RELATÓRIO DE PESQUISA 1994

LEVEL-CONVERGENCE AND FUZZY INTEGRAL

H. Román-Flores, A. Flores-Franulic
and
Rodney C. Bassanezi, Marko A. Rojas-Medar

Junho RP 27/94

INSTITUTO DE MATEMÁTICA
ESTATÍSTICA E CIÊNCIA DA COMPUTAÇÃO

UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS

ABSTRACT – In this paper we define the level-convergence of positive, measurable functions on a fuzzy measure space. We study some of the properties of this convergence and its connections with other classic kinds of convergence, and give conditions for the continuity of fuzzy integral with respect to the level-convergence.

IMECC – UNICAMP
Universidade Estadual de Campinas
CP 6065
13081-970 Campinas SP
Brasil

Junho – 1994

LEVEL-CONVERGENCE AND FUZZY INTEGRAL

H. Román-Flores, A. Flores-Franulic
Departamento de Matemática, Facultad de Ciencias
Universidad de Tarapacá , Casilla 7D Arica, Chile
and
Rodney C. Bassanezi, Marko A. Rojas-Medar
IMECC, UNICAMP
CP 6065, 13081-970, Campinas, SP, Brazil

ABSTRACT –In this paper we define the level-convergence of positive, measurable functions on a fuzzy measure space. We study some of the properties of this convergence and its connections with other classic kinds of convergence, and give conditions for the continuity of fuzzy integral with respect to the level-convergence.

Keywords: Fuzzy measure space, Fuzzy integral, Levelwise convergence.

1. INTRODUCTION

The fuzzy integral is a mathematical tool that has been shown highly efficient in the treatment of certain problems of non-deterministic nature, like prediction and decision in processes with presence of diffuse (fuzzy) information. Some of these applications are described with detail in [7].

Thus, it is important to study the properties of those integrals, and, in particular, the continuity of the fuzzy integral has been exhaustly studied in the last years.

This work is partially supported by DIEXA-Universidad de Tarapacá through Proyect 4742-94.

Ralescu and Adam [3] proved theorems of continuity of fuzzy integral with respect to measure convergence and pointwise convergence for a continuous and subadditive fuzzy measure.

Wang [8] used the concepts of autocontinuity and null-additivity for a continuous fuzzy measure, improving the Ralescu - Adams results.

Finally, Greco and Bassanezi [2] and Flores, Román and Bassanezi [1], by using the concept of F-additivity and autocontinuity, respectively, of a fuzzy measure μ with respect to another fuzzy measure ν, removed the continuity condition and improve the Wang results.

In [3] we essentially prove that:

$$(f_n \xrightarrow{\nu} f \Rightarrow ⨍ f_n d\mu \rightarrow ⨍ f d\mu) \Leftrightarrow \mu \quad \text{is autocontinuous respect to } \nu.$$

Recently in [4], we studied different kinds of multivalued convergences for fuzzy sets on $\mathbb{R}^n$ and their relationships.

The aims of this paper is to analize the continuity of fuzzy integral with respect to multivalued convergences; more precisely we introduce the concept of level-convergence (L-convergence) on a fuzzy measure space, and we give conditions for the continuity of the fuzzy integral with respect to the L-convergence.

2. PRELIMINARIES

Definition 2.1. Let X be a set and $\mathcal{A}$ be a σ-algebra of subsets of X. By a *fuzzy measure* we mean a positive, extended real-valued set function $\mu : \mathcal{A} \rightarrow [0, \infty]$ with the properties:

(FM1) $\mu(\emptyset) = 0$.

(FM2) $A, B \in \mathcal{A}$ and $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

Furthermore, if the following property is true

(FM3) $A_1 \subseteq A_2 \subseteq \ldots, A_n \subseteq \mathcal{A} \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$, we say that the fuzzy measure μ is upper continuous.

Analogously, we say that μ is lower continuous if it has the following property

(FM4) $A_1 \supseteq A_2 \supseteq \ldots, A_n \in \mathcal{A}$ and there exists n_0 such that $\mu(A_{n_0}) < \infty$, then $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

If μ satisfies (FM3) and (FM4) we say that μ is continuous.

Throughout this paper $(X, \mathcal{A}, \mu)$ will be a fuzzy measure space and $\mathbb{M}(X)$ the family of all measurable functions $f : X \rightarrow [0, \infty)$.

If $f \in \mathbb{M}(X)$, the fuzzy integral of f is defined in [6] as:

$$⨍_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap \{f \geq \alpha\})], \quad A \in \mathcal{A},$$

where $\vee, \wedge$ denote respectively the operations of taking the supremum and infimum in $[0, \infty]$.

Also, let us define by $L^1(\mu) = \{f \in \mathbb{M}(X) \ / \ ⨍ f d\mu < \infty\}$.

The following results are well-known:

Theorem 2.4. [3]. If $f : X \rightarrow [0, \infty)$ is a measurable function, then

$$⨍ f d\mu = ⨍_0^{\infty} \mu\{f \geq \alpha\} d\alpha\ , \tag{1}$$

where the integral in the right-side of (1) is the fuzzy integral of $F(\alpha) = \mu\{f \geq \alpha\}$ with respect to the Lebesgue measure in $[0, \infty)$. ■

Theorem 2.5. [3]. If μ is subadditive (i.e. $\mu(A \cup B) \leq \mu(A) + \mu(B)$) and $f_n \rightarrow f$ in measure, then $⨍ f_n d\mu \rightarrow ⨍ f d\mu$. ■

Theorem 2.6. [3]. If μ is subadditive, $\mu(X) < \infty$ and $f_n \rightarrow f$ pointwise, then $⨍ f_n d\mu \rightarrow ⨍ f d\mu$. ■

Other interesting properties and applications of this integral were discussed in [5, 6, 7].

3. LEVEL-CONVERGENCE

Definition 3.1. A sequence of sets (A_n), $A_n \in \mathcal{A}$ is said to converge to $A \in \mathcal{A}$, if $A = \liminf A_n = \limsup A_n$, where $\limsup A_n = \bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} A_k\right]$ and $\liminf A_n = \bigcup_{n=1}^{\infty}\left[\bigcap_{k=n}^{\infty} A_k\right]$. In this case we denoted $A = \lim A_n$ (shortly: $A_n \to A$).

Remark 3.2. It is clear that $\limsup A$ consists of all x which are in infinitely many of the A_n and $\liminf A_n$ consists of all x which are in all but finitely many of the A_n. ∎

Remark 3.3. If (A_n) is an increasing sequence in $\mathcal{A}$, then $\lim A_n$ exists and it is equal to $\bigcup_{n=1}^{\infty} A_n$. Analogously, if (B_n) is a decreasing sequence in $\mathcal{A}$, then $\lim B_n$ exists and is equal to $\bigcap_{n=1}^{\infty} B_n$. ∎

Definition 3.4. Let $f \in M(X)$ and $\alpha \in (0, \infty)$. Then, the α-level of f is defined by $L_\alpha = \{x \in X \ / \ f(x) \geq \alpha\}$.

The support of f is defined by: $\text{supp}(f) = L_0 f = \{x \ / \ f(x) > 0\} = \bigcup_{\alpha > 0} L_\alpha f$.

Definition 3.5. We say that a sequence of functions (f_n), $f_n \in M(X)$, L-converges to $f \in M(X)$ (shortly: $f_n \xrightarrow{L} f$) if, for every $\alpha \geq 0$, $L_\alpha f_n \to L_\alpha f$.

The next proposition shows that L-convergence is stronger than pointwise convergence.

Propositition 3.6. If $f_n \xrightarrow{L} f$, then $f_n \to f$ pointwisely.

Proof. Suppose that $f_n \xrightarrow{L} f$ and let $x_0 \in X$ with $f(x_0) = \alpha_0$. Then, $x \in L_{\alpha_0} f = \lim L_{\alpha_0} f_n = \liminf L_{\alpha_0} f_n$. Consequently, $\exists n_0 \in N$ such that

$x_0 \in L_{\alpha_0} f_n = \{f_n \geq \alpha_0\}$, $\forall n \geq n_0$ (see Remark 3.2). Hence, $f_n(x_0) \geq \alpha_0$, $\forall n \geq n_0$. Thus, $\liminf f_n(x_0) \geq \alpha_0$.

Now suppose that $\beta_0 = \limsup f_n(x_0) > \alpha_0$ and let $\varepsilon > 0$ such that $\beta_0 - \varepsilon > \alpha_0$. Then, $f_n(x_0) \geq \beta_0 - \varepsilon$ for infinite values of n. Hence, $x_0 \in L_{\beta_0 - \varepsilon} f_n$ for infinite values of n. Consequently, $x_0 \in \limsup L_{\beta_0-\varepsilon} f_n = L_{\beta_0-\varepsilon} f$. Thus $f(x) \geq \beta_0 - \varepsilon > \alpha_0$. But this is impossible since $f(x_0) = \alpha_0$. This implies that $\alpha_0 \leq \liminf f_n(x_0) \leq \limsup f_n(x_0) \leq \alpha_0$. Consequently, $\lim_{n\to\infty} f_n(x_0) = f(x_0)$, i.e. $f_n \to f$ pointwisely. ∎

Corollary 3.7. If $f_n \xrightarrow{L} f$, and μ finite, then $f_n \xrightarrow{\mu} f$. ∎

The next examples shows that pointwise (even uniform) convergence does not imply L-convergence and also that L-convergence or pointwise convergence does not imply measure convergence.

Example 3.8. Let $X = \mathbb{R}$ and $\mathcal{A}$ a Lebesgue σ-algebra of measurable sets on X,

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad f_n(x) = \begin{cases} \dfrac{1}{n} + 1 - \dfrac{1}{n} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly $f_n \to f$ uniformly, but (f_n) does not converges levelwise to f.

Example 3.9. Let $(X, \mathcal{A})$ be as in example 3.8 and μ the usually Lebesgue measure on X. Let us define f_n, f by

$$f_n(x) = \begin{cases} \dfrac{|x|}{n} & \text{if } -n \leq x \leq n \\ 1 & \text{elsewhere} \end{cases} \quad \text{and} \quad f(x) = 0, \ \forall x.$$

Clearly $f_n \to f$ pointwise. Moreover $f_n \xrightarrow{L} f$, nevertheless (f_n) does not converges in measure to f. ∎

4. ENDOGRAPHIC CHARACTERIZATION OF L-CONVERGENCE

If $f \in \mathbb{M}(X)$ then the endograph of f, denoted by End(f), is the subset of $X \times [0,\infty)$ defined by $\mathrm{End}(f) = \{(x,\alpha) \; / \; f(x) \geq \alpha\}$.

Definition 4.1. If $f_n, f \in \mathbb{M}(X)$ then we say that (f_n) Γ-converge to f if $\mathrm{End}(f_n) \to \mathrm{End}(f)$.

Proposition 4.2. Let $f_n, f \in \mathbb{M}(X)$, then $f_n \xrightarrow{\Gamma} f \Leftrightarrow f_n \xrightarrow{L} f$.

Proof. Suppose $f_n \xrightarrow{\Gamma} f$ and $x \in X$ such that $f(x) \geq \alpha$. Then $(x,\alpha) \in \mathrm{End}(f) = \liminf \mathrm{End}(f_n) = \bigcup_{n=1}^{\infty}\left[\bigcap_{k=n}^{\infty}\mathrm{End}(f_k)\right]$, and hence, $\exists n_0 \in N$ such that $(x,\alpha) \in \bigcap_{k=n}^{\infty}\mathrm{End}(f_k)$, *i.e.* $f_k(x) \geq \alpha$, $\forall k \geq n_0$. This implies that $x \in \{f_k \geq \alpha\}$, $\forall k \geq n_0$, therefore $x \in \liminf\{f_n \geq \alpha\}$, *i.e.* $\{f \geq \alpha\} \subseteq \liminf\{f_n \geq \alpha\}$.

On the other hand, if $x \in \limsup\{f_n \geq \alpha\}$, then $x \in \{f_n \geq \alpha\}$ for infinite values of n (see Remark 3.2), *i.e.*$(x,\alpha) \in \mathrm{End}(f_n)$ for infinite values of n, consequently $(x,\alpha) \in \limsup \mathrm{End}(f_n) = \mathrm{End}(f)$. Hence $\limsup\{f \geq \alpha\} \subseteq \{f_n \geq \alpha\}$. Consequently, $f_n \xrightarrow{L} f$.

Conversely, let $(x,\alpha) \in \limsup \mathrm{End}(f_n)$. Then, $(x,\alpha) \in \mathrm{End}(f_n)$ for infinite values of n, therefore $(x,\alpha) \in \limsup\{f_n \geq \alpha\} \subseteq \{f \geq \alpha\}$. Thus, $(x,\alpha) \in \mathrm{End}(f)$. This implies that $\limsup \mathrm{End}(f_n) \subseteq \mathrm{End}(f)$. Now, let $(x,\alpha) \in \mathrm{End}(f)$. Then $f(x) \geq \alpha$ and, by hypothesis $x \in \liminf\{f_n \geq \alpha\}$. Hence $(x,\alpha) \in \liminf \mathrm{End}(f_n)$.

Consequently, $\mathrm{End}(f_n) \to \mathrm{End}(f)$ and $f_n \xrightarrow{\Gamma} f$. ∎

5. L-CONVERGENCE AND FUZZY INTEGRAL

Lemma 5.1. If $A_n \to A$, μ continuous, and there exist n_0 such that $\mu\left(\bigcup_{k=n_0}^{\infty} S_l\right) < \infty$, then $\mu(A_n) \to \mu(A)$.

Proof. The classical proof works [7,8]. ∎

Theorem 5.2. If f_n, $f \in \mathbb{M}(X)$ with $\mu\left[\bigcup_{n=1}^{\infty}\mathrm{supp}(f_n)\right] < \infty$, and μ continuous. Then

$$f_n \xrightarrow{L} f \Rightarrow ⨍ f_n d\mu \to ⨍ f d\mu.$$

Proof. If $f_n \xrightarrow{L} f$ then, by definition of L-convergence, it follows that

$$L_\alpha f_n \to L_\alpha f \;, \quad \forall \alpha \geq 0.$$

Hence, by Lemma 5.1, $\mu(L_\alpha f_n) \to \mu(L_\alpha f)$, $\forall \alpha \geq 0$.

So, making use of Theorem 2.5 we obtain that $\int \mu(L_\alpha f_n) d\alpha \to \int \mu(L_\alpha f) d\alpha$. Thus, by Theorem 2.4, we conclude that $⨍ f_n d\mu \to ⨍ f d\mu$. ∎

The hypothesis $\mu\left[\bigcup_{n=1}^{\infty}\mathrm{supp}(f_n)\right] < \infty$ in Theorem 5.2 is essencial, as show the following example:

Example 5.3. Let f_n, f be as in example 3.9. Then $\mathrm{supp}(f_n) = \mathbb{R} - \{0\}$, $\forall n$. So, $\mu\left[\bigcup_{n=1}^{\infty}\mathrm{supp}(f_n)\right] = \infty$.

On the other hand, $f_n \xrightarrow{L} f$ and $⨍ f_n d\mu = 1$ $\forall n$, whereas $⨍ f d\mu = 0$. ∎

Lemma 5.3. Let $A_n, A \in \mathcal{A}$, then: $A_n \to A$ if and only if $\chi_{A_n} \xrightarrow{L} \chi_A$.

Proof. This result is direct consequence of fact that $L_\alpha \chi_{A_n} = A_n$. ∎

Theorem 5.4. Let $(X, \mathcal{A}, \mu)$ a finite fuzzy measure space. Then the following properties are equivalent:

i) μ is continuous

ii) $f_n \xrightarrow{L} f \Rightarrow ⨍ f_n d\mu \to ⨍ f d\mu$.

Proof.

i → ii) By Theorem 5.2.

ii ⇒ i) Let A_n a monotone sequence in $\mathcal{A}$ and $A = \lim A_n$. Then, by Lemma 5.3, $A_n \to A$ implies $\chi_{A_n} \xrightarrow{L} \chi_A$. Thus, by hypothesis, $\int \chi_{A_n} d\mu \to \int \chi_A d\mu$; that is $\mu(A_n) \to \mu(A)$, and therefore, μ is continuous. ■

REFERENCES

[1] A. Flores F., H. Román F., R. C. Bassanezi, On the continuity of fuzzy integral in relation with measure convergence, Proyecciones 12, 137-147 (1993).

[2] G. Greco, R. C. Bassanezi, On the continuity of fuzzy integrals, Fuzzy Sets and Systems 53, 87-91 (1993).

[3] D. Ralescu, G. Adams, The fuzzy integral, J. Math. Anal. Appl., 75, 562-570 (1990).

[4] M. A. Rojas-Medar, H. Román-Flores, On the equivalence of convergences of fuzzy sets, submitted (1992).

[5] H. Román-Flores, Sobre as Entropias Fuzzy, Ph.D. Dissertation, IMECC-UNICAMP, Campinas, SP, Brazil, 1989.

[6] H. Román-Flores, R. Bassanezi, On the continuity of fuzzy entropies, Kybernetes (to appear).

[7] M. Sugeno, Fuzzy measures and fuzzy integrals: A survey, in: M. Gupta et al., Eds., Fuzzy Automata and Decision Process (North-Holland, Amsterdam, 1977), 89-102.

[8] Z. Wang, The autocontinuity of set function and the fuzzy integral, J. Math. Anal. Appl. 99, 165-218 (1984).

RELATÓRIOS DE PESQUISA — 1994

17/94 **On Acyclic Knots** — *Ricardo N. Cruz.*

18/94 **Optimal Stopping Time for a Poisson Point Process** — *Nancy Lopes Garcia.*

19/94 **Field Theory of the Spinning Electron: I – Internal Motions** — *Giovanni Salesi* and *Erasmo Recami.*

20/94 **Field Theory of the Spinning Electron: II – The New, Non-Linear Field Equations** — *Erasmo Recami* and *Giovanni Salesi.*

21/94 **The Weak Solutions and Reproductive Property for a System of Evolution Equations of Magnetohydrodynamic Type** — *Marko A. Rojas-Medar* and *José Luiz Boldrini.*

22/94 **A Fixed Point Theorem of Banach in the Fuzzy Context** — *Marko A. Rojas-Medar, Heriberto Román-Flores* and *Rodney C. Bassanezi.*

23/94 **Spectral Galerkin Approximations for the Equations of Magneto Hydrodynamic Type: Local in Time Error Estimates** — *Marko Rojas-Medar* and *José Luiz Boldrini.*

24/94 **Solving Nonsmooth Equations by Means of Quasi-Newton Methods with Globalization** — *Marcia A. Gomes-Ruggiero, José Mario Martínez* and *Sandra Augusta Santos.*

25/94 **Os Trabalhos de Leopoldo Nachbin (1922-1993)** — *Jorge Mujica.*

26/94 **On a New Class of Polynomials** — *D. Gomes* and *E. Capelas de Oliveira.*