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ON A NEW CLASS OF POLYNOMIALS

D. Gomes
and
E. Capelas de Oliveira

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INSTITUTO DE MATEMÁTICA
ESTATÍSTICA E CIÊNCIA DA COMPUTAÇÃO

UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS

ABSTRACT – We present and discuss the so called $E_N^\ell(\rho)$ and $G_N^\ell(\rho)$ polynomials which appear in the study of the generalized Laplace differential equation in the de Sitter universe.

IMECC – UNICAMP
Universidade Estadual de Campinas
CP 6065
13081-970 Campinas SP
Brasil

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D. Gomes and E. Capelas de Oliveira
Departamento de Matemática Aplicada
IMECC-UNICAMP
13081-970 - Campinas (SP) Brasil

Abstract

We present and discuss the so called $E_N^\ell(\rho)$ and $G_N^\ell(\rho)$ polynomials which appear in the study of the generalized Laplace differential equation in the de Sitter universe.

I. INTRODUCTION

It is known that the special relativity, based on the Poincaré group, can be perfected in an unique way and we obtain the special projective relativity, based on the Fantappié group with ten parameters. In this theory, the de Sitter universe of radius R, is studied using its geodetic representation with the Beltrami metric[1].

The generalized Laplace differential equation which appear in this theory can be study in several ways, e.g., Arcidiacono[2] studied the projective Laplace equation with spherical symmetry; Buzzanca[3] discuss the laplacian on tensors; Kovalyov[4] discuss the d'alembertian in a hyperbolic space and recently, Arcidiacono and Rizzi[5] solved the generalized Laplace equation with a static spherical symmetric field, where the so called $A_N(x)$ and $B_N(x)$ ultraspherical polynomials appear and more recently Arcidiacono and Capelas de Oliveira[6] solved this generalized Laplace equation where they discussed an eigenvalue equation.

Here we discuss the tridimensional generalized Laplace equation in spherical coordinates. Using the method of separation of variables we reduce the original equation in an angular equation and a radial equation. In the study of the radial equation appears a new class of non classical polynomials. The name "non classical polinomials" is because the coeficients of the differential equation depends not only of the independent variable but also of the parameters.

There are many papers where the study of the non classical polynomials appear, e.g. Littejohn and Shore[7] found a second order diferential equation which is different from the so called classical second order equation for Laguerre type and Jacobi type polynomials; Littejohn and Krall[8] developed the eigenfunction expansion theory of a self adjoint operator generated by a symmetric sixth order differential equation; Littejohn[9] and Krall and Littejohn[10] discussed the classification of differential equations having orthogonal polynomial solutions.

Recently two papers[11,12] discussed the continuous Hahn polynomials using the method of finite elements and converts the operator Heisenberg equations that arise from the hamiltonian in a set of operators difference equations on a lattice; the properties and connection with quantum mechanics are presented in terms of Hermite polynomials and, more recently, Micu[13] discuss the continuous Hahn polynomials.

This paper is organized as follow: in section II, we present and discuss the generalized Laplace differential equation in spherical coordinates and we obtain the angular and radial equations; in section III, we present a new class of non classical polynomial, the so called $E_N^\ell(\rho)$ and $G_N^\ell(\rho)$ which generalizes the $A_N(x)$ and $B_N(x)$ ultraspherical polynomials, respectively, which are solutions of the radial Laplace equation, we also present in this section the connection with the hypergeometric functions; in section IV we obtain as a particular case the $A_N(x)$ and $B_N(x)$ polynomials and present the properties of these polynomials and discuss the recurrence relation, the generating functions and the connection with the hypergeometric function completing thus the study of these polynomials. Finally we present our comments.

II. GENERALIZED LAPLACE EQUATION

In this section we present and discuss the generalized Laplace equation in spherical coordinates and we obtain the angular and radial differential equations.

The generalized Laplace differential equation is given by[1]

$$\{A^2(R^2\partial_i^2 + x_ix_j\partial_i\partial_j + 2x_i\partial_i) + N(N+2)\}\psi_N(x_i) = 0 \tag{2.1}$$

where $A^2 = 1 + x_i^2/R^2$ with $i,j = 1,2,3$ and N is the so called the degree of homogeneity of the function $\psi_N(x)$ and R is the radius of the de Sitter universe.

Introducing spherical coordinates,

$$x_1 = R\rho\cos\theta, \quad x_2 = R\rho\cos\theta\sin\phi, \quad x_3 = R\rho\sin\theta\sin\phi$$

in the above equation, we obtain

$$\begin{aligned}(1+\rho^2)\frac{\partial^2\psi}{\partial\rho^2} &+ \frac{2}{\rho}(1+\rho^2)\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial\psi}{\partial\theta} + \\ &+ \frac{1}{\rho^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{N(N+2)}{1+\rho^2}\psi = 0\end{aligned} \tag{2.2}$$

where $\psi \equiv \psi_N(\rho,\theta,\phi)$.

Introducing the function

$$\psi_N(\rho,\theta,\phi) = R_N^\ell(\rho)T_\ell^m(\theta)S_m(\phi)$$

in the above equation and separating the variables we obtain the following ordinary differential equations

$$\frac{d^2}{d\phi^2}S_m(\phi) + m^2S_m(\phi) = 0 \tag{2.3.a}$$

$$\frac{d^2}{d\theta^2}T_\ell^m(\theta) + \cot\theta\frac{d}{d\theta}T_\ell^m(\theta) + [\ell(\ell+1) - \frac{m^2}{\sin^2\theta}]T_\ell^m(\theta) = 0 \tag{2.3.b}$$

$$(1+\rho^2)\frac{d^2}{d\rho^2}R_N^\ell(\rho) + \frac{2}{\rho}(1+\rho^2)\frac{d}{d\rho}R_N^\ell(\rho) + [\frac{N(N+2)}{1+\rho^2} - \frac{\ell(\ell+1)}{\rho^2}]R_N^\ell(\rho) = 0 \tag{2.3.c}$$

where $m = 0, \pm1, \pm2, \ldots$ and $-\ell < m < \ell$ with $\ell = 0,1,2,\ldots$ the parameters m and ℓ are the same as the quantum numbers representing the magnetic quantum number and the angular quantum number, respectively.

Then, the solution of the above equation, angular equations, can be written in terms of the $Y_\ell^m(\theta,\phi)$ harmonic sphericals, as follows

$$S_m(\phi)T_\ell^m(\theta) = Ce^{\pm im\phi}P_\ell^m(\cos\theta) = CY_\ell^m(\theta,\phi)$$

where C is a constant and $P_\ell^m(\cos\theta)$ are the Legendre polynomials*.

* See ref. 15 for the notation of the special functions.

III. THE $E^\ell_N(\rho)$ AND $G^\ell_N(\rho)$ POLYNOMIALS

In this section we present the so called $E^\ell_N(\rho)$ and $G^\ell_N(\rho)$ polynomials which are the solutions of the radial Laplace differential equation and we obtain the relation with the hypergeometric function. Firstly, we consider the solution of the radial equation using a Frobenius type expansion and secondly we introduce an independent change of variable.

Introducing a function defined by

$$R^\ell_N(\rho) = \frac{1}{\rho}(1+\rho^2)^{-N/2} F^\ell_N(\rho) \tag{3.1}$$

in eq. 2.3.c we obtain the following differential equation:

$$(1+\rho^2)\frac{d^2}{d\rho^2}F^\ell_N(\rho) - 2N\rho\frac{d}{d\rho}F^\ell_N(\rho) + [N(N+1) - \frac{\ell(\ell+1)}{\rho^2}]F^\ell_N(\rho) = 0 \tag{3.2}$$

This equation, for $\ell = 0$ is the same obtained by Arcidiacono and Rizzi[5].

To solve this equation we use an expansion of Frobenius type as follows

$$F^\ell_N(\rho) = \sum_{k=0}^{\infty} a_k \rho^{k+s} \tag{3.3}$$

and we obtain for polynomial solutions the following expression for a_k

$$a_{2k} = (-1)^k \frac{(N-\ell)!\Gamma(\ell+3/2)}{(N-\ell-2k)!\Gamma(k+\ell+3/2)} \frac{a_0}{2^{2k}k!} \qquad k = 0,1,2,\dots \tag{3.4}$$

where a_0 is an arbitrary parameter. Making $a_0 = N - \ell + 1$ we obtain

$$E^\ell_N(\rho) \equiv F^\ell_N(\rho) = \rho^{\ell+1}\sum_{k=0}^{[\frac{N-\ell}{2}]}(-1)^k \frac{(N-\ell+1)!\Gamma(\ell+3/2)}{(N-\ell-2k)!\Gamma(k+\ell+3/2)k!}\left(\frac{\rho}{2}\right)^{2k} \tag{3.5}$$

with $N \geq \ell$.

For the another polynomial solution we get

$$G^\ell_N(\rho) \equiv \tilde{F}^\ell_N(\rho) = \frac{1}{\rho^\ell}\sum_{k=0}^{[\frac{N+\ell+1}{2}]}(-1)^k \frac{(N+\ell+1)!\Gamma(-\ell+1/2)}{(N+\ell+1-2k)!\Gamma(k-\ell+1/2)}\frac{1}{k!}\left(\frac{\rho}{2}\right)^{2k} \tag{3.6}$$

Finally, we have for the eq. 2.3.c

$$R^\ell_N(\rho) = \frac{1}{\rho}(1+\rho^2)^{-N/2} F^\ell_N(\rho)[\tilde{F}^\ell_N(\rho)] \tag{3.7}$$

where $F^\ell_N(\rho)[\tilde{F}^\ell_N(\rho)]$ are given above.

Secondly, we introduce in eq. 2.3.c an independent change of variables defined by

$$1+\rho^2 = \frac{1}{x} \tag{3.8}$$

and we obtain

$$x(1-x)\frac{d^2}{dx^2}R^\ell_N(x) + \left(\frac{1}{2} - 2x\right)\frac{d}{dx}R^\ell_N(x) + \left[\frac{N(N+2)}{4} - \frac{\ell(\ell+1)}{4(1-x)}\right]R^\ell_N(x) = 0 \tag{3.9}$$

Now, another change of variables as follow

$$R^\ell_N(x) = (1-x)^{\ell/2}F(x) \tag{3.10}$$

and finally we have

$$x(1-x)\frac{d^2}{dx^2}F(x) + \left[\frac{1}{2} - (\ell+2)x\right]\frac{d}{dx}F(x) + \frac{1}{4}\left[N(N+2) - \ell(\ell+2)\right]F(x) = 0 \tag{3.11}$$

which is an hypergeometric differential equation. The general solution of the above equation is discussed in ref. 14.

We are interested in polynomial solutions and then we introduce $2x = t+1$ and we obtain for eq. 2.3.c the following expressions

$$R^\ell_N(\rho) = \left(\frac{\rho^2}{1+\rho^2}\right)^{\frac{\ell}{2}} P^{(\ell+1/2,-1/2)}_{\frac{N-\ell}{2}}\left(\frac{1-\rho^2}{1+\rho^2}\right) \tag{3.12}$$

and

$$\tilde{R}^\ell_N(\rho) = \left(\frac{\rho^2}{1+\rho^2}\right)^{\frac{-\ell+1}{2}} P^{(-\ell-1/2,-1/2)}_{\frac{N+\ell+1}{2}}\left(\frac{1-\rho^2}{1+\rho^2}\right) \tag{3.13}$$

where $P^{(\alpha,\beta)}_n(x)$ is a Jacobi polynomial.

Using the relation of the Jacobi polynomial and properties of the gama function the above expressions can be done in terms of the Gegenbauer polynomials[14].

The relations among the above expressions and the eq. 3.7 furnish our $E^\ell_N(\rho)$ and $G^\ell_N(\rho)$ polynomials as follow

$$E_N^\ell(\rho) = \rho^{\ell+1}(1+\rho^2)^{\frac{N-\ell}{2}} \frac{\Gamma(N-\ell+2)\Gamma(\ell+3/2)}{\Gamma(N+3/2)} P_{N-\ell}^{(\ell+1/2,\ell+1/2)}\left(\frac{1}{\sqrt{1+\rho^2}}\right) \quad (3.14)$$

with $N-\ell = 0,1,2\dots$ and

$$G_N^\ell(\rho) = \frac{1}{\rho^\ell}(1+\rho^2)^{\frac{N+\ell+1}{2}} \frac{\Gamma(N+\ell+2)\Gamma(-\ell+1/2)}{\Gamma(N+3/2)} P_{N+\ell+1}^{(-\ell-1/2,-\ell-1/2)}\left(\frac{1}{\sqrt{1+\rho^2}}\right) \quad (3.15)$$

with $N+\ell = 1,2,\dots$ the few first $E_N^\ell(\rho)$ and $G_N^\ell(\rho)$ are done in ref. 14.

IV. PARTICULAR CASE

In this section we consider a particular case of the $E_N^\ell(\rho)$ and $G_N^\ell(\rho)$ polynomials when we have $\ell = 0$ and we obtain exactly the $A_N(\rho)$ and $B_N(\rho)$ Arcidiacono and Rizzi polynomial solutions[5].

Taking $\ell = 0$ in the eq. 3.14 we get

$$E_N^0(\rho) = \rho(1+\rho^2)^{\frac{N}{2}} \frac{\Gamma(N+2)\Gamma(3/2)}{\Gamma(N+3/2)} P_N^{(1/2,1/2)}\left(\frac{1}{\sqrt{1+\rho^2}}\right) \quad (4.1)$$

where $N = 0,1,2,\dots$. Using a relation[15] among the Jacobi and Gegenbauer polynomials we obtain

$$E_N^0(\rho) = \rho(1+\rho^2)^{N/2} C_N^1\left(\frac{1}{\sqrt{1+\rho^2}}\right) \quad (4.2)$$

where $C_N^\lambda(x)$ is a Gegenbauer polynomial. Finally, we obtain

$$E_N^0(\rho) = \rho(1+\rho^2)^{N/2} U_N\left(\frac{1}{\sqrt{1+\rho^2}}\right) \quad (4.3)$$

where $N = 0,1,2,\dots$ and $U_N(x)$ are the Tchebischef polynomials of second kind. The eq. 4.3 gives exactly the $B_N(\rho)$ Arcidiacono-Rizzi polynomials[5].

In the same way we obtain

$$G_N^0(\rho) = (1+\rho^2)^{\frac{N+1}{2}} T_{N+1}\left(\frac{1}{\sqrt{1+\rho^2}}\right) \quad (4.4)$$

where $N = 0,1,2,\dots$ and $T_N(x)$ are the Tchebischef polynomials of first kind. The above expression gives exactly the $A_N(\rho)$ Arcidiacono-Rizzi polynomials[5].

To complete the study of the above polynomials we present and discuss the recurrence relations and generating functions for $A_N(\rho)$ and $B_N(\rho)$ polynomials.

Using the definition for $A_N(\rho)$ polynomials and the respective differential equation we obtain the so called pure recurrence relation, as follows

$$A_{N+2}(\rho) - 2A_{N+1}(\rho) + (1+\rho^2)A_N(\rho) = 0 \quad (4.5)$$

where $N = 0,1,2\dots$

Now, differentiating the above relation and using the differential equation for $A_N(\rho)$ we have a relation involving the first derivative as follows

$$(N+2)A_N(\rho) + \rho A'_{N+1}(\rho) - (N+2)A_{N+1}(\rho) = 0 \quad (4.6)$$

where $A'_N(\rho) \equiv \frac{d}{d\rho}A_N(\rho)$ and $N = 0,1,2\dots$.

Finally, we introduce a generating functtion of the type

$$G_A(\rho,t) = \sum_{N=0}^{\infty} t^N A_N(\rho) \quad (4.7)$$

and using the recurrence relations we get

$$G_A(\rho,t) = \frac{1-t(1+\rho^2)}{(1-t)^2+t^2\rho^2} \quad (4.8)$$

and using the Cauchy theorem we have that: the $A_N(\rho)$ polynomials can be generate using the follow expression

$$A_N(\rho) = \frac{1}{N!}\frac{\partial^N}{\partial t^N} G_A(\rho,t)\Big|_{t=0} \quad (4.9)$$

where $G_A(\rho,t)$ is given above.

In the same way we obtain for the $B_N(\rho)$ polynomials the following results:

$$B_N(\rho) = \frac{1}{N!}\frac{\partial^N}{\partial t^N} G_B(\rho,t)\Big|_{t=0} \quad (4.10)$$

where

$$G_B(\rho,t) = \frac{\rho}{(1-t)^2+t^2\rho^2}. \quad (4.11)$$

We note that the eq. 4.5 and eq. 4.6 are the same for $A_N(\rho)$ and $B_N(\rho)$ because the differential equation is the same.

V. COMMENTS

In this paper we discussed the so called $E_N^\ell(\rho)$ and $G_N^\ell(\rho)$ polynomials which are the polynomial solutions of the generalized Laplace differential equation written in spherical coordinates. As a particular case we obtained the $A_N(\rho)$ and $B_N(\rho)$ Arcidiacono-Rizzi polynomials which appear when we study galaxies with spiral orbits[16] and quark confinement[17,18]. We also presented the recurrence relations and generating function for the $A_N(\rho)$ and $B_N(\rho)$ polynomials.

For ours $E_N^\ell(\rho)$ and $G_N^\ell(\rho)$ polynomials we can obtain the recurrence relations and a generating functions and we recall that these polynomials can be applied when studying problems where the angular moment is not zero. This will discussed in a forthcoming paper[19].

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