CLIFFORD ALGEBRA APPROACH TO THE BARUT-ZANGHI MODEL AS A HAMILTONIAN SYSTEM

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ABSTRACT – We use the spacetime algebra to discuss the Barut-Zanghi model and we show that it is in fact a hamiltonian system.

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Abstract. We use the spacetime algebra to discuss the Barut-Zanghi model and we show that it is in fact a hamiltonian system.

Key words: Barut-Zanghi model – spacetime algebra

The Barut-Zanghi (BZ) model (Barut and Zanghi 1984) is a classical model for the Dirac electron. This fact by itself justify its importance, but BZ model has many interesting properties and its study is fundamental for those who wants to understand what is an electron. BZ exhibits the helical motion as the classical analogue of Zitterbewegung and provides a classical action for which one can define a path integral giving the Dirac propagator (Barut and Duru 1984).

The key point of the BZ model consists in recognizing that the pair of conjugated variables (x, p) do not suffice to characterize a Dirac particle; an additional pair of conjugate classical spinor variables $(z, i\bar{z})$ is required. The model consists in the lagrangian $(z = z(\tau),\ x = x(\tau))$

$$L = \frac{i}{2}(\dot{\bar{z}}z - \bar{z}\dot{z}) + p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) + eA_\mu\bar{z}\gamma^\mu z. \tag{1}$$

One sees that p_μ is introduced here as a Lagrange multiplier, implying that the velocity $\dot{x}^\mu = \bar{z}\gamma^\mu z$, with γ^μ being Dirac matrices.

In spite of BZ model being formulated as a lagrangian one, Rawnsley (1992) showed that the theory is actually a *hamiltonian* one by constructing an exact symplectic form on the total space of the spin bundle over spacetime. However, in spite of its beauty, it is our opinion that the mathematical methods used by Rawnsley are not very appropriate to the problem at issue. The fact that BZ model is a hamiltonian system is easily recognized when one formulates it in terms of the spacetime algebra (STA) (Hestenes 1966).

Indeed, the use of STA has many advantages over the traditional tensor and spinor calculus (Hestenes 1966, Hestenes and Sobczyk 1984; see (Vaz and Rodrigues, 1993) as an illustration of how pages of calculus can be performed in few lines by using STA). Our objective in this letter is to show by means of STA that BZ model is a hamiltonian system. Besides being an alternative proof of it, our analysis is useful also as an illustration of how powerful is our method: we shall discuss some aspects of BZ model and show how to give a symplectic structure to phase space using Clifford algebras (Hestenes 1992).

In the following we adopt the notation: $R_{1,3}$ is STA and $R^+_{1,3}$ its even subalgebra; the Clifford (or geometrical) product is denoted by justaposition, while the interior and exterior products are denoted by $\cdot$ and $\wedge$, respectively; tilde denotes reversion, i.e., $\tilde{(AB)} = \tilde{B}\tilde{A}$ with $\tilde{A} = A$ for A scalar or vector. We remark that in STA $\{\gamma^\mu\}$ are interpreted as vectors of TM (the tangent bundle over Minkowski spacetime), in contrast to those $\{\gamma^\mu\}$ in eq.(1) which are matrices (it is well-known that any Clifford algebra can be represented by an appropriate matrix algebra).

First of all, let us clarify the presence of Dirac spinor $z \in C^4$ in BZ lagrangian (Pavšič et al. 1992). One can define z as an element of the minimal left ideal $(C \otimes R_{1,3})f$, where f is the idempotent $f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_1\gamma_2)$; but, as discussed in (Figueiredo êt al. 1990, Rodrigues and Oliveira 1990, Lounesto 1993), an element of $R_{1,3}\frac{1}{2}(1+\gamma_0)$ contains the *same* information as z (this element was called mother spinor by Lounesto (1993)). Moreover, since $R_{1,3}\frac{1}{2}(1+\gamma_0) \simeq R^+_{1,3}\frac{1}{2}(1+\gamma_0)$, one can define a spinor as an element of $R^+_{1,3}$. These kind of spinors are called operator spinors, and for the case in question we call it Dirac-Hestenes spinors.(Hestenes 1967). A non-singular Dirac-Hestenes spinor ψ has a beautiful geometrical interpretation in terms of the canonical decomposition:

$$\psi = \sqrt{\rho}e^{\gamma_5\beta/2}R, \tag{2}$$

that is: R represents a Lorentz rotation, $\sqrt{\rho}$ a dilatation and $e^{\gamma_5\beta/2}$ a duality rotation by the Takabayasi (1957) angle β (we shall assume $\beta = 0$). Now one can understand the presence of z in eq.(1), for a spinning particle must be individuated, besides by (x,p), by the Frenet tetrad $\{e_\mu\}$ given by $e_\mu = R\gamma_\mu\tilde{R}$ (Hestenes and Sobczyk 1984), or:

$$\rho e_\mu = \psi\gamma_\mu\tilde{\psi} \tag{3}$$

for $\psi = \psi(\tau)$, and τ being a parameter that defines the particle's world-line (if τ is identified with proper time then $\rho = 1$).

BZ lagrangian in terms of STA reads (Gull 1991, Pavšič et al. 1992)

$$L = \langle \dot{\tilde{\psi}}\psi\gamma_2\gamma_1 + p(\dot{x} - \psi\gamma_0\tilde{\psi}) + eA\psi\gamma_0\tilde{\psi} \rangle_0 \tag{4}$$

where by $<>_a$ we mean "the a-part of". Euler-Lagrange equations $\partial_X L - \partial_\tau(\partial_{\dot{X}} L) = 0$, where ∂_X is multivector derivative (Hestenes and Sobczyk 1984, Doran et al. 1992), give:

$$\dot{\psi}\gamma_1\gamma_2 + \pi\psi\gamma_0 = 0, \tag{5}$$

$$\dot{x} = \psi\gamma_0\tilde{\psi}, \tag{6}$$

$$\dot{\pi} = eF \cdot \dot{x}, \tag{7}$$

where $\pi = p - eA$. One can extract several informations by analysing this system. Let $\Omega = 2\dot{R}\tilde{R}$ and $S = \frac{\hbar}{2} R\gamma_2\gamma_1\tilde{R}$. Then, using eq.(2) (with $\beta = 0$) into eq.(5) and after splitting the resulting equation into its scalar, bivector and pseudo-scalar parts we get ($\hbar = 1$):

$$\dot{\rho} = 0, \tag{8}$$

$$\Omega = -4\pi \cdot (e_0 \wedge S), \tag{9}$$

$$\pi \wedge e_0 \wedge e_1 \wedge e_2 = 0. \tag{10}$$

Eq.(8) says that ρ = constant, as expected, which we take $\rho = 1$ for τ being the proper time. In this case $v = \dot{x} = e_0$, and eq.(10) says that π is a linear combination of $\{e_0, e_1, e_2\}$, from which we see that we can have solutions of eq.(5) for which v and p are not parallel – which is indeed the case for the solution given by Barut and Zanghi (1984), which in terms of STA is

$$\psi = \cos m\tau\psi(0) + \sin m\tau\gamma_0\psi(0)\gamma_0\gamma_1\gamma_2. \tag{11}$$

The parameter m, identified with mass, comes from kinetic energy of rotation (see also Hestenes 1990, 1991). In fact, from eq.(9) we can show that

$$\Omega \cdot S = \pi \cdot e_0 = m. \tag{12}$$

Another interesting conclusion can be obtained from eq.(5) after multiplying it on the right by $\tilde{\psi}$ and subtracting from the result the reverse of eq.(5) multiplied on the left by ψ, which gives

$$\dot{S} - \pi \wedge e_0 = 0. \tag{13}$$

Now, from Dirac theory one can show (Hestenes 1973) that

$$-\rho[\dot{S} - \pi \wedge e_0] = \gamma_\mu \wedge N^\mu - \partial_\mu M^\mu, \tag{14}$$

where N_μ describes the flow of energy-momentum normal to the velocity streamline and M_μ describes the flow of angular momentum normal to that streamline (we note the presence of a misprint in (Hestenes 1973) that leads to a wrong sign for $\dot{S}$ in that equation in this reference). But for a Weyssenhoff fluid (Weyssenhoff and Raabe 1947) the net flux of energy-momentum and angular momentum through the walls of a comoving volume element vanishes, i.e. (Hestenes 1973):

$$\partial_\mu N^\mu = 0, \qquad \gamma_\mu \wedge N^\mu = \partial_\mu M^\mu, \tag{15}$$

and in this case eq.(14) reduces to eq.(13). We conclude therefore that a BZ fluid is an example of a Weyssenhoff fluid.

Now, let us show that BZ model is a hamiltonian system. First, look at the lagrangian given by eq.(4); it is a first-order lagrangian. It is well-known (Sudarshan and Mukunda 1974) that non-standard lagrangians exhibit some problems to pass to the correspondent hamiltonian; but a first-order lagrangian like our one has a natural hamiltonian structure. In fact, since $< AB >_0 = < BA >_0$, we can write from eq.(4):

$$< (p - eA)\psi\gamma_0\tilde{\psi} >_0 \; = \; < \bar{\psi}\dot{\psi} >_0 - < p\dot{x} >_0 - L, \tag{16}$$

where we defined

$$\bar{\psi} = \gamma_2\gamma_1\tilde{\psi}. \tag{17}$$

Eq.(16) looks like a Legendre transformation; we have p and $\bar{\psi}$ as the momentum canonically conjugate to x and ψ, respectively, and the hamiltonian $H = H(x, p, \psi, \bar{\psi})$ is

$$H = < (p - eA)\psi\gamma_0\tilde{\psi} >_0 \; = \; < (p - eA)\psi\gamma_0\gamma_1\gamma_2\bar{\psi} >_0 . \tag{18}$$

Hamilton equations are:

$$\dot{x} = \partial_p H, \qquad \dot{p} = -\partial_x H, \tag{19}$$

$$\dot{\psi} = \partial_{\bar{\psi}} H, \qquad \dot{\bar{\psi}} = -\partial_\psi H. \tag{20}$$

It is trivial to verify that eq.(19-20) with the hamiltonian (18) give eq.(5-7) of BZ model.

Let us show how to give a symplectic structure to the phase space of BZ model – illustrating therefore the general method given by Hestenes (1992). First, note that an equation like $\dot{x} = \partial_p H$ implies $\gamma^\mu \dot{x}_\mu = \gamma^\mu \partial_{p^\mu} H$, or $\dot{x}_\mu = \partial_{p^\mu} H$. Now take a basis $\{E_0, E_1, E_2, E_3\}$ of R^4 such that $E_a \cdot E_b = \delta_{ab}$ $(a = 0, \ldots, 3)$ and define $X = \sum_a x_a E_a$; take another copy of R^4 and a basis $\{E'_0, E'_1, E'_2, E'_3\}$ with $E'_a \cdot E'_b = \delta_{ab}$ and define $P' = \sum_a p_a E'_a$. Finally, take $R^4 \oplus R^4$ with a basis $\{E_0, \ldots, E_3; E'_0, \ldots, E'_3\}$ such that $E_a \cdot E'_b = 0$ $(\forall a, b)$. We can give a symplectic structure to $R^4 \oplus R^4$ by defining the symplectic bivector J (Hestenes 1992)

$$J = \sum_a J_a = \sum_a E_a \wedge E'_a. \tag{21}$$

Note that $E'_a = E_a \cdot J = -J \cdot E_a$ and $E_a = -E'_a \cdot J = J \cdot E'_a$. Then $X' = X \cdot J = -J \cdot X$, $P = J \cdot P' = -P' \cdot J$, and we can define

$$Q = X' + P = X \cdot J + P, \tag{22}$$

$$\partial_Q = \partial_{X'} + \partial_P, \tag{23}$$

from which we can write Hamilton equations (19) as

$$\dot{Q} = \partial'_Q H \tag{24}$$

where $\partial'_Q = -J \cdot \partial_Q = \partial'_P - \partial_X$.

In order to do the same with eq.(20) remember that $R^+_{1,3} \ni \psi = <\psi>_0 + <\psi>_2 + <\psi>_4$; an equation like $\dot{\psi} = \partial_{\bar{\psi}} H$ gives $<\dot{\psi}>_0 = \partial_{<\bar{\psi}>_0} H$, $<\dot{\psi}>_2 = \partial_{<\bar{\psi}>_2} H$ and $<\dot{\psi}>_4 = \partial_{<\bar{\psi}>_4} H$ – where the second one gives $(<\dot{\psi}>_2)_{\mu\nu} = \partial_{<\bar{\psi}>_2^{\mu\nu}} H$. Now, take a basis $\{F_0, F_1, \ldots, F_7\}$ of R^8 such that $F_m \cdot F_n = \delta_{mn}$ $(m, n = 0, 1, \ldots, 7)$ and define

$$\Psi = <\psi>_0 F_0 + (<\psi>_2)_{01} F_1 + \cdots + (<\psi>_2)_{23} F_6 + <\psi>_4 F_7 \tag{25}$$

take another copy of R^8 with a basis $\{F'_0, F'_1, \ldots, F'_7\}$ such that $F'_m \cdot F'_n = \delta_{mn}$ and define

$$\bar{\Psi}' = <\bar{\psi}>_0 F'_0 + (<\bar{\psi}>_2)_{01} F'_1 + \cdots + (<\bar{\psi}>_2)_{23} F'_6 + <\bar{\psi}>_4 F'_7. \tag{26}$$

Take $R^8 \oplus R^8$ with a basis $\{F_0, \ldots, F_7; F'_0, \ldots, F'_7\}$ such that $F_m \cdot F'_n = 0$ $(\forall m, n)$, and define the symplectic bivector K:

$$K = \sum_m K_m = \sum_m F_m \wedge F'_m, \tag{27}$$

with $F'_m = F_m \cdot K$, etc., just like the previous case. If we define

$$\Phi = \Psi' + \bar{\Psi} = \Psi \cdot K + \bar{\psi} \tag{28}$$

$$\partial_\Phi = \partial_{\Psi'} + \partial_{\bar{\Psi}} \tag{29}$$

then Hamilton equations (20) can be written as

$$\dot{\Phi} = \partial'_\Phi H, \tag{30}$$

where $\partial'_\Phi = -K \cdot \partial_\Phi = \partial'_{\bar{\Psi}} - \partial_\Psi$.

The final step is to take the space $(R^4 \oplus R^8) \oplus (R^4 \oplus R^8)$ with a basis $\{E_0, \ldots, E_3; F_0, \ldots, F_7; E'_0, \ldots, E'_3; F'_0, \ldots, F'_7\}$ with $E_a \cdot F_m = E_a \cdot F'_m = E'_a \cdot F_m = E'_a \cdot F'_m = 0$ $(\forall a, m)$. The symplectic structure is given by the symplectic bivector $\mathcal{J}$:

$$\mathcal{J} = J + K = \sum_a E_a \wedge E'_a + \sum_m F_m \wedge F'_m. \tag{31}$$

After defining

$$\Pi = Q + \Phi \tag{32}$$

we write Hamilton equations (24) and (30) as

$$\dot{\Pi} = \partial'_{\Pi} H \tag{33}$$

where $\partial'_{\Pi} = \partial'_{Q} + \partial'_{\Phi}$.

We observe (Hestenes 1992) that the Poisson brackets are given by

$$\{F, G\} = \mathcal{J} \cdot (\partial_{\Pi} G, \partial_{\Pi} F) \tag{34}$$

in terms of which Hamilton equation (33) can be written as

$$\dot{\Pi} = \{H, \Pi\}. \tag{35}$$

We hope this proof that BZ model is a hamiltonian system has also illustrated the power of Clifford algebras.

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References

Barut, A.O. and Duru, I.H. (1984), Phys. Rev. Lett. **53**, 2355.

Barut, A.O. and Zanghi, N. (1984), Phys. Rev. Lett. **52**, 2009.

Doran, C., Lasenby, A. and Gull, S. (1992), "Grassmann Mechanics, Multivector Derivatives and Geometric Algebra", preprint DAMTP 92-69 Cambridge University.

Figueiredo, V.L., Oliveira, E.C. and Rodrigues, Jr., W.A. (1990), Int. J. Theor. Phys., **29**, 371.

Gull, S.F. (1991), "Charged Particles as Potential Step", in *The Electron*, Hestenes, D. and Weingartshofer, A. (eds.), Kluwer Academic Pub.., Dordrecht.

Hestenes, D. (1966), *Space-Time Algebra*, Gordon and Breach, New York.

Hestenes, D. (1967), J. Math. Phys., **8**, 789.

Hestenes, D. (1973), J. Math. Phys., **14**, 893.

Hestenes, D. (1990), Found. Phys., **20**, 1213.

Hestenes, D. (1991),"Zitterbewegung in Radiative Processes", in *The Electron*, Hestenes, D. and Weingartshofer, A. (eds.), Kluwer Academic Pub.., Dordrecht.

Hestenes, D. (1992), "Hamiltonian Mechanics with Geometric Calculus", preprint Arizona State University.

Hestenes, D. and Sobczyk, G. (1984), *Clifford Algebra to Geometric Calculus*, D. Reidel Pub.. Co., Dordrecht.

Lounesto, P. (1993), "Clifford Algebras and Hestenes Spinors", preprint Helsinki University of Technology, to appear in Found. Phys. (may 1993).

Pavšič, M., Recami, E., Rodrigues, Jr., W.A., Maccarrone, G.D., Raciti, F. and Salesi, G. (1992), "Spin and Electron Structure", preprint RP 45/92 IMECC-UNICAMP, submitted for publication.

Rawnsley, J (1992), Lett. Math. Phys., **24**, 331.

Rodrigues, Jr., W.A. and Oliveira, E.C. (1990), Int. J. Theor. Phys. **397**, 397.
Sudarshan, E.C.G. and Mukunda, N. (1974), *Classical Dynamics: A Modern Perspective*, John Wiley & Sons, Inc., New York.
Takabayasi, T. (1957), Suppl. Prog. Theor. Phys., **4**, 1.
Vaz, Jr., J. and Rodrigues, Jr., W.A., (1993) "On the Equivalence of Dirac and Maxwell Equations, and Quantum Mechanics", preprint RP 57/92 IMECC-UNICAMP, to appear in Int. J. Theor. Phys. (june 1993).
Weyssenhoff, J. and Raabe, A. (1947), Acta Phys. Pol. **9**, 7.

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