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# COMPACTNESS OF TOPOLOGICAL SPACES OF MODELS

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**ABSTRACT** – We study $[k, \lambda]$ compactness of topological spaces in general and spaces of structures associated to abstract logics in particular. A characterization in terms of ultrafilter convergence of preservation of $[k, \lambda]$-compactness by products of general topological spaces is given. The Abstract Compactness Theorem of Makowsky and Shelah becomes then a simple corollary from a general topological result. Reciprocally, several results on compactness shown first for logics are seen to hold true in all topological spaces.

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# Compactness of topological spaces of models

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We discuss various forms of compactness in model theoretical logics, their inter-relations, and consequences, trying to separate clearly purely topological facts from genuine model theoretical facts. We discuss in this context results due to Shelah, Makowsky, and Mundici. Some of them are just special cases of general topological results, others reveal an interesting topological behaviour of the spaces of structures generated by a logic. In particular, inspired by the "Abstract Compactness Theorem" of Makowsky and Shelah [M-Sh], we prove a characterization in terms of ultrafilter convergence of preservation of $[\kappa,\lambda]$-compactness by products of general topological spaces. The Abstract Compactness Theorem becomes then a simple corollary from a general topological result. Reciprocally, several results on compactness shown first for logics are seen to hold true in all topological spaces.

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## 1. TOPOLOGICAL PRELIMINARIES, $[\kappa,\lambda]$-COMPACTNESS

The following generalization of the notion of compactnes of a topological space was considered first by Alexandroff and Urysohn [AU], and studied later by Smirnov [S], Gaal [G] and Vaughan [V1], see also Kunen-Vaughan [Ku-V]. The identity between these notions and those introduced by Makowsky and Shelah [Ma-Sh] in logic was noticed first by Mannila [M].

**Definition 1.1.** A topological space X is said to be $[\kappa,\lambda]$-*compact*, for $\omega \leq \lambda \leq \kappa \leq \infty$, if any covering of X by a family of at most $\kappa$ open sets has a covering subfamily of power less than $\lambda$. Equivalently, any family of at most $\kappa$ closed subsets of X for which every subfamily of power less than $\lambda$ has non- empty intersection (we will call this the *$\lambda$-intersection property)* has itself non-empty intersection.

$[\infty,\omega]$-compactness is ordinary compactness of topological spaces, $[\omega,\omega]$-compactness is usually called *countable compactness* in topology, and $[\infty, \omega_1]$-compactness is the so called *Lindelof property*.

The reader should be aware that the notation utilized for $[\kappa,\lambda]$-compactness in model theory, which we adopt here, reverses the more natural notation utilized in the topological literature.

**LEMMA 1.2.** X *is* $[\kappa,\lambda]$-*compact if and only if it is* $[\mu,\mu]$-*compact for any* $\mu \leq \kappa$.

Proof. $[\kappa,\lambda]$-compactness implies trivially $[\mu,\mu]$-compactness for $\lambda \leq \mu \leq \kappa$. Now, if X is not $[\kappa,\lambda]$-compact, let $\{C_\alpha\}_{\alpha<\mu}$ be a family of closed sets giving a counterexample of minimal cardinality $\mu$ to $[\mu,\lambda]$-compactness. It is in fact a counterexample to $[\mu,\mu]$-compactness because an intersection $\cap_{i<\delta}C_{\alpha_i}$ with $\delta < \mu$ is non-empty: for $\delta < \lambda$ by hypothesis, and for $\lambda \leq \delta < \mu$ because otherwise we would have a counterexample to $[\lambda,\delta]$-compactness contradicting minimality of $\mu$. □

Therefore, we may reduce to the study of $[\kappa,\kappa]$-compactness. This takes a very simple and useful form if $\kappa$ is regular cardinal. First we introduce a stronger notion of $\kappa$-compactness.

**Definition 1.3.** A space X is *$\kappa$-chain compact* if and only if any descending chain $\{C_\alpha\}_{\alpha<\kappa}$ of non-empty closed sets of X has non-empty intersection.

**LEMMA 1.3.** a) *$\kappa$-chain compactness implies $[\kappa,\kappa]$-compactness.*

b) *If* $\mathrm{cof}(\kappa) \geq \lambda$ *and* X *is $[\kappa,\lambda]$-compact then it is $\kappa$-chain compact.*

c) (Alexandroff and Urysohn [AU]) *If $\kappa$ is regular, then* X is *$[\kappa,\kappa]$-compact if and only* X *is $\kappa$-chain compact.*

Proof. a) If $\{C_\alpha\}_{\alpha<\kappa}$ is a counterexample to $[\kappa,\kappa]$-compactness then the family $\{D_\alpha\}_{\alpha<\kappa}$ where $D_\alpha = \cap_{\beta<\alpha}C_\beta$ is a counterexample to $\kappa$-chain compactness.

b) Given a descending chain of closed non-empty closed sets $\{C_\alpha\}_{\alpha<\kappa}$, then the intersection of less than $\lambda$ many $C_\alpha$'s contains a $C_\beta$ which is non-empty by hypothesis; hence, the hypothesis of $[\kappa,\lambda]$-compactness applies and the full intersection is non-empty.

c) From (a) and (b). □

COROLLARY 1.4. *If X is [cof(κ),cof(κ)]-compact then it is [κ,κ]-compact.*

Proof. If X is not [κ,κ] compact, then it is not κ-chain compact. Let $\{C_\alpha\}_{\alpha<\kappa}$ be a descending chain providing a counterexample to κ-chain compactness, then any cofinal descending subchain of length *cof*(κ) provides a counterexample to cof(κ)-chain-compactness, and so to [cof(κ),cof(κ)]-compactness by Lemma 1.3(c), since *cof*(κ) is regular. □

COROLLARY 1.5. a) (Alexandroff and Urysohn [AU]). *X is [κ,ω]-compact if and only if it is [μ,μ]-compact for any infinite regular cardinal μ ≤ κ.*

b) (Vaughan [V2]). *If cof(κ) ≥ λ, then X is [κ,λ]-compact if and only if it is [μ,λ]-compact for any infinite regular cardinal μ ≤ κ.*

Proof. a) One direction is trivial. For the other notice that by hypothesis, X is [cof(μ),cof(μ)]-compact and so [μ,μ]-compact by Corollary 1.4, for any μ ≤ κ. Now apply Lemma 1.2.

b) One direction is trivial. For the other notice that [cof(κ),cof(κ)]-compactness holds by hypothesis; hence, [κ,κ]-compactness holds by Corollary 1.4. Now, let λ ≤ μ < κ, then $[\mu^+,\lambda]$-compactness holds by hypothesis and so [μ,μ]-compactness holds, then apply Lemma 1.2 □

Remark. It follows from the above corollary, part (a), that a space X is compact if and only if it is [μ,μ]-compact for any regular μ ≤ Weigth(X).

Definition 1.6. Given a subset S of a topological space X, a point x ∈ X wil be called a *κ-accumulation point of* S in X if and only if for any open neighborhood V of x we have |V ∩ S| ≥ κ.

THEOREM 1.7. (Alexandroff and Urysohn, [AU]). *Let κ be a regular cardinal. A space X is [κ,κ]-compact if and only if any subset S ⊆ X of power ≥ κ has a κ-accumulation point.*

Proof. Assume X is [κ,κ]-compact. Given $x_\alpha \in S$, $x_\alpha \neq x_\beta$ for $\alpha < \beta < \kappa$, let $C_\alpha = Cl\{x_\beta : \beta \geq \alpha\}$; then the $C_\alpha$ form a descending chain of nonempty closed sets. By Corollary 1.3(c) there is a point x in the intersection of the $C_\alpha$ and so in the adherence of each $\{x_\beta : \beta \geq \alpha\}$ for any $\alpha < \kappa$. Any neigborhood V of x contains then a sequence $\{x_{\beta_\alpha} : \alpha<\kappa\}$ with $\beta_\alpha > \alpha$. By regularity of κ again the sequence $\{\beta_\alpha : \alpha < \kappa\}$ and so the family $\{x_{\beta_\alpha} : \alpha < \kappa\}$ has power κ.

Conversely, assume that [κ,κ]-compactness fails, then by Corollary 1.3(c), there is a descending chain of non-empty closed sets $\{C_\alpha\}_{\alpha<\kappa}$ with empty intersection. We may pick

$x_\alpha \in C_\alpha$ all distinct, otherwise the sequence would be finally constant and the intersection would not be empty. Let x be a $\kappa$-accumulation point of the family $S = \{x_\alpha\}_{\alpha<\kappa}$, then any neigborhood V of x intersects a subsequence of S of power $\kappa$ which evidently must be cofinal. Therefore, V intersects a cofinal sequence of the $C_\alpha$ and so all the $C_\alpha$ 's, which shows x is in the adherence of all $C_\alpha$ and so in their intersection because they are closed. This is a contardicton. □

In some respects $[\kappa,\lambda]$-compacteness has the same good behaviour of full compactness. The following is obvious.

**LEMMA 1.8.** a) *If* f: X → Y *is continuous and* X *is* $[\kappa,\lambda]$*-compact, then* f(X) *is* $[\kappa,\lambda]$*-compact.*
b) *A closed subspace of a* $[\kappa,\lambda]$*-compact space is* $[\kappa,\lambda]$*-compact.*

In others, it behaves badly. Thichonoff's theorem fails. Product of $[\kappa,\lambda]$-compact spaces is rarely $[\kappa,\lambda]$-compact. For example, a product of countably compact spaces is not necessarily countably compact, neither a product of Lindelof spaces nedds to be Lindelof. See Vaughan [V1] for a discussion of this matter.

## II. COMPACTNESS OF SPACES OF STRUCTURES

For the definition of a logic for first order structures see Lindstrom [L] or Ebbinghauss [E]. We will always assume that logics are *regular*, that is they contain first order logic, $L_{\omega\omega}$, and are closed at least under negations, finite conjunctions, substitutions, and relativizations. For a logic L, the *domain of* L, Dom(L), will be the class of vocabularies $\sigma$ where $L(\sigma)$ is defined. $L(\sigma)$ will be always assumed to be a set, that is we consider only *small* logics.

A logic L induces a topology in the class $E_\sigma$ of structures of type $\sigma$, having for basis of open clases the L-elementary classes, for each $\sigma \in$ Dom(L). The closed classes have the form Mod(T) for some theory T $\subseteq L(\sigma)$. We call $E_\sigma(L)$ to the corresponding large topological space. This space is a proper class and the same happens to the open and closed classes of the topology, but the basis is parametrized by the set $L(\sigma)$ and so the topology is also (parametrized by) a set; hence we may apply the ordinary concepts and result of topology without missgivings. This spaces are uniform with the following canonical uniformity: for each finite theory F $\subseteq L(\sigma)$ a basic of the uniformity is given by

Notice a curious connection between both notions: L *is* $[\kappa,\lambda]$*-compact if and only if any closed subclass of* $E_\tau(L)$ *is* $(\kappa,\lambda)$*-compact with respect to the topology inherited form* $E_\tau(L_{\omega\omega})$. This is just a reformulation of Lemma 2.1 above.

The following is a very useful characterization of $[\kappa,\kappa]$-compactness for regular logics, noticed first by Väänänen [V] and implicit in Lindström [L] for the countable case (see Th. 1.2.2 in [Ma]).

THEOREM 2.3. *The following are equivalent for a regular logic* L *and a regular cardinal* $\kappa$*:*

*i)* L *is* $[\kappa,\kappa]$*-compact.*

*ii) Any structure* $\mathcal{M} = (M, <, \ldots)$ *where* $<$ *is a linear order of* M *of cofinality* $\kappa$*, has an elementary extension* $\mathcal{M}' = (M', <', \ldots)$ *such that* M *is not cofinal in* $(M', <')$.

Proof. (i) $\Rightarrow$ (ii). Assume L is $[\kappa,\kappa]$-compact. Given $\mathcal{M} = (A, <, \ldots)$ as in (ii), let $(a_\alpha)_{\alpha<\kappa}$ be a strictly increasing cofinal sequence in $(A, <)$ and

$$T = Th_L[(A, <, \ldots, a_\alpha, a)_{a \in A,\ \alpha < \kappa}] .$$

If c is a new constant, then each theory $T_\delta = T \cup \{ c > a_\alpha \mid \alpha < \delta\}$, $\delta < \kappa$, has a model (interpret c by $a_\delta$). The family $\{Mod(T_\alpha)\}_{\alpha<\mu}$ forms a descending chain of closed classes; by regularity of $\kappa$ and Lemma 1.3(c), the intersection of the family is non-empty, yielding a model of $T \cup \{c > a_\beta \mid \beta < \kappa\}$ that is the desired extension.

(ii) $\Rightarrow$ (i) if L is not $[\kappa,\kappa]$-compact, it is not chain-compact. Let $\{T_\alpha \mid \alpha < \kappa\}$ be a family of theories in $L(\tau)$ such that each $\cup_{\alpha<\gamma} T_\alpha$ has a model $A_\gamma$ for $\gamma < \kappa$, but $\cup_{\alpha<\kappa} T_\alpha$ does not. Choose an ordering $<_\gamma$ of each $A_\gamma$ with first element $a_\gamma$ and let

$$\mathcal{M} = ( \cup_{\gamma<\kappa} A_\gamma,\ <,\ \cup_{\gamma<\kappa} R^{A_\gamma}, \ldots, S )_{(R \in \tau)}$$

where $<$ is the ordered union of the $<_\gamma$ and

$$S = \cup_{\gamma<\kappa} (A_\gamma \times A_\gamma).$$

Evidently $(a_\gamma)_{\gamma<\kappa}$ is cofinal in $(M, <)$. Moreover, if $x \geq a_\gamma$ then $\mathcal{M} \restriction \{ y : S(y,x)\} \restriction \tau \approx A_{\gamma'}$, with $\gamma' > \gamma$ and so:

$$\mathcal{M} \models \forall x\, (x \geq a_\gamma \rightarrow \varphi^{\{y : S(y,x)\}}) \qquad (1)$$

for any $\varphi \in T_\gamma$. If $\mathcal{M}' > \mathcal{M}$, $\mathcal{M}'$ must satisfy these sentences. Hence, it can not contain b such that $b > a_\gamma$ for all $\gamma < \kappa$, because then we would have by (1):

$$\mathcal{M}' \models \varphi^{\{y \mid S(y,b)\}}$$

and so $\mathcal{M}' \restriction \{y : S(y,b)\} \restriction \tau$ would be a model of all $\varphi \in \bigcup_{\gamma<\kappa} T_\gamma$. We conclude that $(a_\gamma)_{\gamma<\kappa}$, and so M, is cofinal in $(M', <')$. □

**COROLLARY 2.4.** *The following are equivalent for a regular logic* L:

i) L *is compact.*

*ii) Any infinite structure* $\mathcal{M} = (M, <, ...)$ *where* $<$ *is a linear order of* M *has an elementary extension* $\mathcal{M}' = (M', <', ...)$ *such that* M *is not cofinal in* $(M', <)$.

**COROLLARY 2.5** L *is* $[\omega,\omega]$ *compact if and only if* $(\omega, <)$ *is not* $RPC_\delta$ *characterizable in* L.

Proof. "⇒" obvious. "⇐" If L is not $[\omega,\omega]$-compact then there is $\mathcal{M} = (M, <, ...)$ of cofinality $\omega$ with $(M, <)$ cofinal in all its L-elementary extensions. Take a strictly increasing cofinal sequence $(a_n)_{n\in\omega}$ of $(M, <)$. The theory:

$$\begin{aligned} Th_L[(M, <, ..., a, a_n)_{a\in A, n\in\omega}] &\cup \{\forall x \exists y(P(y) \wedge x < y)\} \cup \{\neg\exists x\, (P(x) \wedge x < c_0)\} \\ &\cup \{P(c_n) \wedge \neg\exists x\, (P(x) \wedge c_n < x < c_{n+1}) : n \in \omega\} \end{aligned}$$

gives a $RPC_\delta$ characterization of $(\omega, <)$, since the interpretation of P in any model must consist of the $a_n$ alone. □

It follows from the previous corollary that the failure of $[\omega,\omega]$-compactness for L implies that $(\omega,<)$ is $RPC_\delta$ characterizable in L. In fact any structure (of power below the first measurable cardinal if there is any) is $RPC_\delta$-characterizable in L under failure of countable compactness, as we will se next. This may be shown via Thorem 2.3, proving first that any ordinal below the first measurable is $RPC_\delta$, but the following reformulation of the Rabin-Keisler theorem allows a more direct and elegant proof.

**Definition 2.6.** A structure A will be said to be L-*full* (also *complete*) if any relation $R \subseteq A^n$ is the interpretation of a predicate in the vocabulary of A.

Given an full structure and an extension $A <_L A^*$, we will denote by $R^*$ the canonical extension in $A^*$ of any relation R in A (that is, if R is the interpretation of $P_R$ in A, then $R^*$ is the interpretation of $P_R$ in $A^*$) . Notice that if $a \in A^m$ and $R = \{x \in A^n: A \models \phi(x, a) \}$ with $\phi \in L$, then $R^* = \{x \in (A^*)^n: A^* \models \phi(x, a) \}$, because the sentence $\forall x(P_R(x) \leftrightarrow \phi(x, a))$ holds in A if and only it holds in $A^*$. Hence, any relation holding between the R's, expressible in L holding in A, holds for the corresponding $R^*$'s in $A^*$. The following is a reformulation of the Rabin-Keisler theorem [BS], [Ch-K].

**LEMMA 2.7.** *Let A be a L-full structure, $A <_{L_{\omega\omega}} A^*$, $P \subseteq A$, and $b \in P^* - P$. Define* $U_b = \{S \subseteq P : b \in S^*\}$.

i) $U_b$ *is a non principal ultrafilter over* P.

ii) *If* $|P| <$ *first measurable cardinal then any infinite* $Q \subseteq A$ *has* $Q^* - Q$ *no empty.*

Proof. i) To show that it is an ultrafilter, utilize that the extension is full and the previous remarks; for example, $S, T \in U_b \Rightarrow b \in S^*, T^* \Rightarrow b \in S^* \cap T^* = \{x \in A^*: P_S(x) \wedge P_T(x) \} = \{x \in A^*: P_{S \cap T}(x) \}^* = (S \cap T)^*$, where the last equation holds beacuse the sentence $\forall x((P_S(x) \wedge P_T(x) \Leftrightarrow P_{S \cap T}(x))$ holds in A ands so in $A^*$. Now, if the ultrafillter where principal, say $U_b = (\{a\})$ with $a \in A$, then we would have $b \in \{a\}^* = \{a\}$ and so $b = a \in P$.

ii) It is enough to show that any countably infinite subset Q of A is extended. Suposse Q is not extended. Given a countable family of elements of $U_b$ , utilize Q as subindex set: $\{S_q: q \in Q\}$. Let $S = \cap_q S_q$ and $F(q,x)$ be the binary relation $x \in S_q$ in A. The following sentences hold trivially in A where we use the names of the sets as predicates:

$$\forall x\, [\, S_q(x) \to F(q, x)] \quad \text{for each } q \in Q;$$
$$\forall x\, [\, \forall q\, (Q(q) \to F(q, x)\,) \to S(x)\,]$$

hence, it holds for b in $^*A$ that:

$$^*S_q(b) \to {}^*F(q, b) \quad \text{for each } q \in Q$$
$$\forall q\, ({}^*Q(q) \to {}^*F(q, b)\,) \to {}^*S(b)$$

Since $^*S_q(b)$ holds by hypothesis for $q \in Q^*$ then $^*F(q,b)$ holds in $^*A$ for all $q \in Q$. As $Q = Q^*$ also by hypothesis then the antecedent of the last implication holds and so $S^*(b)$ holds, showing $S \in U_b$. We conclude that $U_b$ is an $\omega$-complete ultrafilter. But it is well known that as set carrying an $\omega$-complete no principal ultrafilter must have power at least the first measurable cardinal [BS]. □

**COROLLARY 2.8.** *If L is not $[\omega,\omega]$-compact then any structure of power less than the first measurable is $RPC_\delta$-characterizable in L.*

**Proof.** Let T(Q,R) be a theory such that $B \models T \Leftrightarrow B|Q^B|\{R\} \approx (\omega,<)$, and fix a model B of T. Let C be the full expansion of [B, A] where A has power smaller than the first measurable, and call P be the universe of A in C. We may asume the vocabularies of T and A are disjoint. Any L-extension of C contains an L-extension of B which must be a model of T (thanks to relativizations), and so $(Q^B,<) \cong (\omega,<)$ can not be extended in C by hypothesis. By the previous lemma P can not be extended either. This means that any model of $Th_L((C,c)_{c \in |C|})$ relativized to P is isomorphic to A, yielding a $RPC_\delta$ characterization of A in L. □

The expresive power of theories in a non-countably compact logic is therefore quite strong.

**COROLLARY 2.9.** (Makowsky and Shelah, [Ma-Sh]). *If $\kappa$ is a regular cardinal smaller than the first measurable cardinal (or arbitrary if there are not such cardinals), then $[\kappa,\kappa]$-compactness of a logic implies $[\omega,\omega]$-compactness.*

**Proof.** If L is not $[\omega,\omega]$-compact, $\kappa$ is $RPC_\delta$-characterizable which contradicts $[\kappa,\kappa]$-compactness. □

**Remark.** If we had the stronger hypothesis in Corollary 2.8 that $(\omega,<)$ is $PC_\delta$ characterizable, then any structure (of power < first measurable) would be $PC_\delta$ characterizable, by the L-theory of its full expansion (since in the infinite case it would contain a copy of a structure capturing $\omega$). This is the case for example of $L_{\omega\omega}(Q_0)$. If $(\omega,<)$ has a characterization with models of power less than the first measurable, then any structure above certain cardinality is also $PC_\delta$. Could it happen that any theory $RPC_\delta$ characterizing $(\omega,<)$ in L only has models of cardinality above the first measurable? In that case there would not be counterexample to $[\omega,\omega]$-compactness in the form: if each finite subfamily of theories has models of power less than the first measurable, then it has models.

## III. COMPACTNESS AND ROBINSON'S LEMMA

It is well known that in the presence of compactness interpolation implies Robinson's lemma for first order logic, this being true for any logic. Mundici and Makowsky discovered independently the remarkable result that in any logic with small dependence number, Robinson's theorem implies compactness, yielding the equations Robinson = Compactness + Interpolation.

**Definition 3.1.** A logic L satisfies the *Robinson property* if given any pair of vocabularies and structures $\sigma_i$, $A_i$ , i = 1, 2, with $\mu = \sigma_1 \cap \sigma_2$ such that $A_1|\mu \equiv_L A_2|\mu$ , there is a a third structure B such that $A_1 \equiv_L B|\sigma_1$ and $A_2 \equiv_L B|\sigma_2$.

Of course, we may strength $\equiv_L$ to $<_L$ above. In terms of theories the property may be expressed: if we have satisfiable theories $T_i \subseteq L(\sigma_1)$, $T_1$, $T_2 \supseteq T$ with T complete in $L(\tau_1 \cap \tau_2)$ then $T_1 \cup T_2$ is satisfiable. The following is an equivalent characterization of Robinson's property which follows also by definition.

**LEMMA 3.2.** L *satisfies the Robinson property if and only if any pair of* $PC_\delta$ *classes* $K_1$, $K_2$ *of* L *having structures* $A_i \in K_i$ *such that* $A_1 \equiv_L A_2$ , *must have non-empty intersection.*

**THEOREM 3.3** (see Mundici, [Mu]). *For logics with finite dependence, Robinson's property implies compactness. In general, for small logics with dependence number at most* $\kappa$, *Robinson's property implies* $[\infty, \kappa]$-*compactness.*

<u>Proof.</u> Suppose L is not $[\mu,\mu]$ compact for some regular $\mu \geq \kappa$ and let $\mathcal{A} = (A, <, ...)$ a expansion of a linear order of cofinality $\mu$, having all its elementary extensions cofinal. Let $(a_\alpha)_{\alpha < \mu}$ be a well ordered cofinal sequence in $(A, <)$, introduce predicates $P_\alpha = \{x \in A \mid x \leq a_\alpha\}$ and let:

$$\mathcal{A}_1 = (A, <, ... a_\alpha, P_\alpha)_{\alpha<\mu}$$
$$\mathcal{A}_2 = (A \cup \{c\}, c, P_\alpha)_{\alpha<\mu}$$

where $c \notin A$. Let $\tau_1 = \tau(\mathcal{A}_1)$, $\tau_2 = \tau(\mathcal{A}_2)$ and $\tau = \tau_1 \cap \tau_2 = \{P_\alpha\}_\alpha$ then for $\delta < \kappa \leq \mu$ and $\tau_\delta = \{P_\alpha : \alpha < \delta\}$ we have

$$\mathcal{A}_1 \upharpoonright \tau_\delta = (A, P_\alpha)_{\alpha < \delta} \approx (A \cup \{c\}, P_\alpha)_{\alpha < \delta} = \mathcal{A}_2 \upharpoonright \tau_\delta .$$

As any sentence of L depends in less than $\kappa$ symbols, then $\mathcal{A}_1 \upharpoonright \tau \equiv_L \mathcal{A}_2 \upharpoonright \tau$ by the isomorphism axiom of logics. By Robinson's theorem there is

$$\mathcal{A}' = (A', <', ..., c', a'_\alpha, P'_\alpha)_{\alpha < \mu}$$

of type $\tau_1 \cup \tau_2$ such that $\mathcal{A}' \upharpoonright \tau_1 \equiv_L \mathcal{A}_1$, $\mathcal{A}' \upharpoonright \tau_2 \equiv \mathcal{A}_2$. But

$$\mathcal{A}_1 \models \forall x \,[\neg P_\alpha(x) \to x > a_\alpha], \quad \text{for all } \alpha < \mu,$$
$$\mathcal{A}_2 \models \neg P_\alpha(c),$$

hence $\mathcal{A}' \models c' > a'_\alpha$ for all $\alpha < \mu$, and so $\mathcal{A}' \upharpoonright \tau(\mathcal{A})$ gives a non cofinal elementary extension of $\mathcal{A}$, a contradiction. □

The finite dependence hypothesis of the first part of the above theorem may be weakened, thanks to a strong result due to Shelah [M-Sh] that we will not prove. It says that a $[\omega,\omega]$-compact logic has finite dependence for each sentence (notice that this is easy if the logic is fully compact, but it is a deep theorem if we have only $[\omega,\omega]$-compactness).

COROLLARY 3.4. *For small logics with dependence number smaller than the first measurable cardinal, we have: Robinsons's lemma ⇒ compactness.*

Proof. Let $\mu$ be the dependence number of L and let $\kappa$ be regular such that $\mu < \kappa <$ first measurable. By Theorem 3.3, L is $[\kappa, \kappa]$-compact. By Corollary 2.7 it must be $[\omega,\omega]$-compact. By the finite dependence theorem of Shelah (Th. 2.2.1. in [Ma]), L has finite dependence. Hence, by Theorem 3.3 again we have full compactness. □

For the next theorem we need a topological observation first. Define in a uniform space $x \equiv y$ if $(x,y) \in U$ for any element U of the uniformity.

LEMMA 3.5 . *Let M and N be disjoint compact sets in a uniform space inseparable by finite unions of basic open sets (from any given basis), then there exists* $x \in N$, $y \in M$ *such that* $x \equiv y$.

Proof. Let $E = \{ (x,y) \in X \times X : x \equiv y \}$, then $E \cap N \times M$ is the intersection of the closed sets $C_{p,n} = \{(x,y) \in N \times M : p(x,y) \leq 1/n \}$ of $M \times N$, where p runs through the system of pseudometrics defining the uniformity. Now, for each pseudometric p and number $n \in \omega$, the coverings $\{V_{p,1/2n}(x) : x \in N \}$ and $\{V_{p,1/2n}(y) : y \in M \}$, must have some $x \in N$, $y \in M$ with $V_{p,1/2n}(x) \cap V_{p,1/2n}(y)$ non empty. Otherwise, N and M would be separable by opens, which could be reduced to finite unions of basics by compactness. Therefore $p(x,y) \leq 2(1/2n) = 1/n$. Since the pseudoemtrics are directed by $\leq$ (see [ ]), this shows that $E \cap N x M$ has the f.i.p. By compactness of the product topology we have that E is non empty.

Remark. The above lemma holds for regular spaces.

COROLLARY 3.6 . *The following are equivalent if L is small and has dependence number less than the first measurable cardinal:*

*i) Robinson's property,*

*ii) $Int_\delta$ (interpolation of disjoint $PC_\delta$ classes),*

*iii) Interpolation + Compactness.*

Proof. (i) ⇒ (ii, iii). Assume Robinson's property, then we have compactness by Theorem 3.3. Let $K_1$, $K_2$ be disjoint $PC_\delta$ classes in $E_\mu(L)$ and assume they are not separable by a sentence. Since the $K_i$ are compact, the Lemma above implies the existence of $A_i \in K_i$ with $A_1 \equiv_L A_2$. Then Robinson's lemma provides a structure in the intersection of this two classes by Lemma 3.1.

(ii) ⇒ (i). Robinson's property is just a case of interpolation of $PC_\delta$. If $K_i$ are $PC_\delta$ classes of L as in Lemma 3.1, i = 1,2, they must intersect, otherwise by $Int_\delta$ they would be separable by a sentence $\phi$ contradicting that $A_1 \equiv_L A_2$.

(iii) ⇒ (ii). Trivial, because disjoint $PC_\delta$ classes are reduced by compactness to disjoint PC classes. □

Finally we relate Robinson's property to characterization of structures.

LEMMA 3.7. *Let L satisfy Robinson's property, if A and B are structures $PC_\delta$-characterisable in L then $A \equiv_L B$ implies $A \approx B$.*

Proof. Assume $T_i \subseteq L(\mu_i)$ is the theory characterizing $A_i$, $\sigma \subseteq \mu_i$, and let $(A_i, R_i)$ be model of $T_i$, i = 1, 2. Assume $A_1 \equiv_L A_2$, then by Robinson,'s property there is a model $(A, R_1^*, R_2^*)$ with $(A, R_i^*) \equiv_L (A_i, R_i)$; hence, $A \approx A_1 \approx A_2$ . □

If the logic satisfies the pair preservation property, PPP (cf. [Ma]), then the Robinson's property implies the following stronger property (see [C2]):

Definition 3.8. A logic L satisfies the *relativized Robinson property* if given any pair of vocabularies $\sigma_i$, $P_i \in \sigma_i$, and structures $A_i$ , i = 1, 2, with $\mu = \sigma_1 \cap \sigma_2$ such that $A_1|P_1^{A_1}|\mu \equiv_L A_2|P_2^{A_2}|\mu$ , there is a a third structure B such that $A_1 \equiv_L B|\sigma_1$ and $A_2 \equiv_L B|\sigma_2$.

Therefore, under PPP, Lemma 3.7 holds for structures $RPC_\delta$-characterizables in L. There is some confusion in the literature on relation to Robinson's property. What is usually called Robinson's property in the context of many sorted logics is in fact equivalent to the above stronger property.

**COROLLARY 3.9.** (compare with Mundici [Mu]). *If* L *satisfies* PPP *and Robinson's theorem but is not* $[\omega,\omega]$*-compact then for structures of power less than the first non measurable cardinal,* $A \equiv_L B$ *implies* $A \approx B$.

Proof. If the logic is not $[\omega,\omega]$-compact, then any structure of power less than the first measurable is $RPC_\delta$ in L by Corollary 2.10. Apply the previous lemma and observations. □

Therefore, under PPP, Lemma 3.7 holds for structures $RPC_\delta$-characterizables in L. There is some confusion in the literature on relation to Robinson's property. What is usually called Robinson's property in the context of many sorted logics is in fact equivalent to the above stronger property.

**COROLLARY 3.9.** (compare with Mundici [Mu]). *If* L *satisfies* PPP *and Robinson's theorem but is not* $[\omega,\omega]$*-compact then for structures of power less than the first non measurable cardinal,* $A \equiv_L B$ *implies* $A \approx B$.

Proof. If the logic is not $[\omega,\omega]$-compact, then any structure of power less than the first measurable is $RPC_\delta$ in L by Corollary 2.10. Apply the previous lemma and observations. □

## IV. $[\kappa,\lambda]$-COMPACTNESS AND ULTRAFILTER CONVERGENCE

It is well known full compactness of a topological space is equivalent to the convergence of any ultrafilter of subsets of the space (cf. [W]). This in turn may be expressed in terms of convergence of I-families of elements of the space with repect to ultrafilters over I. It is natural to ask if we have characterizations via ultrafilters of $[\kappa,\lambda]$-compactness. In certain sense this is true as we show in this section. In fact, we get more than we sougth: a characterization by ultrafilter convergence of families of $[\kappa,\lambda]$-compact spaces closed under products, and of spaces with $[\kappa,\lambda]$-compact powers.

**Definition 4.1.** Let U be an ultrafilter over a set I. We will say that a family $\{a_i : i \in I\}$ in a topological space X, U-*converges to* x (or that x is an U-*limit* of the family), if $\{i \in I : a_i \in V\} \in U$ for any open neigborhood V of x .

Evidently, $\{a_i : i \in I\}$ U-converges to x if and only if x is an adherence point in X of the ultrafilter $a(U) = \{ S \subseteq X : \{i \in I : a_i \in S\} \in U \}$ in the ordinary sense of topology (cf. [W]). Therefore, X is compact if and only if any I-family U-converges in X for any set I and ultrafilter U over I. We show next that $[\kappa,\lambda]$-compactness corresponds to U-convergency with respect to certain ultrafilters.

**Definition 4.2.** Let U be an ultrafilter over a set I, a space X will be called U-*compact* if and only if if any I-family of X U-converges.

The following are obvious properties of U-convergence and U-compactness:

**LEMMA 4.3.** i) *If* $f: X \to Y$ *is continuous and* $\{a_i : i \in I\}$ U-*converges in* X *to* a, *then* $\{f(a_i) : i \in I\}$ U-*converges in* Y *to* f(a).

ii) $\{ (a_{i,\alpha})_\alpha : i \in I\}$ U-*converges in* $\prod_\alpha X_\alpha$ *to* $(a_\alpha)_\alpha$ *if and only if* $\{a_{i,\alpha} : i \in I\}$ U-*converges in* $X_\alpha$ *to* $a_\alpha$, *for each* $\alpha$.

iii) *If each* $X_\alpha$ *is* U-*compact, then* $\prod_\alpha X_\alpha$ *is* U-*compact.*

Proof. i) Given a neigborhood V of f(a), then $\{i \in I : f(a_i) \in V\} \supseteq \{i \in I : a_i \in f^{-1}(V)\} \in U$.

ii) One direction follows from (i) by continuity of the projections. For the other, use that each basic neigborhood W of $(a_\alpha)_\alpha$ is a finite intersection of sets $\pi_\alpha^{-1}(V_\alpha)$ with $V_\alpha$ a neigborhood of $a_\alpha$ in $X_\alpha$. For each such set $\{i \in I : a_{i,\alpha} \in \pi_\alpha^{-1}(V_\alpha)\} \in U$; hence, $\{i \in I : (a_{i,\alpha})_\alpha \in W\} = \{i \in I : a_{i,\alpha} \in \cap \pi_\alpha^{-1}(V_\alpha)\} = \cap \{i \in I : a_{i,\alpha} \in \pi_\alpha^{-1}(V_\alpha)\} \in U$ .

iii) Immediate from (ii). □

**Definition 4.4.** An ultrafilter U over a set I is *$(\kappa,\lambda)$-regular* if and only if there is a family $\{I_\alpha\}_{\alpha<\kappa}$ such that $I_\alpha \in U$ and $\cap_{i<\lambda} I_{\alpha_i} = \emptyset$ for any family $\{\alpha_i : i < \lambda\} \subseteq \kappa$. U is *$\kappa$-uniform* if $|S| \geq \kappa$ for any $S \in U$, see [Ch-K] or [Ma].

PROPOSITION 4.5. *If X is U-compact for a $(\kappa,\lambda)$-regular ultrafilter U, then X is $[\kappa,\lambda]$-compact.*

Proof. Let $\{I_\alpha\}_{\alpha<\kappa}$ be a family of elements of U such that the intersection of any $\lambda$ of $I_\alpha$'s is empty. We may assume $I = I_0$. Given a family $\{F_\alpha\}_{\alpha<\kappa}$ of closed sets in X with the $\lambda$-intersection property, define $F_t = \cap_{t \in I_\alpha} F_\alpha$ for each $t \in I$. This set is non-empty, because t belongs to less than $\lambda$ sets $I_\alpha$ by hypothesis, and by hypothesis the intersection of less than $\lambda$ sets $F_\alpha$ is non empty. Choose $a_t \in F_t$, then $J_\alpha = \{ t \in I : a_t \in F_\alpha \} \in U$ because $t \in I_\alpha$ implies $a_t \in F_t \subseteq F_\alpha$ by construction, and so $J_\alpha \supseteq I_\alpha$. By hypothesis, $\{a_t\}_{t \in I}$ U-converges to some x of X; and given an open neigborhood V of x, then $J = \{ t \in I : a_t \in V \} \in U$. Therefore, $\{ t : a_t \in V \cap F_\alpha \} = J \cap J_\alpha \in U$ for any $\alpha$, and so this set is non-empty, showing that x belongs to adherence of any $F_\alpha$, that is to any $F_\alpha$. □

**Definition 4.6.** Let $P(\kappa,\lambda) = P_\lambda(\kappa) = \{ S \subseteq \kappa : |\kappa| < \lambda \}$.

PROPOSITION 4.7. i) *If X is $[\kappa,\lambda]$-compact, then every I-family in X, with $I = P(\kappa,\lambda)$, U-converges for some $(\kappa,\lambda)$-regular ultrafilter U over I (which may depend on the family).*

ii) *Moreover, if $\kappa > \lambda$ then U may be chosen $\kappa$-uniform, and if $\kappa = \lambda$ then U may be chosen $\mathrm{cof}(\kappa)$-uniform.*

Proof. Given $\{ a_t : t \in I \}$, let $A_t = \{ a_s : t \subseteq s \}$. The family of closed sets $\{\mathrm{cl}(A_t) : t \in I, |t|<\omega \}$ has the $<\lambda$-intersection intersection property, because

$$\cap_{i<\delta} \mathrm{cl}(A_{t_i}) \supseteq \mathrm{cl}(\cap_{i<\delta} A_{t_i}) = \mathrm{cl}(A_{\cup_{i<\delta} t_i}),$$

and $t = \cup_{i<\delta} t_i \in I$ if $\delta < \lambda$ and the $t_i$ are finite, so that $A_t$ is non-empty. By $[\kappa,\lambda]$-compactness there is an element $a \in \cap_{t \in I, |t|<\omega} \mathrm{cl}(A_t)$. Hence, $V \cap A_t \neq \emptyset$ for any neigborhood V of a and any $t \in I$, $|t|<\omega$. This implies that the family $F = \{ V \cap A_t : V$ open, $a \in V$, $t \in I$, $|t| < \omega \}$ has the finite intersection property. Then the family $F' = \{a^{-1}(W) : W \in F\} = \{ \{s \in I : a_s \in W\} : W \in F \}$ has also the finite intersection property. Extend F' to an ultrafilter U over I. By construction, given an open neigborhood V of a, then $\{ s \in I : a_s \in V \} \supseteq \{s \in I : a_s \in V \cap A_{\{\alpha\}} \} = a^{-1}(V \cap A_{\{\alpha\}}) \in F' \subseteq U$. Therefore $\{ a_t : t \in I\}$ U-converges to a. Moreover, $I_\alpha = \{ s \in I : \alpha \in s \} \supseteq \{s \in I : a_s \in V \cap A_{\{\alpha\}} \}$, and so

$I_\alpha \in U$. But the intersection of $\lambda$-many of these $I_\alpha$'s is empty because no $s \in I$ may contain $\lambda$ many ordinals. This shows that U is $(\kappa,\lambda)$ regular.

It remains to show that U may be chosen to be $\kappa$-uniform (repectively cof($\kappa$)-uniform). Fix $t_0 \in P(\omega,\kappa)$. For each finite $t \subseteq \kappa$ with $t \supseteq t_0$ the set $\{ s \supseteq t : a_s \in V \} = a^{-1}(V \cap A_t)$ is in U and so we may pick $a_t \in a^{-1}(V \cap A_t)$; hence $a_t \supseteq t$. This gives a function $f$: $B = \{t \in P(\omega,\kappa)$: $t \supseteq t_0\} \rightarrow I$ with the property that for any $r \in I$, $|f^{-1}(r)| = |\{ t \in B : a_t = r\}| \leq |\{ t \in B$: $t \subseteq r\}| < \lambda$, because $|r| < \lambda$ and so r has less than $\lambda$ finite subsets. Since $|B| = \kappa$, the number of non-empty distinct and so disjoint $f^{-1}(r)$ must be at least cof($\kappa$) in case $\kappa = \lambda$, and $\kappa$ in case $\kappa > \lambda$. Picking one $a_t$ from each non-empty $f^{-1}(r)$ gives $\kappa$ (respectively cof($\kappa$)) elements of $a^{-1}(V \cap A_{t_0})$. Therefore, F' may be extended to a $\kappa$-uniform (respectively cof($\kappa$)-uniform) ultrafilter. □

**COROLLARY 4.8.** i) *If X is $[\kappa,\lambda]$-compact and $\kappa > \lambda$ then any $P(\kappa,\lambda)$-family of X has a $\kappa$-accumulation point.*

ii) *If X is $[\kappa,\kappa]$-compact then any $P(\kappa,\kappa)$-family of X has a cof($\kappa$)-accumulation point.*

Proposition 4.7 does not guarantee a converse to Proposition 4.6, since the ultrafilter depends on the family. In fact, a converse is impossible because U-compactness corresponds to a strong form of $[\kappa,\lambda]$-compactness, as we will see next.

**THEOREM 4.9.** *The following are equivalent for any class T of topological spaces:*

i) *Any product of spaces in T is $[\kappa,\lambda]$-compact.*

ii) *There exists a $(\kappa,\lambda)$-regular ultrafilter U (which may be taken over $I = P(\kappa,\lambda)$) such that all the spaces in T are U-compact.*

Proof. Assume that any product of space in T is $[\kappa,\lambda]$-compact and there is no $(\kappa,\lambda)$-regular ultrafilter U over $I = P(\kappa,\lambda)$ such that all the elements of T are U-compact. Let $\Sigma$ be the family of such ultrafilters and choose for each $U \in \Sigma$ a I-family $\{a_{U,i} : i \in I\}$ in some space $X_U \in T$ which does not U-converge. For each i, let $\sigma_i = (a_{U,i})_U \in \Pi_{U \in \Sigma} X_U = X^*$. As this space is $[\kappa,\lambda]$-compact, then by Proposition 4.8, there is an ultrafilter $W \in \Sigma$ such that $\{\sigma_i$ : $i \in I\}$ W-converges to some $\sigma = (a_U)_U \in X^*$. By continuity of the W-projection, then $\{a_{W,i}: i \in I\}$ W-converges to $a_W$ in $X_W$, a contradiction.

Conversely, if each $X_r$ is U-compact then $\Pi_r X_r$ is U-compact by Lemma 4.3, and by $(\kappa,\lambda)$-regularity of U it follows from Proposition 4.6 that $\Pi_r X_r$ is $[\kappa,\lambda]$-compact. □

**THEOREM 4.10.** *The following are equivalent for any topological space X:*

i) $X^{\beta}$ *is* $[\kappa,\lambda]$ *compact for any cardinal* $\beta$.

ii) X *is* U-*compact for some* $(\kappa,\lambda)$-*regular ultrafilter* U *(which may be taken over* $I = P(\kappa,\lambda)$*)*.

Proof. Take T = { X } in Theorem 1. □

Definition 4.11. Call a space X *strongly* $[\kappa,\lambda]$-*compact* (in short, *s*-$[\kappa,\lambda]$-*compact*) if $X^{\beta}$ is $[\kappa,\lambda]$-compact for any cardinal $\beta$.

**COROLLARY 4.12.** X *is s*-$[\kappa,\lambda]$-*compact if and only if* $X^{2^{2^{|P(\kappa,\lambda)|}}}$ *is* $[\kappa,\lambda]$-*compact.*

Proof. In the proof of (i) ⇒ (ii) in Theorem 4.9 we utilized $[\kappa,\lambda]$-compactness of $X^{\beta}$ for $\beta = 2^{2^{|P(\kappa,\lambda)|}}$ only. □

**COROLLARY 4.13.** *If* X *is s*-$[\kappa,\lambda]$-*compact, then* $X \times Y$ *is* $[\kappa,\lambda]$-*compact for any compact space* Y.

Proof. A compact space is U-compact for any U. Apply Theorem 4.10 and Lemma 4.3. □

**Example.** Under the continuum hypothesis, plus $\omega_2 = 2^{\omega_1}$, if $X^{\omega_2}$ is countably compact, then $X^{\beta} \times Y$ will be countably compact for any $\beta$, and any compact space Y.

**COROLLARY 4.14.** *Let* $\kappa$ *be smaller than the first measurable cardinal (or arbitrary if no such cardinal exists). If* X *is s*-$[\kappa,\kappa]$-*compact, then* X *is* $[\omega,\omega]$-*compact.*

Proof. Let U be the ultrafilter given by Theorem 4.10, such that X is U-compact. By $(\kappa,\kappa)$-regularity, U is non principal. If U is not $(\omega,\omega)$-regular, then it is $\omega$-complete. But it is well known that the smallest $\omega$-complete no principal ultrafilter is measurable (see [BS]). □

Notice that the above fails strongly for plain compactness, $X = \omega$ with the discrete topology is $[\kappa,\kappa]$ compact for any $\kappa \geq \omega_1$ and however it is not $[\omega,\omega]$-compact.

For $[\kappa,\kappa]$-compactness with $\kappa$ regular, theorems 4.9 and 4.10 take a more beautiful form. The condition on $(\kappa,\lambda)$-regularity of the ultrafilter U may be changed to $\kappa$-uniformity. We state the case of one single space.

**THEOREM 4.15.** *Let* $\kappa$ *be a regular cardinal then* X *is s*-$[\kappa,\kappa]$-*compact if an only if* X *is* U-*compact for some* $\kappa$-*uniform ultrafilter* U.

Proof. A $\kappa$-uniform ultrafilter is $(\kappa,\kappa)$-regular. On the other hand, Proposition 4.8 allows to choose the ultrafilter U being cof($\kappa$)-uniform. But for $\kappa$ regular, cof($\kappa$) = $\kappa$. □

Of course, the ultrafilter U may be chosen over $P_\kappa(\kappa)$. Could it be chosen over $\kappa$?

**COROLLARY 4.16.** *If X is s-$[\kappa^+, \kappa^+]$-compact, then X is s-$[\kappa,\kappa]$-compact.*
Proof. A $\kappa^+$-uniform ultrafilter is $(\kappa,\kappa)$-regular by results of Kanamori [K] and Kunen-Pikry [K-P]. □

**COROLLARY 4.17.** *Let $\kappa$ be a regular cardinal or $\infty$, then X is s-$[\kappa,\lambda]$-compact if and only if X is s-$[\mu,\mu]$-compact for any regular cardinal $\mu$, $\lambda \leq \mu \leq \kappa$.*
Proof. Let $\mu$, $\lambda \leq \mu \leq \kappa$, be singular, then $\lambda \leq \mu^+ \leq \kappa$. Apply then the previous corollary. □

The previous results do not hold for plain $[\kappa,\lambda]$-compactness, $X = \aleph_\omega$ discrete is $[\infty, \aleph_\omega^+]$-compact but not $[\aleph_\omega, \aleph_\omega]$-compact.

## V. THE ABSTRACT COMPACTNESS THEOREM REVISITED

Let us apply the results of the previous section to spaces of structures. We start with the following simple observation which follows from the zero-dimensionality of spaces of structures.

**FACT 5.1.** $\{A_i : i \in I\}$ *U-converges to* $A^*$ *in* $E_\sigma(L)$ *if and only if for any sentence* $\phi \in L(\sigma)$ *we have:* $A^* \models \phi \iff \{ i \in P : A \models \phi \} \in U$.

If we use "$\Rightarrow$" instead of "$\Leftrightarrow$", the last equivalence, this is just the definition of U-convergence expressed in terms of a basis, because the basics neigborhoods are the classes Mod($\phi$). The arrow becomes a double arrow since by applying it to the negation of $\phi$ and so: $A^* \not\models \phi \Rightarrow \{i \in P : A \models \neg\phi \} \in U \Rightarrow \{ i \in P : A \models \phi \} \notin U$.

Let us recall the following definition form [Ma-Sh].

**Definition 5.2.** An ultrafilter U over a set I is *related* to a logic L if and only if for any structure A of type $\sigma$ there is an extension $A^*$ of the ultrapower $A^I/_U$ satisfying for any formula $\phi(x, \ldots) \in L(\sigma)$ and functions f, ... $\in A^I$:

$$A^* \models \phi(f/_U, \ldots) \text{ iff } \{ i \in P : A \models \phi(f(i), \ldots) \} \in U. \quad (1)$$

This definition says that $A^*$ behaves as the ultrapower would if Los theorem where true for L. However, $A^*$ can not be chosen to be the ultrapower itself unless $L = L_{\omega\omega}$. The following fact reduces the relation (1) to pure topology.

Given a structure A, of type $\sigma$ consider the expande vocabulary $\sigma_{A,I} = \sigma + \{c_f: f \in A^I\}$, where each $c_f$ is a constant symbol, and define for each fixed $j \in I$ the following expansion of A of type $\sigma_{A,I}$:

$$A^*_j = (A, f(j), \ldots)_{f \in A^I}$$

where $c_f$ is interpreted by f(j).

**LEMMA 5.3.** $A^*$ *is an extension of* $A^I/_U$ *satisfying* (1) *if and only if* $A^*$ *may be expanded to an U-limit of* $\{ A^*_j : j \in I \}$.

Proof. (1) implies that $(A^*, f/_U, \ldots)_{f \in A^I}$ is an U limit of the family by Fact 5.1. Conversely, an U-limit $(A^*, a_f, \ldots)$ of this family will have interpretations $a_f$ for the constants $c_f$, $f \in A^I$. Moreover, for any formula $\phi(x, \ldots)$ of type $\sigma$: $A^*|\sigma \models \phi(a_f, \ldots)$ if and only if $A^* \models \phi(c_f, \ldots)$ if and only if $\{ j \in I: A^*_j \models \phi(c_f, \ldots) \} \in U$ if and only if $\{ i \in I: A \models \phi(f(i), \ldots)\} \in U$. Applying this to the atomic formulae, we get an isomorphism $a_f \mapsto f/_U$ between the substructure of $A^*|\sigma$ induced by $\{ a_f: f \in A^I\}$ and the ultrapower $A^I/_U$. Via this identification, (1) holds. □

**PROPOSITION 5.4.** *The following are equivalent for any logic* L *closed under relativizations, and any ultrafilter* U:

i) U *is related to* L .

ii) *The spaces* $E_\sigma(L)$ *are U-compact for any* $\sigma \in \mathrm{Dom}(L)$.

Proof. Assume U is related to L. Given a family $\{A_i: i \in I\} \subseteq E_\sigma$ code it in a single structure $A = (\cup_i A_i, \cup_i Q^{A_i}, \ldots, I, R)$ where Q runs through the vocabulary $\sigma$, and $R = \cup_i \{i\} \times A_i$ so that $A_i = A|\{x: R(i,x)\}$. If $A^I/_U \subseteq A^*$ where $A^*$ is given by Def. 5.2, let $g_0(i) = i$ be the identity function and $P^* = \{x \in A^*: A^* \models R(g_0/_U, x) \}$. Utilizing that the logic has relativizations and property (1), we obtain for any sentence $\phi \in L(\sigma)$:

$$A^*|P^* \models \phi \text{ iff } A^* \models \phi^{\{x: R(g_0/_U, x)\}}$$
$$\text{iff } \{ i \in I : A \models \phi^{\{x: R(i,x)\}} \} \in U$$

$$\text{iff } \{ i \in I : A|\{x: R(i,x)\} \models \phi \} = \{ i \in I : A_i \models \phi \} \in U.$$

Hence, $\{A_i : i \in I\}$ U-converges to $A^*|P^*$.

Conversely, assume any I-family U-converges in the spaces of structures. Given a structure A, there is an U-limit $(A^*, a_f, \dots)$ of the associated family $\{A^*_j\}$ which will satisfy (1) by Lemma 5.3 showing that U is related to L. □

**Remark.** Notice that in the first part of the above proof, the existence of $A(\mu)^*$ satisfying (1) for the complete extension $A(\mu)$ of power $\mu \geq |I|$, will guarantee the U-convergence of all I-families of structures of power at most $\mu$, because they may all be accomodated in A, via appropriate codings and renamings. Therefore, the U-convergence of a single I-family of structures of power $\mu$ in each cardinal $\mu \geq |I|$ yields U-compactness.

To apply our topological results, we need a further observation on the spaces of structures. Given vocabularies $\sigma_i$, $i \in I$, let $\oplus_i \sigma_i$ be the disjoint union of the vocabularies $\sigma_i + \{P_i\}$ where $P_i$ is a monadic symbol not in $\sigma_i$. Then the cartesian product $\Pi_i E_{\sigma_i}$ may be identified with the class of structures of type $\oplus_i \sigma_i$ of the form:

$$[A_i]_{i \in I} = ( \cup_{i \in I} |A_i| , \underline{A}_i , \dots )_{i \in I} , \qquad A_i \in E_{\sigma_i} ,$$

where the universe is the disjoint union of the universes of the $A_i$, and the disjoint renaming of $\sigma_i + \{P_i\}$ is interpreted by the disjoint copy $\underline{A}_i$ of $A_i$, with $P_i$ interpreting the universe (cf. [C1]).

**LEMMA 5.4.** *If* L *is closed under relativizations then for any product space* $\Pi_i E_{\sigma_i}(L)$ *there is a uniformly continuous onto operation* $F: C \to \Pi_i E_{\sigma_i}(L)$ *where* C *is a closed subclass of* $E_{\oplus_i \sigma_i}(L)$.

Proof. Let C be the class of models of the sentences

$$\forall x \neg (P_i(x) \wedge P_j(x)), \;\; i \neq j; \qquad \forall x_1 \dots \forall x_n (R(x_1, \dots, x_n) \to P_i(x_1) \wedge \dots \wedge P_i(x_n)), \quad R \in \sigma_i.$$

The operation $A \mapsto A|\cup_i P_i^A = [A|P_i^A]$ from C to $\Pi_i E_{\sigma_i}$ is evidently onto. The basic clopens of the product topology in $\Pi_i E_{\sigma_i}(L)$ are given by finite conjunctions $\theta = \wedge_r \phi_r^{P_{i_r}}$ where $\phi_r \in L(\sigma_{i_r})$; hence, $A \models \theta \Leftrightarrow A|(\cup_r P_{i_r}^A) \models \theta$ for any A. As $(A|\cup_i P_i^A)|(\cup_r P_{i_r}^A) = A|(\cup_r P_{i_r}^A)$, we have $A|\cup_i P_i^A \models \theta \Leftrightarrow A \models \theta$, showing uniform continutity (cf. [C2]). □

COROLLARY 5.5. *If L is closed under relativizations and $[\kappa,\lambda]$-compact then any product space $\Pi_i E_{\sigma_i}(L)$ is $[\kappa,\lambda]$-compact.*

Proof. Lemma 1.8, Sect 1, and the previous Lemma. □

It follows that a logic is $[\kappa,\lambda]$-compact if and only if all the spaces $E_\sigma(L)$ are strongly $[\kappa,\lambda]$-compact. Hence all the results of Section III apply. For example Corollary 2.7 becomes a case of Corollary 4.14. Similarly, the theorem of Makowsky and Shela [Ma-Sh] that $[\kappa^+,\kappa^+]$-copmactness of a logic implies $[\kappa,\kappa]$ compactness is a case of Corollary 4.16. Moreovoer, utilizing Theorem 4.9 and Proposition 5.4 we have:

THEOREM 5.6 (ABSTRACT COMPACTNESS THEOREM, [M-Sh]). *L is $[\kappa,\lambda]$-compact if and only if there is a $(\kappa,\lambda)$ regular ultrafilter U related to L (which may be taken over $P_\mu(\kappa)$).*

Proof. L is $[\kappa,\lambda]$-compact if and only if the spaces $\Pi_i E_{\sigma_i}(L)$ are all $[\kappa,\lambda]$-compact by the previous Corollary. By Theorem 4.9., Sect. IV applied to the class $T = \{ E_\sigma(L) : \sigma \in \mathrm{dom}(L)\}$ this is equivalent to the existence of a single $(\kappa,\lambda)$-regular ultrafilter U such that all $E_\sigma(L)$ are U-compact, which means is U is related to L by Proposition 5.4. □

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