ON FRAISSE'S PROOF OF COMPACTNESS

Xavier Caicedo
and
A. M. Sette

Janeiro RP 02/93

Relatório de Pesquisa

**Instituto de Matemática
Estatística e Ciência da Computação**

**UNIVERSIDADE ESTADUAL DE CAMPINAS
Campinas - São Paulo - Brasil**

ABSTRACT – Appart from four trivial cases, a universal Horn class of graphs generated by finitely many finite graphs can not be finitely axiomatizable.

IMECC – UNICAMP
Universidade Estadual de Campinas
CP 6065
13081 Campinas SP
Brasil

Janeiro – 1993

ON FRAISSE'S PROOF OF COMPACTNESS

Xavier Caicedo
and
A. M. Sette
Campinas, November 1992

"Viajando por caminhos que não sei onde vão dar"

(Brazilian popular song)

Fraisse's proof of countable compactness in [F], Th. 1.5.3, ammounts to the construction of a sheaf of structures over the one point compactification $\omega \cup \{\infty\}$ of the discrete space ω of natural numbers, so that the fiber in ∞ is generic (forcing and truth coincide). This gives perhaps the simplest non-boolean sheaves with all their fibers being generic.

If τ is a countable type of structures, then the space E_τ of structures of type τ topologized by the elementary classes as basic open classes is pseudometric. In fact the following is a totally bounded metric generating the elementary topology. Choose first a sequence $\tau_1 \subseteq \tau_2 \subseteq \ldots$ of finite subtypes of τ such that $\tau = \cup_n \tau_n$ and define a ***modified rank*** for any sentence of type τ by

$$mr(\varphi) = \min\{n \ : \ \varphi \in L_{\omega\omega}(\tau_n), \ \ n \geq qr(\varphi)\}$$

where qr is the ordinary quantifier rank. In this context, $\mathcal{A} \equiv^n_* \mathcal{C}$ will denote elementary equivalence with respect to sentences of modified rank less than n (hence, $\mathcal{A} \equiv^0_* \mathcal{C}$ always holds). Finally define the metric by

$$d(\mathcal{A}, \ \mathcal{C}) = \inf \ \{\frac{1}{n+1} \ : \ \mathcal{A} \equiv^n_* \mathcal{C}\}.$$

It is easy to see that this pseudo-metric is totally bounded (see [W]); hence, to show compactness it is enough to show that every Cauchy sequence converges.

Fraisse's proof may be construed as extracting first a Cauchy sequence from an arbitrary sequence (this may be done because the pseudometric is totally bounded) and then showing the Cauchy sequence converges. He assumes τ is finite, but with a slight modification

Supported by FAPESP, Fundação de Amparo à Pesquisa do Estado de São Paulo.

of the notion of n-partial isomorphism his construction works for countable τ.

Following Fraissé, construct a subsequence $\{\mathcal{A}_n\}_n$ together with finite subsets $E_n \subseteq A_n$ and injective functions $f_n : E_n \to E_{n+1}$ such that

i) $f_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ is an n-partial isomorphism for each n.
ii) For any $a \in A_n$ there is $g \supseteq f_n$ such that g is a $(n-1)$-partial isomorphism, $a \in dom(g)$, and $g(a) \in E_{n+1}$.

It may be shown that if $f_{nk} = f_k \circ \ldots \circ f_m$ for $k > n$ and $\mathcal{A}_\infty =_{\underrightarrow{\lim}} \{f_{nk} : \mathcal{A}_n|E_n \to \mathcal{A}_k|E_k\}_{n \leq k}$ then the limit morphisms $f_{m\infty} : \mathcal{A}_n \to \mathcal{A}_\infty$ are n-partial isomorphisms for any n. Here, a n-partial isomorphism $f : \mathcal{A} \to \mathcal{L}$ is defined as in [F], except that it will be assumed to be isomorphism only with respect to the relation symbols in τ_n. In particular, we have

$$(\mathcal{A}_n, a_1, \ldots, a_m) \equiv^n_* (\mathcal{A}_k, f_{nk}(a_1), \ldots, f_{nk}(a_m)) \tag{1}$$

for any $a_1, \ldots, a_m \in E_n$ and $k = n, n+1, \ldots, \infty$.

Let $\omega \cup \{\infty\}$ be the one point (or Alexandroff) compactification of discrete ω. Hence, the topology of $\omega \cup \{\infty\}$ is discrete in the open subspace ω and a neighborhood basis for ∞ consists of the sets:

$$[n, \infty] = \{j \in \omega : j \geq n\} \cup \{\infty\}$$

we will write also:

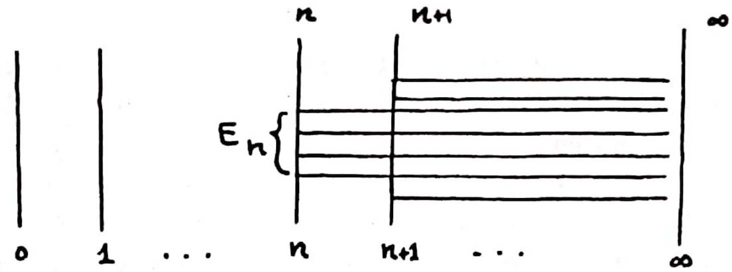
$$[n, \infty) = \{j \in \omega : j \geq n\}.$$

Define a sheaf $\mathcal{A}$ over $\omega \cup \{\infty\}$ as follows:

$$\mathcal{A}(S) = \Pi_{i \in S} \mathcal{A}_i \quad \text{if} \quad S \subseteq \omega$$

$$\mathcal{A}([n, \infty]) = \{(f_{nk}(a))_{k \in [n, \infty]} \mid a \in E_n\}, \quad n \in \omega;$$

the projections $\mathcal{P}_{ST} : \mathcal{A}(S) \to \mathcal{A}(T)$, $S \supseteq T$ are simply the restrictions. It is readily seen that this is a sheaf where the fiber over $n \in \omega$ is $\mathcal{A}_n$ and the fiber over ∞ is $\mathcal{A}_\infty$:



The sections over $[n, \infty]$ are in one to one correspondence with the elements of E_n, so that there are finitely many only. If σ is such a section then

$$\begin{aligned} \sigma(k) &= f_{nk}(a)\,, \quad k \in [n, \infty) \\ \sigma(\infty) &= f_{n\infty}(a) = \hat{a}. \end{aligned}$$

For the semantics of forcing in sheaves we refer to [E]. A sheaf of structures wilt be said to be *generic* at a point x of the base space if forcing at that point coincides with classical truth in the fiber. More, precissely, if one has

$$\mathcal{A} \Vdash_x \varphi(\sigma_1, \ldots, \sigma_k) \leftrightarrow \mathcal{A}_x \models \varphi(\sigma, (x), \ldots, \sigma_n(x)) \tag{2}$$

for each formula $\varphi(x_1, \ldots, x_n)$ and (partial) sections $\sigma_1, \ldots, \sigma_n$ of $\mathcal{A}$ defined in x. Here, $\mathcal{A}_x$ is *fiber of* $\mathcal{A}$ *at* x.

Clearly, Fraisse's sheaf is generic at any point $n \in \omega$, because $\{n\}$ is open. Less obviously, it is generic at ∞. First we need:

LEMMA. *Let* $n \in \omega$, $n \geq mr(\varphi)$, *then the following are equivalent for* $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \mathcal{A}([n, \infty])$:
i) $\mathcal{A} \Vdash_n \varphi(\sigma)$
ii) $\mathcal{A}_j \models \varphi(\overline{\sigma}(j))$ *for some* $j \in [n, \infty]$.

Proof. We have already noticed that $\mathcal{A} \Vdash_n \varphi(\overline{\sigma})$ iff $\mathcal{A}_n \models \varphi(\overline{\sigma})$. Now, if $j \in [n, \infty]$ let $\overline{a} = \overline{\sigma}(n)$, then $\overline{\sigma}(j) = f_{nj}(\overline{a})$ and by (1), $\mathcal{A}_n \models \varphi(\overline{\sigma}(n))$ iff $\mathcal{A}_j \models \varphi(\overline{\sigma}(j))$, including $j = \infty$, because $n \geq mr(\varphi)$. □

PROPOSITION. $\mathcal{A}$ *is generic at* ∞.

Proof. We show (2) by induction in formulas

φ *atomic.* Follows by definition of forcing.

$\varphi = \psi \wedge \theta,\ \ \psi \vee \theta$. Trivial inductive step.

$\varphi = \psi \rightarrow \theta$. If $\mathcal{A} \Vdash_{\infty} (\psi \rightarrow \theta)(\overline{\sigma})$ and $\mathcal{A}_{\infty} \models \psi(\overline{\sigma}(\infty))$ then $\mathcal{A} \Vdash_{\infty} \psi(\overline{\sigma})$ by induction hypothesis and so $\mathcal{A} \Vdash_{\infty} \theta(\overline{\sigma})$. By induction again, $\mathcal{A}_{\infty} \models \theta(\overline{\sigma}(\infty))$. Conversely, assume $\mathcal{A}_{\infty} \models (\psi \rightarrow \theta)(\overline{\sigma}(\infty))$ and let $n \in \omega$ be large enough so that $n \geq qr(\varphi)$ and $\overline{\sigma}$ is defined in $[n,\ \infty]$. If $\phi \neq S \subseteq [n,\ \infty]$ is such that $\mathcal{A} \Vdash_{S} \psi(\overline{\sigma})$ then by genericity in $j \in \omega$ and by induction hypothesis in ∞, we have $\mathcal{A}_j \models \psi(\overline{\sigma}(j))$ for any $j \in S$. By the Lemma, $\mathcal{A}_{\infty} \models \psi(\overline{\sigma}(\infty))$ and so $\mathcal{A}_{\infty} \models \theta(\overline{\sigma}(\infty))$. By the lemma again $\mathcal{A}_j \models \theta(\overline{\sigma}(\sigma))$ for any $j \in S$. By induction hypothesis, $\mathcal{A} \Vdash_{S} \theta(\overline{\sigma})$. This shows $\mathcal{A} \Vdash_{\infty} (\psi \rightarrow \theta)(\overline{\sigma})$.

$\varphi = \exists \mu \theta(\mu,\ \overline{x})$. Trivial induction.

$\varphi = \forall \mu \theta(\mu,\ \overline{x})$. If $\mathcal{A}_{\infty} \Vdash \forall \mu \theta(\mu,\ \overline{\sigma})$ then for any τ defined in ∞, $\mathcal{A} \Vdash_{\infty} \theta(\tau,\ \overline{\sigma})$ and by induction hypothesis $\mathcal{A}_{\infty} \models \theta(\tau(\infty),\ \overline{\sigma}(\infty))$. As any element of $\mathcal{A}_{\infty}$ is to the form $\tau(\infty)$, we have $\mathcal{A}_{\infty} \models \forall \mu \theta(\mu,\ \overline{\sigma}(\infty))$.

Conversely, if $\mathcal{A}_{\infty} \models \forall \mu \theta(\mu,\ \overline{\sigma})$, let $n = mr(\theta) + 1$ We must show that for any τ defined in $j \in [n,\ \infty]$,

$$\mathcal{A} \Vdash_{j} \theta(\tau,\ \overline{\sigma}). \tag{3}$$

For $j = \infty$ this follows by induction since by hypothesis $\mathcal{A}_{\infty} \models \theta(\tau(\infty),\ \overline{\sigma}(\infty))$. For $j \in [n,\ \infty)$, let $a = \tau(j) \in \mathcal{A}_j$. Since $f_{j\infty}$ is an n-partial isomorphism and $f_{j\infty}(\overline{\sigma}(j)) = \overline{\sigma}(\infty)$, then there is $\hat{b} \in \mathcal{A}_{\infty}$ such that

$$(\mathcal{A}_j,\ a,\ \overline{\sigma}(\sigma)) \equiv^{n-1} (\mathcal{A}_{\infty},\ \hat{b},\ \overline{\sigma}(\infty)).$$

As $\mathcal{A}_{\infty} \models \theta(\hat{b},\ \overline{\sigma}(\infty))$ by hypothesis and $mr(\theta) = n - 1$, then $\mathcal{A}_j \models \theta(a,\ \overline{\sigma}(j))$ and so $\mathcal{A}_j \Vdash \theta(\tau,\ \overline{\sigma})$. With this we have shown (3). □

Although Fraisse's performs his limit construction with a subsequence of the given Cauchy sequence, it is clear that we may from a sheaf as above with the original sequence.

In this way, the Cauchy sequences of E_τ are extendible to sheaves of structures of type τ over $\omega \cup \{\infty\}$. If we identify a structure $\mathcal{A}$ with the constant sheaf $\overline{\mathcal{A}}$ over $\omega \cup \{\infty\}$, we have an embedding

$$E_\tau \overset{I}{\hookrightarrow} Sh_\tau(\omega \cup \{\infty\})$$

where $Sh_\tau(\omega \cup \{\infty\})$ is the category of sheaves of structures of type τ over $\omega \cup \{\infty\}$. Now, $Sh_\tau(\omega \cup \{\infty\})$ has the natural topology induced by intuitionist logic where the basic closed classes are of the form

$$\mathcal{O}_\varphi = \{S \in Sh_\tau(\omega \cup \{\infty\})\ :\ S \Vdash \varphi\}.$$

Under this topology, the embedding is continuous, because for the constant sheaf: $\overline{\mathcal{A}} \Vdash \varphi \Leftrightarrow \mathcal{A} \models \varphi$.

Questions: 1) Is $Sh_\tau(\omega \cup \{\infty\})$ compact?
2) Is $\{S \in Sh_\tau(\omega \cup \{\infty\})\ :\ S \text{ generic at } \infty\}$ a closed compact subspace of Sh_τ?
3) May every sheaf over $\omega \cup \{\infty\}$ with generic limit be construed as a Fraisse's limit?

BIBLIOGRAPHY.

[F] R. Fraissé, *Cours de Logique Mathematique*, Gauthier-Villars, Paris, 1972.

[E] D. P. Ellerman, *Sheaves of Structures and Generalized ultraproducts.* Ann. Math. Logic 7 (1974) 163-195

[W] S. Willard, *General Topology*, Addison Wesley, 1968.

RELATÓRIOS DE PESQUISA — 1993

01/93 **On the Convergence Rate of Spectral Approximation for the Equations for Nonhomogeneous Asymmetric Fluids** — *José Luiz Boldrini* and *Marko Rojas-Medar.*