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A FIXED POINT THEOREM OF BANACH IN THE FUZZY CONTEXT

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ABSTRACT - The classic fixed point theorem of Banach stablish that if (X, d) is a complete metric space and $f : X \longrightarrow X$ is a contractive function, then f has a unique fixed point.

In this work we present a "Banach theorem" type for a function $F : X \longrightarrow \mathbb{F}(X)$, where $\mathbb{F}(X)$ denote the metric space of fuzzy sets.

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A FIXED POINT THEOREM OF BANACH IN THE FUZZY CONTEXT*

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Abstract. The classic fixed point theorem of Banach stablish that if (X,d) is a complete metric space and $f : X \to X$ is a contractive function, then f has a unique fixed point.

In this work we present a "Banach theorem" type for a function $F : X \to \mathbb{F}(X)$, where $\mathbb{F}(X)$ denote the metric space of fuzzy sets.

1. Preliminaries.

Let (X,d) be a metric space, we denote by

$$\begin{aligned} \mathbf{C}(X) &= \{A \subseteq X \mid A \text{ is closed, bounded and nonempty}\} \\ \mathbb{K}(X) &= \{A \subseteq X \mid A \text{ is compact and nonempty}\} \\ N(A,r) &= \{x \in X \mid d(x,a) < r \text{ for some } a \in A\}. \end{aligned}$$

The *Hausdorff metric* on $\mathbf{C}(X)$ is defined by

$$H(A,B) = \inf\ \{r > 0 \ / \ B \subseteq N(A,r) \text{ and } A \subseteq N(B,r)\}$$

$A, B \in \mathbf{C}(X)$.

Definition 1.1. Let (X,d) and (Y,d') metric spaces; an application $\Gamma : X \to \mathbf{C}(Y)$ is a multivalued contraction if

$$H(\Gamma(x),\Gamma(y)) \leq \alpha d(x,y)\ ,\ \forall x,y \in X, 0 \leq \alpha < 1;$$

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where α is constant.

Remark. A multivalued contraction Γ is H-continuous.

Definition 1.2 Let $\Gamma : X \to \mathcal{C}(X)$. Then $x \in X$ is a fixed point of Γ if $x \in \Gamma(x)$.

To prove the main result of this work we need of the following results:

Lemma 1.3 Let $A, B \in \mathcal{C}(x), a \in A$ and $\eta > 0$. Then there exist $b \in B$ such that

$$d(a,b) \leq H(A,B) + \eta. \qquad \blacksquare$$

Remark. If $A, B \in \mathbb{K}(X)$, then we can take $\eta = 0$ (see [1]).

The following theorem give a generalization for the Banach theorem in the multivalued case.

Theorem 1.4 (Nadler [1]). Let (X,d) be a complete metric space and $\Gamma : X \to \mathcal{C}(X)$ be a multivalued contraction with constant α, $0 \leq \alpha < 1$. Then Γ has a fixed point.

Proof. Let $p_0 \in X$, we choose $p_1 \in \Gamma(p_0)$. Since $\Gamma(p_0)$ and $\Gamma(p_1)$ belong to $\mathcal{C}(X)$ and $p_1 \in \Gamma(p_0)$ the Lemma 1.2 implies that there exist $p_2 \in \Gamma(p_1)$ such that $d(p_1,p_2) \leq H(\Gamma(p_0),\Gamma(p_1)) + \alpha$. Analogously, since $\Gamma(p_1),\Gamma(p_2) \in \mathcal{C}(X)$ and $p_2 \in \Gamma(p_1)$, there exist $p_3 \in \Gamma(p_2)$ such that $d(p_2,p_3) \leq H(\Gamma(p_1),\Gamma(p_2)) + \alpha^2$. By induction we generate a sequence (p_n) in X such that $p_{n+1} \in \Gamma(p_n)$ and $d(p_n,p_{n+1}) \leq H(\Gamma(p_{n-1}),\Gamma(p_n)) + \alpha^n$ for any $n \geq 1$. We observe that

$$\begin{aligned}
d(p_n,p_{n+1}) &\leq H(\Gamma(p_{n-1}),\Gamma(p_n)) + \alpha^n \\
&\leq \alpha d(p_{n-1},p_n) + \alpha^n \\
&\leq \alpha[H(\Gamma(p_{n-2}),\Gamma(p_{n-1})) + \alpha^{n-1}] + \alpha^n \\
&\leq \alpha^2 d(p_{n-2},p_{n-1}) + 2\alpha^n \\
&\ \ \vdots \\
&\leq \alpha^n d(p_0,p_1) + n\alpha^n, \ \forall n \geq 1.
\end{aligned}$$

So,

$$\begin{aligned}
d(p_n,p_{n+j}) &\leq d(p_n,p_{n+1}) + d(p_{n+1},p_{n+2}) + \cdots + d(p_{n+j-1},p_{n+j}) \\
&\leq \left(\left[\sum_{k=n}^{n+j-1}\right] d(p_0,p_1) + \sum_{k=n}^{n+j-1} k\alpha^k\right) \to 0
\end{aligned}$$

if $n, j \to \infty$. Therefore, $\{p_n\}$ is a Cauchy-sequence; since (X,d) is complete, the limit of this sequence is a point p^* in X. Since Γ is continuous, $\Gamma(p_n) \to \Gamma(p^*)$ as $n \to \infty$.

By definition, we have that given $\varepsilon > 0$, there exist $N = N(\varepsilon)$ such that $H(\Gamma(p_n),\Gamma(p^*)) < \varepsilon$. Moreover, $\Gamma(p_n) \subseteq N(\Gamma(p^*),\varepsilon)$, consequently $d(p_{n+1},q) < \varepsilon$ for some $q \in \Gamma(p^*)$, so $d(p_{n+1},\Gamma(p^*)) < \varepsilon \quad \forall n \geq N$.

Being $\varepsilon > 0$ sufficiently small, we conclude that $d(p^*,\Gamma(p^*)) = 0$. But, $\Gamma(p^*)$ is closed. Hence $p^* \in \Gamma(p^*)$. $\blacksquare$

Remark. The uniqueness of the fixed point is not guaranted in this context (see example 1), unless one uses additionals properties (Fisher [3], Kaneko [4]).

Example 1. Let $X = [0,1]$ be a metric space with the usual metric. We consider $u : X \to [0,1]$ defined by

$$u(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/n \\ 0 & \text{if } 1/n < x \leq 1 \end{cases}$$

Then $L_\alpha u = \{x \in X | u(x) \geq \alpha\} = [\alpha/n, 1/n]$ is the *α-level of* u, $\forall \alpha \in [0,1]$

We consider now the multifunction $\Gamma : X \to \mathbb{K}(X)$ define by $\Gamma(\alpha) = L_\alpha u = [\alpha/n, 1/n] \ \forall\alpha$.

We observe that

$$H(\Gamma(\alpha),\Gamma(\beta)) = H([\alpha/n,1/n],[\beta/n,1/n]) = \frac{1}{n}|\alpha - \beta|.$$

Then, Γ is a multivalued contraction for $n \geq 2$.

But,

$$\alpha \in \Gamma(\alpha) \Leftrightarrow \alpha \in [\alpha/n, 1/n] \Leftrightarrow \alpha/n \leq \alpha \leq 1/n$$

and this inequality is verified for all $\alpha \in [0, 1/n]$.

Consequently, every point in $[0,1]$ is a fixed point of Γ.

This example shows that, in the multivalued case, the fixed point is not unique necessarely.

2. Fixed point theorem in the Fuzzy-Multivalued case.

Let (X,d) be a metric space and $\mathbb{F}(X) = \{u : X \to [0,1] / L_\alpha u \in \mathcal{C}(X), \ \forall \alpha \in [0,1]\}$, where $L_0 u \equiv \text{supp}(u) = cl\{x \in X \mid u(x) > 0\}$ is the *support of u*.

We observe that $L_\alpha u \neq \phi$, $\forall \alpha$, is equivalent to $u(x) = 1$, for some $x \in X$.

In the following, we will use the notation $[u]^\alpha = L_\alpha u$.

Definition 2.1. Let $u, v \in \mathbb{F}(X)$ and $\alpha \in [0,1]$. We define

$$h_\alpha(u,v) = \inf\{d(x,y) | x \in [u]^\alpha, y \in [v]^\alpha\},$$

$$H_\alpha(u,v) = H([u]^\alpha, [v]^\alpha),$$

$$H^*(u,v) = \sup_{\alpha \in [0,1]} H_\alpha(u,v).$$

It is easy to show that h_α is increasing monotonically function in α and that H^* is a metric on $\mathbb{F}(X)$. If X is a complete metric space then the metric space $(\mathbb{F}(X), H^*)$ is also complete. But, $(\mathbb{F}(X), H^*)$ is not a separable metric space again if X is.

The space $(\mathbb{F}(X), H^*)$ is called a *fuzzy metric space*.

Moreover, we can define a partial order, $\subseteq_F$, on $\mathbb{F}(X)$ by setting

$$u \subseteq_F v \Leftrightarrow u(x) \le v(x), \ \forall x \in X$$

$$\Leftrightarrow [u]^\alpha \subseteq [v]^\alpha, \ \forall \alpha \in [0,1].$$

Remark. It is easy to see that

$$(X,d) \hookrightarrow (\mathcal{C}(X), H) \hookrightarrow (\mathbb{F}(X), H^*).$$

In fact, we observe that, for every $A \in \mathcal{C}(X)$ we can associate the characteristic function $\chi_A : X \to \{0,1\}$ defined by $\chi_A(x) = 0$ if $x \notin A$ and $\chi_A(x) = 1$ if $x \in A$; then we have

$$H^*(A,B) \equiv H^*(\chi_A, \chi_B) = H(A,B).$$

If $A = \{x\}$ and $B = \{y\}$, then

$$H^*(\{x\},\{y\}) \equiv H^*(\chi_{\{x\}}, \chi_{\{y\}}) = H(\{x\},\{y\}) = d(x,y).$$

Also, we will denote $\chi_{\{x\}}(x)$ by χ_x.

Definition 2.2. Let X, Y be metric spaces. A fuzzy-multimapping is an application $F : X \to \mathbb{F}(Y)$

Lemma 2.3. Let $x \in X$ and $u \in \mathbb{F}(X)$. Then $\chi_x \subseteq_F u \Leftrightarrow h_\alpha(\chi_x, u) = 0$, $\forall \alpha \in [0,1]$.

Proof. If $\chi_x \subseteq_F u$, then $u(x) = 1$, consequently, $x \in [u]^\alpha$, $\forall \alpha \in [0,1]$. Hence,

$$h_\alpha(\chi_x, u) = \inf\{d(x,y) \mid y \in [u]^\alpha\} = 0.$$

The reciproque is evidenty. ■

Lemma 2.4. $h_\alpha(\chi_x, u) \le d(x,y) + h_\alpha(\chi_y, u)$ for any $x, y \in X$ and $u \in \mathbb{F}(X)$.

Proof. Since $[\chi_x]^\alpha = \{x\}$, $\forall \alpha$, we have

$$\begin{aligned} h_\alpha(\chi_x, u) &= \inf\{d(x,z)/z \in [u]^\alpha\} \\ &\le \inf\{d(x,y) + d(y,z)/z \in [u]^\alpha\} \\ &= d(x,y) + h_\alpha(\chi_y, u) \end{aligned}$$ ■

Lemma 2.5. If $\chi_x \subseteq_F u$, then $h_\alpha(\chi_x, v) \le H_\alpha(u,v)$, $\forall \alpha \in [0,1], \forall v \in \mathbb{F}(X)$.

Proof.

$$\begin{aligned} h_\alpha(\chi_x, v) &= \inf\{d(x,y)/y \in [v]^\alpha\} \\ &\le \sup_{X \in [u]^\alpha} \inf\{d(x,y)/y \in [v]^\alpha\} \\ &\le H([u]^\alpha, [v]^\alpha) = H_\alpha(u,v). \end{aligned}$$ ■

Definition 2.6. Let (X,d) and (Y,d') be two metric spaces. An application $F : X \to \mathbb{F}(Y)$ is a H^*-contraction if

$$H^*(F(x),F(y)) \leq \alpha d(x,y),\ \forall x,y \in X,\ 0 \leq \alpha \leq 1.$$

Definition 2.7. Let $F : X \to \mathbb{F}(X)$. Then $x^* \in X$ is a fixed point of F if $\chi_{x^*} \subseteq_F F(X^*)$.

Remark. The definitions 2.6 and 2.7 generalize the correspondente definition of the multivalued-mapping.

We can give now the fixed point theorem in the fuzzy multivalued case:

Theorem 2.8. Let (X,d) a complete metric space and $F : X \to \mathbb{F}(X)$ a H^*-contraction. Then, F has a fixed point in the sense of the definition 2.7.

Proof. Let $x_0 \in X$ and we choose $x_1 \in [F(x_0)]^1$. This implies that $\chi_{x_1} \subseteq_F F(x_0)$. By using the Lemma 1.3, we can to find $x_2 \in [F(x_1)]^1$ such that

$$\begin{aligned} d(x_1,x_2) &\leq H_1(F(x_0),F(x_1)) + \alpha \\ &\leq H^*(F(x_0),F(x_1) + \alpha, \end{aligned}$$

therefore, $\chi_{x_2} \subseteq_F F(x_1)$ and $d(x_1,x_2) \leq \alpha d(x_0,x_1) + \alpha$.

Analogously, we can find $x_3 \in X$ such that $\chi_{x_3} \subseteq_F F(x_2)$ and

$$\begin{aligned} d(x_2,x_3) &\leq H_1(F(x_2),F(x_1)) + \alpha^2 \\ &\leq \alpha d(x_2,x_1) + \alpha^2 \\ &\leq \alpha^2 d(x_0,x_1) + 2\alpha^2 \end{aligned}$$

and by recursion we generated a sequence $\{x_k\}$ in X, such that

$$\begin{aligned} \chi_{x_k} &\subseteq_F F(x_{k-1}) \quad \text{and} \\ d(x_k,x_{k+1}) &\leq H_1(F(x_k),F(x_{k-1})) + \alpha^k \\ &\leq \alpha^k d(x_0,x_1) + k\alpha^k. \end{aligned}$$

Exactly as in the proof of Nadler - Theorem 1.4, we prove that the sequence $\{x_k\}$ is a Cauchy-sequence in X and, since X is a complete metric space, then there exist $x^* \in X$ such that $x_k \to x^*$ as $k \to \infty$.

Let us now to show that x^* is a fixed point of F.

Indeed, we have

$$\begin{aligned} h_\alpha(\chi_{x^*},F(x^*)) &\leq d(x^*,x_k) + h_\alpha(\chi_{x_k},F(x^*)) && \text{(by Lemma 2.4)} \\ &\leq d(x^*,x_k) + H_\alpha(F(x_{k-1}),F(x^*)) && \text{(by Lemma 2.5)} \\ &\leq d(x^*,x_k) + H^*(F(x_{k-1}),F(x^*)) \\ &\leq d(x^*,x_k) + \alpha d(x_{k-1},x^*) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus, $h_\alpha(\chi_{x^*},F(x^*)) = 0$, $\forall \alpha \in [0,1]$, this implies that $\chi_{x^*} \subseteq_F F(x^*)$. ■

Example 2. Let $X = [0,1]$ be the metric space with the usual metric and $F : X \to \mathbb{F}(X)$ is defined by

$$F(t)(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ t & \text{if} \quad x \in (0,1] \end{cases}$$

In this case, we have that

$$\begin{aligned} \chi_{t^*} \subseteq_F F(t^*) &\Leftrightarrow F(t^*)(t^*) = 1 \\ &\Leftrightarrow t^* = 0 \text{ or } t^* = 1. \end{aligned}$$

Remark. The principals ideas of this work are based in the Heilpern' paper [2], without to suppose that X have a linear structure and F have compact-convexes levels.

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