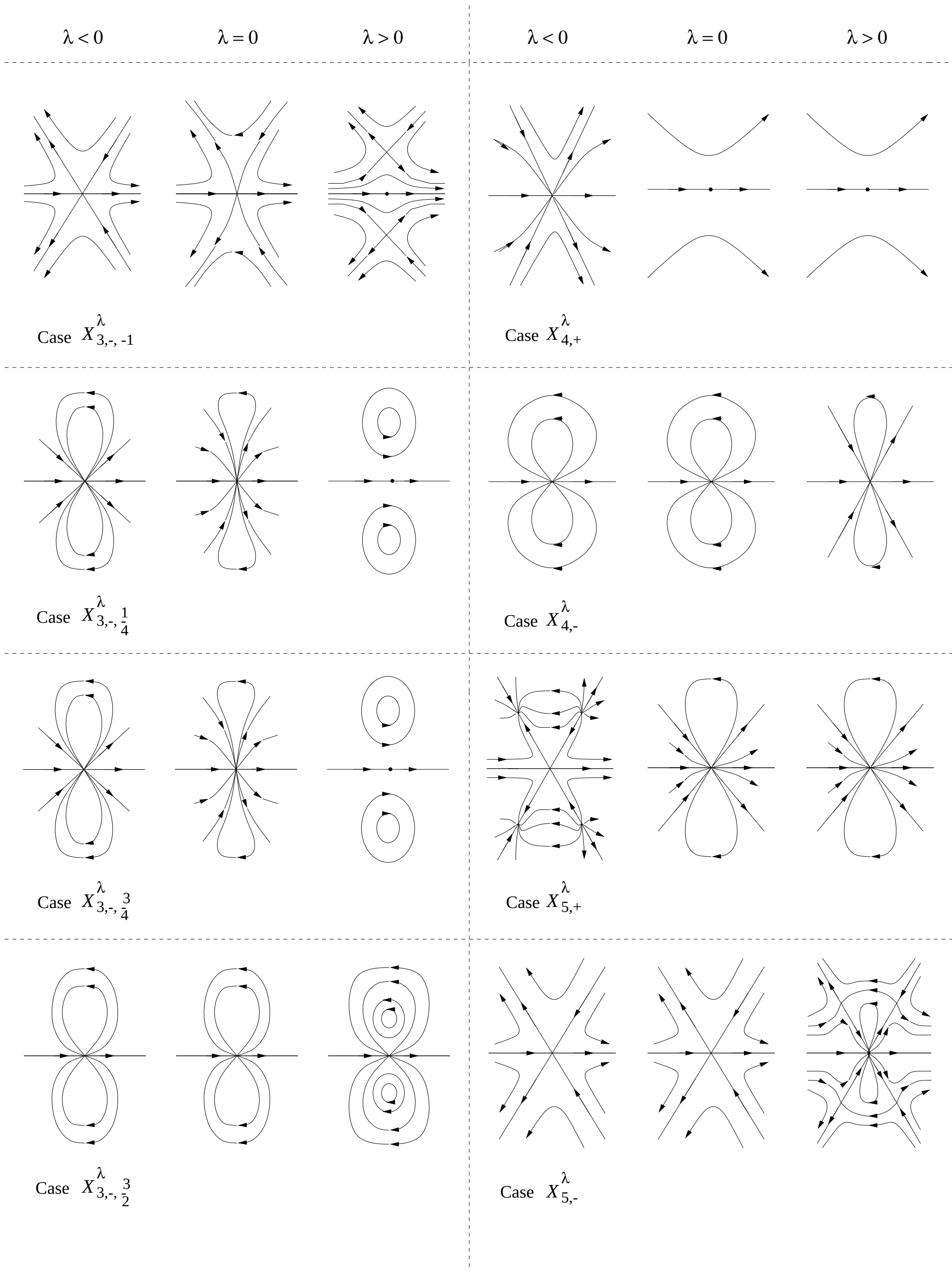

Figure 3: Bifurcation Diagrams II

6 Appendix

In this appendix, we give a brief discussion on the unfoldings of (3,1)-type reversible vector fields where the singular point disappears. To illustrate the main ideas, we shall consider, in form of examples, one possibility in each case. The other possibilities can be treated in a similar way. For the terminology we use here, please refer to [15].

Example A.1 Let X_α be a (3,1)-type reversible vector field given by

$$X_\alpha = (\alpha + x^2 - (y^2 + z^2), -z + xy, y + xz).$$

Then in the cylindrical polar coordinates it takes the form

$$X_\alpha = (\alpha + x^2 - r^2, xr, 1), \quad r \geq 0.$$

Thus the associated planar vector field is

$$\tilde{X}_\alpha = (\alpha + x^2 - r^2, xr), \quad r \geq 0,$$

which is (2,1)-type reversible with respect to $\varphi(x, r) = (-x, r)$. By [15] we perform the coordinate change $u = x^2$, $v = r$ in the region $\{x \geq 0\}$ to transform $\tilde{X}_\alpha$ to the auxiliary vector field

$$F_\alpha(u, v) = (\alpha + u - v^2, \frac{v}{2}), \quad u \geq 0, v \geq 0.$$

Denote $f(u, v) = u$. Then the following facts about F_α hold.

The critical points of F_α: If $\alpha > 0$, then F has no critical point; If $\alpha \leq 0$, then F has one critical point $P_0 = (-\alpha, 0)$ which is symmetric iff. $\alpha = 0$.

The boundary singularities of F_α: The boundary singularities are given by $u = 0$ and $v^2 = \alpha$, observing that $F_\alpha f = 0$ iff. $\alpha + u - v^2 = 0$. Thus we have the following two cases.

(i) $\alpha > 0$: In this case there is one external quadratic tangency point $P = (0, \sqrt{\alpha})$ between F_α and $u = 0$. Therefore $\tilde{X}_\alpha$ has a symmetric critical point of center type, which precisely means that X_α has a periodic orbit of center type (this vector field has no critical points).

(ii) $\alpha < 0$: In this case there is no tangency point between the vector field and $u = 0$. Thus $\tilde{X}_\alpha$ has an asymmetric critical point of nodal type, which equivalently means that X_α has two critical points of nodal type (an atractor and a repeller).

Example A.2 Let $X_{\alpha,\lambda}$ be a (3,1)-type reversible vector field given by

$$X_{\alpha,\lambda} = (\alpha + \lambda x^2 - (y^2 + z^2) + x^4, -z - xy, y - xz)$$

Then in the cylindrical polar coordinates it takes the form

$$X_{\alpha,\lambda} = (\alpha + \lambda x^2 - r^2 + x^4, -xr, 1), \quad r \geq 0.$$

Thus the associated planar vector field is

$$\tilde{X}_{\alpha,\lambda} = (\alpha + \lambda x^2 - r^2 + x^4, -xr), \quad r \geq 0.$$

As in Example A.1, we denote $f(u, v) = u$ and perform the same change of coordinates to obtain the corresponding auxiliar vector field

$$F_{\alpha,\lambda}(u, v) = (\alpha + \lambda u - v^2 + u^2, -\frac{v}{2}), \quad u \geq 0, v \geq 0.$$

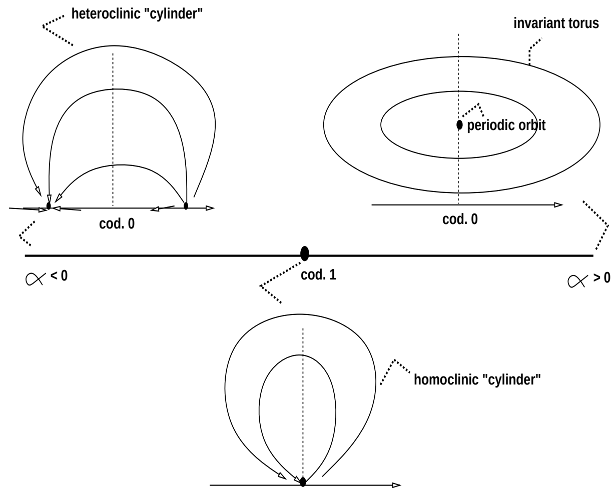

Figure 4: Bifurcation diagram of $X_0 = (x^2 - (y^2 + z^2), -z + xy, y + xz)$

Below we describe the singularity distribution of $F_{\alpha,\lambda}$, according to all possible values of α and λ. These cases can be easily located in Fig. 5, where the horizontal axis is $\alpha = 0$, the vertical axis is $\lambda = 0$, and the curve stands for $\Delta := \lambda^2 - 4\alpha = 0$.

The critical points of $F_{\alpha,\lambda}$:

(i) If $\Delta < 0$, then $F_{\alpha,\lambda}$ has no critical points in the region $\Omega := \{u \geq 0, v \geq 0\}$.

(ii) Let $\Delta \geq 0$:

(a) If $\alpha < 0$, then $F_{\alpha,\lambda}$ has one saddle critical point in Ω: $P_{\alpha,\lambda} = ((-\lambda + \sqrt{\Delta})/2, 0)$;

(b) If $\alpha > 0$ and $\lambda < 0$, then $F_{\alpha,\lambda}$ has two critical points in Ω: $P_{\alpha,\lambda} = ((-\lambda \pm \sqrt{\Delta})/2, 0)$ (one is saddle and the other is attractor node);

(c) If $\alpha > 0$ and $\lambda > 0$, then $F_{\alpha,\lambda}$ has no critical point in the region;

(d) If $\alpha = 0$, then $F_{\alpha,\lambda}$ has two critical points in Ω: $P_{0,\lambda} = (-\lambda, 0)$ and $P_0 = (0,0)$ provided $\lambda < 0$; if $\lambda > 0$, then P_0 is the unique critical point of $F_{\alpha,\lambda}$.

(e) If $\Delta = 0$ and $\lambda < 0$, then $F_{\alpha,\lambda}$ has a unique critical point $P_{\alpha,\lambda} = (-\lambda/2, 0)$ in Ω, which is a degenerate saddle-node (see Figure 5).

The boundary singularities of $F_{\alpha,\lambda}$

Note that $F_{\alpha,\lambda} f = 0$ iff. $\alpha + \lambda u - v^2 + u^2 = 0$. Therefore the boundary singularity characterized by $u = 0$ and $v^2 = \alpha$ occurs only when $\alpha > 0$ (the case $\alpha = 0$ is not considered here) and is given by $Q_{\alpha,\lambda} = (0, \sqrt{\alpha})$. It follows that $F^2_{\alpha,\lambda} f(Q_{\alpha,\lambda}) = (\lambda + 1)\alpha > 0$. Thus if $\alpha > 0$, then $Q_{\alpha,\lambda} = (0, \sqrt{\alpha})$ is an external quadratic tangency point between $F_{\alpha,\lambda}$ and $u = 0$, whereas if $\alpha < 0$, then there is no tangency point between the vector field and $u = 0$.

Remark 6.2 *All the eigenspaces of all critical points treated here are transverse to the lines* $\{u = 0\}$ *and* $\{v = 0\}$.

Now, as above, we can derive the bifurcation diagram of $X_{\alpha,\lambda}$ which it is illustrated in Figure 5.

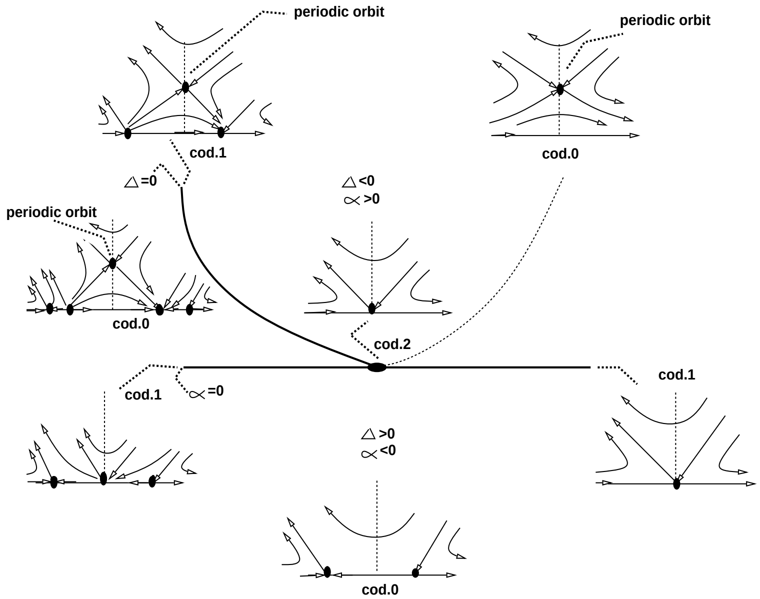

Figure 5: Bifurcation diagram of $X_0 = (-(y^2 + z^2) + x^4, -z - xy, y - xz)$

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