STATIONARY SOLUTIONS FOR GENERALIZED BOUSSINESQ MODELS

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ABSTRACT – We study the existence, regularity and conditions for uniqueness of solutions of a generalized Boussinesq model for thermally driven convection. The model allows temperature dependent viscosity and thermal conductivity

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Key words: Boussinesq, thermally driven convection, stationary solutions, temperature dependent viscosity.

1. Introduction

We study the stationary problem for the equations governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are temperature dependent. The equations are

$$(1.1) \qquad \begin{cases} - & div(\nu(T)\nabla u) + u.\nabla u - \alpha T g + \nabla p = 0, \\ & div\ u = 0 \\ - & div(k(T)\nabla T) + u.\nabla T = 0 \quad \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n, n = 2$ or 3 throughout the paper.

Here $u(x) \in \mathbb{R}^n$ denotes the velocity of the fluid at a point $x \in \Omega$; $p(x) \in \mathbb{R}$ is the hydrostatic pressure; $T(x) \in \mathbb{R}$ is the temperature; $g(x)$ is the external force by unit of mass; $\nu(\cdot) > 0$ and $k(\cdot) > 0$ are kinematic viscosity and thermal conductivity, respectively; α is a positive constant associated to the coefficient of volume expansion. Without loss of generality, we have taken the reference temperature as zero. For a derivation of the above equations, see for instance Drazin and Reid [5].

The expressions ∇, Δ and *div* denote the gradient, Laplacian and divergence operators, respectively (we also denote the gradient by *grad*); the i^{th} component of $u.\nabla u$ is given by

$(u.\nabla u)_i = \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j}$; $u.\nabla T = \sum_{j=1}^{n} u_j \frac{\partial T}{\partial x_j}$.

The boundary conditions are as follows

$$\begin{cases} u = 0 \\ T = T_0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where T_0 is a given function on $\partial\Omega$ (the boundary of Ω).

The classical Boussinesq equations correspond to the special case where ν an k are positive constants. This case has been much studied (see for instance Morimoto [8], [9] and Rabinowitz [12] and the references there in).

Equations (1.1) are much less studied, and they correspond to the following physical situation. For certain fluids we can not disregard the variation of viscosity (and thermal conductivity) with temperature, this being important in the determination of the details of the flow. It is found, for example, that a liquid is usually rises in a middle of a polygonal convection cell, while a gas falls. Graham [6] suggested that this is because the viscosity of a typical liquid decreases with temperature whereas that of a typical gas increases. This suggestion was subsequently confirmed by Tippelkirch's experiments (see [14]) on convection of liquid sulphur, for which the viscosity has a minimum near 153°C. He found that the direction of the flow depended on the temperature being above or below 153°C. Palm [10] was the first to analyse the effect of the variation of viscosity with temperature; other papers on the subject are, for instance, Busse [3] and Palm, Ellingsen, Gjevik [11]. All these papers have the Theoretical Fluid Dynamics flavour. A rigorous mathematical analysis is more difficult in this case than that in the case of the classical Boussinesq equations due to the stronger nonlinear coupling between the equations.

For this, in this paper we will use an spectral Galerkin method combined with fixed point arguments; we will need more estimates than the ones required in the classical case in order to handle the nonlinearity in the heigher order terms of the equations.

We will show the existence of weak and strong solutions of problem (1.1), (1.2) under certain conditions on the temperature dependency of the viscosity and thermal conductivity; we allow more general external forces than the usual one (constant gravitacional field) because this could be useful in certain geophysical models. Properties of regularity and uniqueness are also studied. Questions concerning stability and bifurcation are left for future work.

We observe that if we take $u \equiv 0$ in (1.1) and $g = (0,0,1)$, the usual approximation for the gravitacional acceleration, we are left with grad $p = \alpha T g = (0,0,\alpha T)$. Consequently, curl $(0,0,\alpha T) = 0$ in Ω, and so $\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = 0$ in Ω. Therefore, we see that an arbitrary temperature on the boundary will require, in general, motion. That is, in general the solution of (1.1), (1.2) are not trivial.

Finally, we would like to comment that analogous questions can be considered for the corresponding evolution problem. Results along these lines will appear elsewhere.

The paper is organized as follows:

In section 2 we describe the notation and the basic facts to be used later on; we also state the our main results. In Section 3, we make a technical preparation by proving certain a priori estimates that will be useful for the proofs of the main results. The proofs of Theorem 2.1 (existence of a weak solution), Theorem 2.2 (existence of strong solution), Theorem 2.3 (regularity) and Theorem 2.4 (uniqueness) are done in Sections 4, 5, 6 and 7, respectively.

2. Preliminaries and results

In this article the functions are either R or R^n valued ($n = 2$ or 3) and we will not distinguish them in our notations; this being clear from the context. The $L^2(\Omega)$-product and norm are denoted by $(\ ,\ )$ and $|\ |$, respectively, the $L^p(\Omega)$ norm by $|\ |_p, 1 \le p \le \infty$; the $H^m(\Omega)$ norm is denoted by $\|\ \|_m$ and the $W^{k,p}(\Omega)$ norm by $|\ |_{k,p}$. Here $H^m(\Omega) = W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev spaces (see Adams [1] for their properties).

Let $D(\Omega) = \{v \in (C_0^\infty(\Omega))^n\ div\ v = 0 \text{ in } \Omega\}$, $H =$ completion of $D(\Omega)$ in $L^2(\Omega)$, and $V =$ completion of $D(\Omega)$ in $H^1(\Omega)$.

Let P be a orthogonal projection of $L^2(\Omega)$ onto H obtained by the usual Helmholtz decomposition. We shall denote by v^k and α^k respectively the eigenfunctions and eigenvalues of the Stokes operator $\tilde{\Delta} = -P\Delta$ defined in $V \cap H^2(\Omega)$. It is well known that v^k are orthogonal in the inner products $(\ ,\ ), (\nabla, \nabla)$ and $(\tilde{\Delta}, \tilde{\Delta})$ and are complete in the spaces H, V and $H^2(\Omega) \cap V$ (see, for example, Temam [13]).

Similar considerations are true for the Laplacian operator Δ; we will denote by ψ^k and λ_k respectively the eigenfunctions and eigenvalues of the operator $-\Delta$ defined in $H_0^1(\Omega) \cap H^2(\Omega)$.

Let $W^{1-\frac{1}{p},p}(\partial\Omega)$ be the trace space obtained as the image of $W^{1,p}(\Omega)$ by the boundary value mapping on $\partial\Omega$, equipped with the norm

$$|\gamma|_{1-\frac{1}{p},p,\partial\Omega} = \inf_{\substack{v \in W^{1,p}(\Omega) \\ v=\gamma \text{ on } \partial\Omega}} |v|_{k,p}.$$

Similarly, when $\partial\Omega$ is sufficiently smooth, we can define the trace spaces $W^{k-\frac{1}{p},p}(\partial\Omega)$ with norm $|\ |_{k-\frac{1}{p},p,\partial\Omega}$. When $p = 2$, we denote $H^{k-1/2}(\partial\Omega) = W^{k-1/2,2}(\partial\Omega)$ and $\|\ \|_{k-1/2,\partial\Omega} = |\ |_{k-1/2,2,\partial\Omega}$ (see Adams [1]).

We assume that we can find a function S defined in Ω satisfying $S = T_0$ on $\partial\Omega$; then, we can transform the equations (1.1), (1.2), by introducing the new variable $\varphi = T - S$ to obtain

$$(2.1) \qquad \begin{cases} -\ div(\nu(\varphi + S)\nabla u) + u.\nabla u - \alpha\varphi g - \alpha S g + \nabla p = 0, \\ div\ u = 0 \\ -\ div(k(\varphi + S)\nabla\varphi) + u.\nabla\varphi - div(k(\varphi + S)\nabla S) + u.\nabla S = 0 \ \text{ in } \Omega \\ u = 0 \ \text{ and } \ \varphi = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Suppose that $S \in H^1(\Omega)$, then we can reformulate (2.1) in weak form as follows: to find $u \in V$ and $\varphi \in H_0^1(\Omega)$ satisfying

$$(2.2) \qquad \begin{cases} (\nu(\varphi + S)\nabla u, \nabla v) + B(u,u,v) - \alpha(\varphi g, v) - \alpha(Sg; v) = 0, \\ \text{for all } v \text{ in } V \\ (k(\varphi + S)\nabla\varphi, \nabla\psi) + b(u,\varphi,\psi) + (k(\varphi + S)\nabla S, \nabla\psi) + b(u,S,\psi) = 0, \\ \text{for all } \psi \text{ in } H_0^1(\Omega) \end{cases}$$

where

$$B(u,v,w) = (u.\nabla v, w) = \int_\Omega \sum_{i,j=1}^n u_j(x) \frac{\partial v_i}{\partial x_j}(x) w_i(x) dx$$

and

$$b(u,\varphi,\psi) = (u.\nabla\varphi, \psi) = \int_\Omega \sum_{j=1}^n u_j(x) \frac{\partial \varphi}{\partial x_j}(x)\psi(x) dx$$

Now, we define weak solutions of (1.1) - (1.2).

Definition. A pair of functions $(u,T) \in V \times H^1(\Omega)$ is called a weak solution of (1.1), (1.2) if there exists a function S in $H^1(\Omega)$ such that $u \in V$, $\varphi = T - S \in H_0^1(\Omega)$, $S = T_0$ on $\partial\Omega$, and, (u,φ) is a solution of (2.2)

Based on physical assumptions, throughout the paper we will suppose that

(A.1.) $\quad \nu(\tau) > 0;\ k(\tau) > 0 \qquad$ for all $\quad \tau \in \mathbb{R}$

We observe that assumption (A.1) allows the cases $\lim \inf_{T \to +\infty} \nu(T) = 0$ or $\lim \sup_{T \to +\infty} \nu(T) = +\infty$ (the same holds for $k(\cdot)$).

Our first result concerns the existence of weak solutions.

Theorem 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with Lipschitz continuous boundary; let the functions ν, k be continuous $g \in L^2(\Omega)$ and $T_0 \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$. Then, there exists a weak solution of (1.1), (1.2). In case that $\inf\{\nu(\tau), k(\tau); \tau \in \mathbb{R}\} > 0$ and $\sup\{\nu(\tau), k(\tau); \tau \in \mathbb{R}\} < \infty$ the result is true under the weaker assumption $T_0 \in H^{1/2}(\partial\Omega)$.

If we have stronger assumptions, we are able to prove the following.

Theorem 2.2. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with C^1 boundary; we suppose that ν, k are of class C^1, $g \in L^3(\Omega)$ and $T_0 \in H^{3/2}(\partial\Omega)$. Then, if $\|T_0\|_{3/2,\partial\Omega}$

is small enough, there exists a strong solution of (1.1), (1.2), that is, there exists a pair $(u,T) \in (V \cap H^2(\Omega)) \times H^2(\Omega)$ such that

$$\begin{aligned} P(-div(\nu(T)\nabla u) + u.\nabla u - \alpha T g) &= 0 \quad \text{in } L^2(\Omega), \\ -div(k(T)\nabla T) + u.\nabla T &= 0 \quad \text{in } L^2(\Omega), \\ T &= T_0 \quad \text{on } \partial\Omega. \end{aligned}$$

We observe that there exists a unique function p (the pressure) in $H^1(\Omega) \cap L_0^2(\Omega)$, with $L_0^2(\Omega) = \{f \in L^2(\Omega)/(f,1) = 0\}$, such that

$$-div(\nu(T)\nabla u) + u.\nabla u - Tg = -grad\, p. \tag{2.3}$$

For this, see Temam [14]. We also observe that Theorem 2.2 is true if we take small α instead of small $\|T_0\|_{3/2,\partial\Omega}$.

The next result is concerned with the regularity of (u,T,p).

Theorem 2.3 Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with a $C^{k+1,1}$ boundary; let the functions ν, k be of class C^{k+1}, $g \in W^{k,3}(\Omega)$ and $T_0 \in W^{k+7/4,4}(\partial\Omega)$. Then a strong solution (u,T) satisfies $u \in H^{k+2}(\Omega)$ and $T \in W^{k+2,4}(\Omega)$. Moreover, the associated pressure satisfies $p \in H^{k+1}(\Omega) \cap L_0^2(\Omega)$.

The following is an uniqueness result for "small" weak solutions.

Theorem 2.4. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3), with a Lipschitz continuous boundary and ν, k, k' Lipschitz continuous. There exists $\varepsilon > 0$ such that, if there exists a weak solution (u,T) of (1.1), (1.2) satisfying $|\nabla u| + \|T\|_2 < \varepsilon$, then it is unique.

Finally we state two lemmas for convenience of reference.

By Hölder's inequality and Sobolev imbeddings, we have

Lemma 2.5. There exists a constant C_B depending on Ω such that

$$\begin{aligned} |B(u,v,w)| &\le C_B |\nabla u||\nabla v||\nabla w| \quad \forall u \in V, \forall v \in H^1(\Omega), \forall w \in H_0^1(\Omega) \\ |b(u,\varphi,\psi)| &\le C_B |\nabla u||\nabla \varphi||\nabla \psi| \quad \forall u \in V, \forall \varphi \in H^1(\Omega), \forall w \in H_0^1(\Omega) \end{aligned}$$

By density arguments and integration by parts (see Temam [13]) we have

Lemma 2.6 (i) $\quad B(u,v,w) = -B(u,w,v) \quad \forall u \in V, \quad \forall v,w \in H^1(\Omega)$

$\qquad b(u,\varphi,\psi) = -b(u,\psi,\varphi) \quad \forall u \in V, \quad \forall \psi,\varphi \in H^1(\Omega)$

$$\text{(ii)} \qquad B(u,v,v) = 0 \quad \forall u \in V, \quad \forall v \in H^1(\Omega)$$
$$b(u,\varphi,\varphi) = 0 \quad \forall u \in V, \quad \forall \varphi \in H^1(\Omega)$$

In what follows we will use C as a generic positive constant which depends only on Ω, through constants appearing in the Poincaré inequality and Sobolev inequalities.

3. A Priori Estimates.

In this section we will show that problem (1.1), (1.2) satisfies a weak maximum principle. Also, we will obtain a priori estimate for weak solutions.

Lemma 3.1. Let $\{u, T\}$ be a weak solution of (1.1), (1.2). Then we have

$$\inf_{\partial\Omega} T_0 \le T(x) \le \sup_{\partial\Omega} T_0 \ \ a.e. \ \text{in} \ \overline{\Omega} \tag{3.1}$$

Proof. Assume $l = \sup_{\partial\Omega} T_0 < \infty$ (If $l = \infty$ we are done). We take $\psi = T^+$ in (2.2), where $T^+ = \sup\{T - l, 0\}$ to obtain

$$(\nu(T)\nabla T, \nabla T^+) = -b(u, \nabla T, T^+).$$

On easy computation shows that

$$(\nu(T)\nabla T^+, \nabla T^+) = (\nu(T)\nabla T, \nabla T^+) = -b(u, \nabla T, T^+) = -b(u, \nabla T^+, T^+)$$

Therefore, using Lemma 2.6 (ii), we have $\int_\Omega \nu(T)|\nabla T^+|^2 = 0$

Thus, $\nu(T)|\nabla T^+|^2 = 0$ a.e in Ω, and consequently $|\nabla T^+|^2 = 0$ a.e. in Ω.

This last equality implies that $T^+ = 0$ since $T^+ \in H_0^1$; thus, the right hand side of (3.1) follows. The left hand side (3.1) is similarly obtained □

An interesting consequence of previous lemma, is that we can transforme problem (1.1), (1.2) into an equivalent one. Suppose that inf $\{k(t); t \in \mathbb{R}\} = 0$ or $\sup\{k(t), t \in \mathbb{R}\} = +\infty$ and $\sup_{\partial\Omega} |T_0| < \infty$; then, we consider the modified function $\tilde{k}$, with the same regularity as k and satisfying

$$\tilde{k}(\tau) = k(\tau) \quad \text{for all} \quad |\tau| \le \sup_{\partial\Omega} T_0$$

and

$$\frac{1}{2}\inf\{k(\tau); |\tau| \le \sup_{\partial\Omega} |T_0|\} \le \tilde{k}(\tau) \le 2\sup\{k(\tau); |\tau| \le \sup_{\partial\Omega} |T_0|\} \quad \text{for all} \ \ \tau \in \mathbb{R}$$

Analogous considerations can be done for $\nu(\cdot)$.

Clearly, a pair (u, T) is a weak solution of (1.2) (.2) if and only if it is a weak solution of the following problem

$$(3.2)\qquad \begin{cases} -div(\tilde{\nu}(T)\nabla u) + u.\nabla u + \nabla p &= \alpha T g, \\ div\, u &= 0, \\ -div(\tilde{k}(T)\nabla T) + u.\nabla T &= 0 \text{ in } \Omega \end{cases}$$

$$(3.3)\qquad T = T_0 \quad , \quad u = 0 \qquad \text{on } \partial\Omega.$$

Therefore, hereafter we can suppose that the functions $\nu(\cdot)$ and $k(\cdot)$ satisfy

$$(3.4)\qquad \begin{array}{l} 0 < \nu_0(T_0) \leq \nu(\tau) \leq \nu_1(T_0) \\ 0 < k_0(T_0) \leq k(\tau) \leq k_1(T_0) \quad \text{for all } \tau \in I\!R. \end{array}$$

where $\nu_0(T_0) = \frac{1}{2}\inf\{\nu(t); |t| \leq \sup_{\partial\Omega}|T_0|\}$, $\nu_1(T_0) = 2\sup\{\nu(t), |t| \leq \sup_{\partial\Omega}|T_0|\}$ with analougous definitions for $k_0(T_0)$ and $k_1(T_0)$

Remark 3.2 Obviously, if we assume

$$(3.5)\qquad \inf\{\nu(t), k(t); t \in I\!R\} > 0 \text{ , } \sup\{\nu(t), k(t), t \in I\!R\} < +\infty$$

then, the above modification is unnecessary.

Now, we prove an a priori estimate. Let $\{u, \varphi\}$ be a weak solution of (1.1), (1.2). Thus, by taking $v = u$ and $\psi = \varphi$ in (2.2), we have

$$(3.6)\qquad (\nu(\varphi + S)\nabla u, \nabla u) + B(u, u, u) - \alpha(\varphi g, u) - \alpha(Sg, u) = 0,$$

$$(3.7)\qquad (k(\varphi + S)\nabla\varphi, \nabla\varphi) + b(u, \varphi, \varphi) + (k(\varphi + S)\nabla, \nabla\varphi) + b(u, S, \varphi) = 0.$$

From Lemma 2.6, Holder's inequality and (3.4), we obtain

$$\nu_0(T_0)|\nabla u|^2 \leq \alpha(\varphi g, u) + \alpha(Sg, u) \leq \alpha|g|(|\varphi|_3 + |S|_3)|u|_6$$

by Sobolev imbeddings, we find

$$(3.8)\qquad |\nabla u| \leq \alpha\frac{C}{\nu_0(T_0)}(|g||\nabla\varphi| + |g|\, ||S||_1)$$

Similarly, we have

$$\begin{aligned} k_0(T_0)|\nabla\varphi|^2 &\leq -b(u, S, \varphi) - (k(\varphi + S)\nabla S, \nabla\varphi) \\ &= b(u, \varphi, S) - (k(\varphi + S)\nabla S, \nabla\varphi) \\ &\leq |u|_6|\nabla\varphi||S|_3 + k_1(T_0)|\nabla S||\nabla\varphi| \\ &\leq C|\nabla u||S|_3|\nabla\varphi| + k_1(T_0)|\nabla S||\nabla\varphi| \end{aligned}$$

Thus,

$$|\nabla\varphi| \le \frac{C}{k_0(T_0)}|S|_3|\nabla u| + \frac{k_1(T_0)}{k_0(T_0)}||S||_1 \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$\left(1 - \frac{\alpha C}{\nu_0(T_0)k_0(T_0)}|g||s|_3\right)|\nabla u| \le \frac{\alpha C}{\nu_0(T_0)k_0(T_0)}||S||_1(1 + k_0(T_0))$$

If we assume

$$\alpha\frac{C}{\nu_0(T_0)k_0(T_0)}|g||S|_3 < \frac{1}{2}, \tag{3.10}$$

then we have

$$\begin{aligned} |\nabla u| &\le \frac{2\alpha C}{\nu_0(T_0)k_0(T_0)}|g|\,||S||_1(k_1(T_0) + k_0(T_0)) = F_1(||S||_1), \\ |\nabla\varphi| &\le \left(\frac{2k_1(T_0)}{k_0(T_0)} + 1\right)(||S||_1) = F_2(||S||_1). \end{aligned} \tag{3.11}$$

4. Existence of Weak Solutions

We start by proving the following result (compare with Morimoto [8]).

Lemma 4.1. Let Ω be a bounded domain in $\mathbb{R}^n, n = 2$ or 3, with Lipschitz continuous boundary. If T_0 is a function in $H^{1/2}(\partial\Omega)$, then for any positive numbers ε and $1 \le p \le 6$ if $n = 3$ or any finite $p \ge 1$ if $n = 2$, there exists an extension $S \in H^1(\Omega)$ of T_0 such that $|S|_p < \varepsilon$

Proof By definition of $H^{1/2}(\partial\Omega)$, we can obtain an extension $\tilde{T}_0 \in H^1(\Omega)$, of T_0. For any $\delta > 0$, we consider $\partial\Omega_\delta = \{x \in \mathbb{R}^n; d(x, \partial\Omega) < \delta\}$ and $\alpha(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \alpha(x) \le 1$, $\alpha(x) \equiv 1$ in $\partial\Omega_{\delta/2}$, $\alpha(x) \equiv 0$ in $\mathbb{R}^n \setminus \partial\Omega_\delta$ (we can obtain such function, by applying a differential version of Urysohn's Lemma).

Define $S(x) = \alpha(x)\tilde{T}_0(x)$. Then S is a required extension, because $S \in H^1(\Omega)$ and

$$|S|_p \le \left(\int_{\Omega\cap\partial\Omega_\delta}|\tilde{T}_0(x)|^p dx\right)^{1/p}$$

Since by Sobolev embedding, $H^1(\Omega) \subset L^p(\Omega)$, with p satisfying the stated conditions, $\int_\Omega|\tilde{T}_0(x)|^p dx < +\infty$, and, therefore, we can choose $\delta > 0$ sufficiently small so that the right hand side of the above inequality is less than ε. $\square$

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. According to the Lemma 4.1, with $p = 3$, we can choose an extension S of T_0 such that $S \in H^1$ and satisfies (3.10).

As n^{th} aproximate solution of equation (2.1) we choose functions $u^n(x) = \sum_{k=1}^{n} c_{n,k} v^k$ and $\varphi^n(x) = \sum_{k=1}^{n} d_{n,k}\psi^k$ satisfying the equations

$$(\nu(\varphi^n + S)\nabla u^n, \nabla v^j) + B(u^n, u^n, v^j) - \alpha(\varphi^n g, v^j) - \alpha(Sg, v^j) = 0, \tag{4.1}$$

$$(k(\varphi^n + S)\nabla \varphi^n, \nabla \psi^j) + b(u^n, \varphi^n, \psi^j) + (k(\varphi^n + S)\nabla S, \nabla \psi^j) + b(u^n, S, \psi^j) = 0, \tag{4.2}$$

for $1 \le j \le n$.

First we assume the existence of (u^n, φ^n) for any $n \in \mathbb{N}$. Note that the solutions (u^n, φ^n) must satisfy estimate (3.11). In fact, the identity (3.8) for u^n is obtained by multiplying (4.1) by $c_{n,j}$ and summing over j from 1 to n. Similarly, we have estimate (3.9) for φ^n.

Estimates (3.11) follow for estimates (3.8), (3.9) as in the Section 3. Therefore, the sequence (u^n, φ^n) is bounded in $V \times H_0^1$.

Since V (respectively H_0^1) is compactly imbedded in H (respectively $L^2(\Omega)$) we can choose subsequences, which we again denote by (u^n, φ^n), and elements $u \in V$, $\varphi \in H_0^1$ such that the following convergences hold

$$u^n \longrightarrow u \text{ , weakly in } V \text{, strongly in } H$$
$$\varphi^n \longrightarrow \varphi \text{ , weakly in } H_0^1 \text{ , strongly in } L^2 \text{ and almost everywhere in } \Omega$$

Furthermore, we can suppose

$$\nabla u_m \longrightarrow \nabla u \text{ , weakly in } L^2$$
$$\nabla \varphi_m \longrightarrow \nabla \varphi \text{ , weakly in } L^2$$

Thus, we can take the limit as n goes to ∞ in (4.1), (4.2), obtaining

$$\begin{aligned} &(\nu(\varphi + S)\nabla u, \nabla v^j) + B(u, u, v^j) - \alpha(\varphi g, v^j) - \alpha(Sg, v^j) = 0, \quad \forall j. \\ &(k(\varphi + S)\nabla \varphi, \nabla \psi^j) + b(u, \varphi, \psi^j) + (k(\varphi + S)\nabla S, \nabla \psi^j) + b(u, S, \psi^j) = 0, \quad \forall j. \end{aligned} \tag{4.3}$$

In fact, in taking this limit, there is no difficulty with the nonlinear term. It is easy to see that.

$$B(u^m, u^m, v) \longrightarrow B(u,u,v) \quad , \forall v \in V$$
$$b(u^n, \varphi^n, \psi) \longrightarrow b(u, \varphi, \psi) \quad , \forall \psi \in H^1(\Omega)$$

(see, for example, Temam [13]). Also, we observe that

$$(\nu(\varphi^n + S)\nabla u^n, \nabla v^j) = (\nabla u^n, \nu(\varphi^n + S)\nabla v^j) \longrightarrow (\nabla u, \nu(\varphi + S)\nabla v^j) = (\nu(\varphi + S)\nabla u, \nabla v^j)$$

because $\nu(\varphi^n + S)\nabla v^j \longrightarrow \nu(\varphi + S)\nabla v^j$, strongly in $L^2(\Omega)$ due to Lebesgue Dominated Convergence Theorem.

Similarly,

$$(k(\varphi^n + S)\nabla \varphi^n, \nabla \psi^j) \longrightarrow (k(\varphi + S)\nabla \varphi, \nabla \psi^j)$$

As the system $\{v^k\}$ (respectively $\{\psi^k\}$) is complete in V (respectively $H_0^1(\Omega)$), (4.3) implies that (u, φ) satisfies (2.1). Therefore, $(u, \varphi + S)$ is a required weak solution.

In order to prove the solvability of the system (4.1), (4.2) for any $n \in \mathbb{N}$, we follow Heywood [7] in applying Brouwer's Fixed Point Theorem.

Let V_n be the subspace of V spanned by $\{v^1, \dots, v^n\}$, and let M_n be the subspace spanned by $\{\psi^1, \dots, \psi^n\}$. For every $(w, \xi) \in V_n \times M_n$ we consider the unique solution $L(w, \xi) = (u, \varphi) \in V_n \times M_n$ of the linearized equations

(4.4) $\qquad (\nu(\xi + S)\nabla u, \nabla v^j) + B(w, u, v^j) - \alpha(\varphi g, v^j) - \alpha(Sg, v^j) = 0,$

(4.5) $\qquad (k(\xi + S)\nabla \varphi, \nabla \psi^j) + b(w, \varphi, \psi^j) + (k(\xi + S)\nabla S, \nabla \psi^j) + b(w, S, \psi^j) = 0,$

for $1 \leq j \leq n$. This is a system of $2n$ linear equations for the coefficients in the expansions

$$u = \sum_{k=1}^{n} c_k v^k \quad , \quad \varphi = \sum_{k=1}^{n} d_k \psi^k$$

The equations (4.4), (4.5) have a unique solution because the associated homogeneous system ($S = 0$) has an unique solution. In fact, if (u, φ) is a solution of the homogeneous system, proceeding as before, we multiply (4.4) by c_j, (4.5) by d_j and sum over j from 1 to n, to obtain

$$\nu_0(T_0)|\nabla u|^2 = 0 \ , \ \ k_0(T_0)|\nabla \varphi|^2 = 0$$

Hence $u = 0$, $\varphi = 0$.

The continuity of L follows by applying the arguments similar to the ones used to take the limit in (4.1), (4.2).

We also have the following estimates

$$(4.6) \qquad |\nabla u| \leq \frac{\alpha C}{\nu_0(T_0)}(|g|\,|\nabla\varphi| + |g|\,\|S\|_1)$$

$$(4.7) \qquad |\nabla\varphi| \leq \frac{C}{k_0(T_0)}|S|_3|\nabla w| + \frac{k_1(T_0)}{k_0(T_0)}\|S\|_1$$

which are shown in exactly the same way as it was done for a solution (u^n, φ^n) of (4.1), (4.2).

Substituting (4.7) into (4.6), we find

$$|\nabla u| \leq \frac{\alpha C}{\nu_0(T_0)k_0(T_0)}|g|\,|S|_3|\nabla w| + \frac{\alpha C}{\nu_0(T_0)k_0(T_0)}|g|\,\|S\|_1(k_1(T_0) + k_0(T_0))$$

Thus,

$$(4.8) \qquad |\nabla u| \leq \frac{1}{2}|\nabla w| + \alpha\frac{C}{\nu_0(T_0)k_0(T_0)}|g|\,\|S\|_1(k_1(T_0) + k_0(T_0))$$

because (2.10) holds.

If we assume $|\nabla w| \leq F_1(\|S\|_1)$ (see (3.11)), then (4.7) and (4.8) imply that (u, φ) satisfies (3.11), that is,

$$(4.9) \qquad |\nabla u| \leq F_1(\|S\|_1) \text{ and } |\nabla\varphi| \leq F_2(\|S\|_1).$$

Thus, (4.4), (4.5) define a continuous mapping L from the closed and convex set $M = \{(w,\xi) \in V_n \times M_n / |\nabla w| \leq F_1(\|S\|_1) \text{ and } |\nabla\xi| \leq F_2(\|S\|_1)\}$ into itself. Using Brower's Fixed Point Theorem, we conclude that the map L has at least one fixed point, which is a solution of (4.1), (4.2). Thus, the proof os Theorem 2.1 is complete. □

5. Existence of Strong Solutions

In this section we will prove Theorem 2.2; for this we follows Temam [13] in using the equivalence of the norm given by the Stokes Operator (respectively Laplacian operator) and the $V \cap H^2(\Omega)$ norm (respectively $H_0^1(\Omega) \cap H^2(\Omega)$ norm) in smooth domain. The main difficult here is to estimate the nonlinearity in the heigher order terms in the velocity equation; for this we will need an estimate for the associated pressure in the Stokes' Problem.

Proof of Theorem 2.2.

We choose the extension S of T_0 such that S is solution of the problem

$$\begin{aligned} -\Delta S &= 0 \quad \text{in } \Omega \\ S &= T_0 \quad \text{on } \partial\Omega \end{aligned}$$

We know that S is a function in $H^2(\Omega)$ and satisfies

(5.1)
$$\|S\|_2 \le C\|T_0\|_{3/2,\partial\Omega}$$

According to the proof of Theorem 2.1, we have a sequence (u^n, φ^n) satisfying (4.1), (4.2) provided (3.10) holds. Since $|S|_3 \le C\|S\|_1 \le C\|T_0\|_{3/2,\partial\Omega}$, we conclude the existence of this sequence for $\|T_0\|_{3/2,\partial\Omega}$ sufficiently small. We need only to show that this sequence is bounded in $H^2(\Omega)$.

For this, we multiply the equation (4.4) by $\alpha_j c_j$, and then sum over j from 1 to n, to obtain

$$(div(\nu(\xi + S)\nabla u), \tilde{\Delta} u) - B(u, u, \tilde{\Delta} u) + \alpha(Sg, \tilde{\Delta} u) + \alpha(\varphi g, \tilde{\Delta} u) = 0$$

Using the identity

$$div(\nu(\xi + S)\nabla v) = \nu(\xi + S)\Delta v + \nu'(\xi + S)\nabla(\xi + S)\nabla v,$$

where $\nabla(\xi + S)\nabla v$ denotes the vector field which i^{th} component is given by $[\nabla(\xi+S)\nabla v]_i = (\nabla(\xi + S), \nabla u_i)_{\mathbb{R}^n}$, where $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the cannonical inner product in $\mathbb{R}^n$, we find

(5.2)
$$\begin{aligned}(\nu(\xi + S)\Delta u, \tilde{\Delta} u) &= B(u, u, \tilde{\Delta} u) - \alpha(\varphi g, \tilde{\Delta} u) - \alpha(Sg, \tilde{\Delta} u) \\ &- (\nu'(\xi + S)\nabla(\xi + S)\nabla u, \tilde{\Delta} u).\end{aligned}$$

Since $\Delta u \neq \tilde{\Delta} u$, we need the following decomposition

$$\Delta u + grad\, q = \tilde{\Delta} u$$

It is well known that (see Teman [13])

(5.3)
$$\|q\|_1 \le C|\tilde{\Delta} u|.$$

Now, we can rewrite (5.2) as

$$\begin{aligned}(\nu(\xi + S)\tilde{\Delta} u, \tilde{\Delta} u) &= B(u, u, \tilde{\Delta} u) - \alpha(\varphi g, \tilde{\Delta} u) - \alpha(Sg, \tilde{\Delta} u) \\ &- (\nu'(\xi + S)\nabla(\xi + S)\nabla u, \tilde{\Delta} u) + (\nu(\xi + S)\nabla q, \tilde{\Delta} u)\end{aligned}$$

By Holder's inequality, Sobolev imbedding and (3.4), we have

(5.4)
$$\begin{aligned}|\tilde{\Delta} u| &\le \frac{C}{\nu_0(T_0)}(|\nabla u| + \nu_1'(T_0)(|\Delta\xi| + \|S\|_2))|\tilde{\Delta} u| \\ &+ \frac{C}{\nu_0(T_0)}|g|_3(|\nabla\varphi| + \|S\|_1) + \left|\left(\frac{\nu(\xi + S)}{\nu_0(T_0)}\nabla q, \tilde{\Delta} u\right)\right|\end{aligned}$$

where

$$\nu_1'(T_0) = 2\sup\{\nu'(t); |t| \leq \sup_{\partial\Omega} |T_0|\}$$

We observe that

$$(\nu(\xi+S)\nabla q, \tilde{\Delta}u) = -(q, div(\nu(\xi+S)\tilde{\Delta}u)) = -(q, (\nu'(\xi+S)\nabla(\xi+S), \tilde{\Delta}u)_{R^n})$$

because $\tilde{\Delta}u \in V_n$. Therefore,

$$(5.5) \qquad \begin{aligned} |(\frac{\nu(\xi+S)}{\nu_0(T_0)}\nabla q, \tilde{\Delta}u)| &\leq \frac{\nu_1'(T_0)}{\nu_0(T_0)}|q|_4|\nabla(\xi+S)|_4|\tilde{\Delta}u| \\ &\leq C\frac{\nu_1'(T_0)}{\nu_0(T_0)}(|\Delta\xi| + \|S\|_2)\|q\|_1|\tilde{\Delta}u| \end{aligned}$$

Combining estimates (5.4), (5.6) and (5.1), we have

$$(5.6) \qquad \begin{aligned} |\tilde{\Delta}u| &\leq \frac{\overline{C}}{\nu_0(T_0)}(|\tilde{\Delta}w| + 2\nu_1'(T_0)(|\Delta\xi| + \|T_0\|_{\frac{3}{2},\partial\Omega}))|\tilde{\Delta}u| \\ &+ \alpha\frac{\overline{C}}{\nu_0(T_0)}|g|_3(|\Delta\varphi| + \|T_0\|_{\frac{3}{2},\partial\Omega}) \end{aligned}$$

Similarly, the following estimate holds

$$(5.7) \qquad \begin{aligned} |\Delta\varphi| &\leq \frac{\overline{C}}{k_0(T_0)}(|\tilde{\Delta}w| + k_1'(T_0)(|\Delta\xi| + \|T_0\|_{\frac{3}{2},\partial\Omega}))(|\Delta\varphi| + \|T_0\|_{\frac{3}{2},\partial\Omega}) \\ &+ \frac{k_1(T_0)}{k_0(T_0)}\|T_0\|_{\frac{3}{2},\partial\Omega}, \end{aligned}$$

where $k_1'(T_0) = 2\sup\{k'(t); |t| \leq \sup_{\partial\Omega}|T_0|\}$ and $\overline{C}$ is a positive constant. Now, we take (w,ξ) such that $|\Delta w| \leq 1$ and $|\Delta\xi| \leq (1 + 2\frac{k_1(T_0)}{k_0(T_0)})\|T_0\|_{\frac{3}{2},\partial\Omega}$ and $\|T_0\|_{3/2,\partial\Omega}$ sufficiently small in such way that

$$(5.8) \qquad \frac{\overline{C}}{k_0(T_0)}\left[1 + 2k_1'(T_0)(1 + \frac{k_1(T_0)}{k_0(T_0)})\|T_0\|_{\frac{3}{2},\partial\Omega}\right] < \frac{1}{2}$$

and

$$(5.9) \qquad \frac{\overline{C}}{\nu_0(T_0)}\left[1 + 4\nu_1'(T_0)(1 + \frac{k_1(T_0)}{k_0(T_0)})\|T_0\|_{\frac{3}{2},\partial\Omega}\right] < \frac{1}{2}$$

Observe that this is possible because $\lim_{\delta\to 0+} k(\delta) > 0$ and $\lim_{\delta\to 0+} \nu(\delta) > 0$.

Then, estimates (5.6), (5.7) imply

$$
\begin{aligned}
|\tilde{\Delta}u| &\leq 4\frac{\alpha\overline{C}}{\nu_0(T_0)}|g|_3(1+\frac{k_1(T_0)}{k_0(T_0)})||T_0||_{3/2,\partial\Omega},\\
|\Delta\varphi| &\leq (1+2\frac{k_1(T_0)}{k_0(T_0)})||T_0||_{3/2,\partial\Omega}.
\end{aligned}
$$

Consequently, if $||T_0||_{3/2,\partial\Omega}$ is small enough, L map the closed and convex set $\{(w,\xi)\in V_n\times M_n; |\tilde{\Delta}w|\leq 1$ and $|\Delta\varphi|\leq(1+2\frac{k_1(T_0)}{k_0(T_0)})||T_0||_{3/2,\partial\Omega}\}$ into it self. Then we can choose a sequence (u^n,φ^n) bounded in $H^2(\Omega)$ satisfying (4.1), (4.2), and Theorem 2.2 follows.

6. Regularity

We first state some lemmas necessary for proving Theorem 2.3.

Lema 6.1. Let h be any function of class C^k such that $\sup\{|\frac{d^i h}{dt^i}(t)|, t\in I\!R\}\leq C$, $i=1,\ldots,k$. Then, there exists constants $C(k)$ and $C_1(k)$ such that

(i) $||h(T)||_{k,\infty}\leq C(k)||T||^k_{k+1,4}$

(ii) $||h(T)||_{k,4}\leq C_1(k)||T||^k_{k,4}$

for all $T\in W^{k,4}(\Omega)$.

Proof. We only proof (i); the other inequality can be proved in the same way. We proceed by induction on k.

If $k=0$, the result is trivial. So, suppose the result is true for any $j\in I\!N$ such that $0\leq j<k$, and take $\beta=(\beta_1,\ldots,\beta_n),\beta_i\in I\!N,|\beta|=\beta_1+\cdots+\beta_n=k$. Then, if $i\in\{1,\ldots,n\}$ is such that $\beta_i>0$, we have.

$$
\begin{aligned}
\partial^\beta(h(T)) &= \partial^{\beta'}(\partial_{x_i}(h(T)) = \partial^{\beta'}(h'(T).\partial_{x_i}T)\\
&= \sum_{\gamma+\delta=\beta'} c(j)\partial^\gamma(h'(T))\partial^\delta(\partial_{x_i}T),
\end{aligned}
$$

where $\beta'=\beta-e_i, e_i$ is the i^{th} vector in the cannonical basis of $I\!R^n$, $c(j)$ are positive constants and $\gamma=(\gamma_1,\ldots,\gamma_n)$, $\delta=(\delta_1,\ldots,\delta_n)$, γ_j, $\delta_j\in I\!N$.

Thus, by the inductive hyposthesis and Sobolev imbeddings

$$
|\partial^\beta(h(T))|_\infty \leq \sum_{\gamma+\delta=\beta'} c(j)|\partial^\gamma(h'(T))|_\infty|\partial^\delta(\partial_{x_i}T)|_\infty
$$

$$\leq \sum_{\gamma+\delta=\beta'} c(j)M(|\gamma|)\|T\|_{|\gamma|+1}^{|\gamma|}\|T\|_{|\delta|+1,\infty}$$
$$\leq \sum_{\gamma+\delta=\beta'} c(j)M(|\gamma|)C\|T\|_{|\gamma|+1}^{|\gamma|}\|T\|_{|\delta|+2,4}$$

Now, we observe that $|\gamma| \leq k-1$, $|\delta| \leq k-1$ and so $|\gamma|+1 \leq k$ and $|\delta|+2 \leq k+1$. Thus,

$$|\partial^\beta(h(T))|_\infty \leq C(k)\|T\|_{k+1,4}^k$$

and the Lemma is proved. □

Lemma 6.2: If h satisfies the conditions of Lemma 6.1, then

$$\|h(T)f\|_{k,p} \leq C_1(k)\|T\|_{k+1,4}^k\|f\|_{k,p}$$

for all T in $W^{k+1,4}(\Omega)$ and all f in $W^{k,p}(\Omega)$.

Proof. Let $\beta = (\beta_1, \ldots, \beta_n), \beta_i \in I\!N\, |\beta| = \beta_1 + \cdots + \beta_n = k$ As before

$$\partial^\beta(h(T)f) = \sum_{\gamma+\delta=\beta} c(j)\partial^\gamma(h(T))\partial^\delta f$$

then by Lemma 6.1, we have

$$\begin{aligned} |\partial^\beta(h(T)f)|_p &\leq \sum_{j=0}^{k} c(j)|\partial^\gamma(h(T))|_\infty|\partial^\delta f|_p \\ &\leq \sum_{j=0}^{k} c(j)(C(j)\|T\|_{j+1,4}^j|\partial^\delta f|_p \\ &\leq C_1(k)\|T\|_{k+1,4}^k\|f\|_{k,p} \end{aligned}$$

This proves the Lemma. □

Lemma 6.3: Let (u, T) be a strong solution of (1.1), (3.2). Assume

$$T_0 \in W^{7/4,4}(\partial\Omega), \quad \text{then} \quad T \in W^{2,4}(\Omega).$$

Proof. According to section 3, we can suppose that (3.4) holds. We observe that $T \in H^2(\Omega)$ satisfies

$$(6.1) \qquad \begin{cases} -\Delta T + \dfrac{k'(T)}{k(T)}|\nabla T|^2 + \dfrac{1}{k(T)}u.\nabla T = 0 \quad \text{in} \quad \Omega \\ T = T_0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Since $T_0 \in W^{7/4,4}(\partial\Omega) \subseteq W^{5/3,3}(\partial\Omega)$ and

$$|\frac{k'(T)}{k(T)}|\nabla T|^2|_3 \leq \frac{k_1'(T_0)}{k_0(T_0)}|\nabla T|_6^2 \leq \frac{k_1'(T_0)}{k_0(T_0)}C|\Delta T|^2$$

$$|\frac{u\nabla T}{k(T)}|_3 \leq \frac{1}{k_0(T_0)}|u|_6|\nabla T|_6 \leq \frac{C}{k_0(T_0)}|\nabla u||\Delta T|,$$

we can apply the well-know L^p-regularity properties of Laplacian operator obtaining T in $W^{2,3}(\Omega)$. By using Sobolev imbedding we have that $T \in L^q(\Omega)$, for all $1 \leq q < \infty$. Consequently, $\left|\frac{k'(T)}{k(T)}|\nabla T|^2\right|_4 < \infty$ and $\left|\frac{1}{k(T)}u.\nabla T\right|_4 < \infty$.

Applying L^p-regularity once again, we find $T \in W^{2,4}(\Omega)$. □

Proof of Theorem 2.3. We proceed inductively on k. if $k = 0$ the result follows by Lemma 6.3.

Now, we suppose the result is true for $k-1$ (that is, $T \in W^{k+1,4}(\Omega)$). Note that if $\beta = (\beta_1, \ldots, \beta_n), \beta_i \in I\!N, |\beta| = \beta_i + \cdots + \beta_n = k$, then

$$\partial^\beta(u\nabla T) = \sum_{\gamma+\delta=\beta} c(j)\partial^\gamma u\partial^\delta(\nabla T) + u\partial^\beta(\nabla T),$$

where $c(j)$ are positive constants, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\delta = (\delta_1, \ldots, \delta_n)$, $\gamma_i, \delta_i \in I\!N$. Thus,

$$|\partial^\beta(u\nabla T)|_4 \leq \sum_{\gamma+\delta=\beta} c(j)|\partial^\gamma u|_4|\partial^\delta(\nabla T)|_\infty + |u|_\infty|\partial^\beta(\nabla T)|_4$$

Sobolev imbeddings, together with the inductive hypothesis imply

$$||u\nabla T||_{k,4} \leq C_{\Omega,k}||u||_{k+1,2}||T||_{k+1,4} + C_{\Omega,k}||u||_2||T||_{k+1,4} < \infty$$

By using Lemma 6.2, we have $\frac{1}{\nu(T)}u\nabla T \in W^{k,4}(\Omega)$.

Similarly as before

$$\begin{aligned}|\partial^\beta|\nabla T|^2|_4 &\leq \sum_{\gamma+\delta=\beta} c(j)|\partial^\gamma\nabla T|_\infty|\partial^\delta\nabla T|_4 + |\partial^\beta\nabla T|_4|\nabla T|_\infty \\ &\leq C_{\Omega,k}||T||^2_{k+1,4} + C_{\Omega,k}||T||_{k+1,4}||T||_{2,4} < \infty\end{aligned}$$

Therefore $\frac{k'(T)}{k(T)}|\nabla T|^2 \in W^{k,4}(\Omega)$

Applying L^p-regularity for problem (6.1), we see, that $T \in W^{k+2,4}(\Omega)$.

As in the proof of Lemma 6.3 we have that u is solution of the following Stokes Problem

$$
(6.2)\begin{cases} -\ \Delta u + grad\left(\dfrac{p}{\nu(T)}\right) = -\dfrac{\nu'(T)}{\nu(T)^2}p\nabla T - \dfrac{1}{\nu(T)}u.\nabla u + \alpha\dfrac{T}{\nu(T)}g + \dfrac{\nu'(T)}{\nu(T)}\nabla T.\nabla u \\ div\ u = 0 \\ u = 0 \quad \text{on } \partial\Omega \end{cases}
$$

where p satisfies (2.2).

As above, we show by induction that the right hand side terms in the first equation of (6.4), are in $H^k(\Omega)$. By Cattabriga's Theorem (see [4] and [2]) applied to (6.2), we find that $u \in H^{k+2}(\Omega)$, $\dfrac{p}{\nu(T)} \in H^{k+1}(\Omega)$.

Applying Lemma 6.2 we conclude that $p \in H^{k+1}(\Omega)$. This completes the proof. □

7. Uniqueness

In this section we will prove Theorem 2.4.

Let $(u_1, T_1), (u_2, T_2)$ be a weak solutions of (1.1), (1.2) such that T_1 and T_2 are in $H^2(\Omega)$. Put $w = u_1 - u_2, \xi = T_1 - T_2$. Then $w \in V, \xi \in H_0^1(\Omega) \cap H^2(\Omega)$ and satisfy.

$$
\begin{aligned}
&(\nu(T_1)\nabla w, \nabla v) + B(w, u_1, v) + B(u_2, w, v) - \alpha(\xi g, v) + ((\nu(T_1) - \nu(T_2))\nabla u_2, \nabla v) = 0,\\
-\ &(div(k(T_1)\nabla\xi), \psi) + b(w, T_1, \psi) + b(u_2, \xi, \psi) - (div((k(T_1) - k(T_2))\nabla T_2), \psi) = 0,\\
&\forall v \in V,\ \forall\psi \in L^2(\Omega)
\end{aligned}
$$

We take $v = w$ and $\psi = \Delta\xi$ in these last equalities, thus obtaining

$$
(7.1) \qquad |\nabla w| \ \le\ \frac{\alpha C}{\nu_0(T_0)}|g||\Delta\xi| + \frac{C_B}{\nu_0(T_0)}|\nabla u||\nabla u_1| + \frac{C_1}{\nu_0(T_0)}|\xi|_\infty|\nabla u_2|
$$

$$
\begin{aligned}
(7.2) \qquad |\Delta\xi| \ \le\ & \frac{C}{k_0(T_0)}||T_1||_2|\nabla w| + \frac{C}{k_0(T_0)}|\nabla u_2||\Delta\xi|\\
&+ \frac{C_1}{\nu_0(T_0)}|\xi|_\infty|\Delta T_2| + \frac{k_1'(T_0)}{k_0(T_0)}C||T_2||_2|\Delta\xi| + \frac{C_1}{k_0(T_0)}||T_2||_2^2|\xi|_\infty
\end{aligned}
$$

where C_1 is a constant such that $|k'(t) - k'(s)| + |k(t) - k(s)| + |\nu(t) - \nu(s)| \le C_1|t - s|$ for all t, s in $\mathbb{R}$. This is shown in exactly the same way as in the case of the n^{th} aproximate solutions (u^n, φ^n) in Sections 4 and 5.

We note that $|\xi|_\infty \le C|\Delta\xi|$. Thus, (7.2) implies

$$
|\Delta\xi| \le \frac{C}{k_0(T_0)}||T_1||_2|\nabla w| + \frac{C}{k_0(T_0)}\left[|\nabla u_2| + (C_1 + k_1'(T_0))||T_2||_2 + C_1||T_2||_2^2\right]|\Delta\xi|
$$

Assume that $\dfrac{C}{k_0(T_0)}\left[|\nabla u_2| + (C_1 + k_1'(T_0))||T_2||_2 + C_1||T_2||_2^2\right] < \dfrac{1}{2}$, then

$$|\Delta\xi| \leq \frac{2}{k_0(T_0)}||T_1||_2|\nabla w| \tag{7.3}$$

substituting (7.3) into (7.1), we obtain

$$|\nabla w| \leq \left[\frac{2}{\nu_0(T_0)k_0(T_0)}||T_1||_2(\alpha C|g| + C_1|\nabla u_2|) + \frac{C_B}{\nu_0(T_0)}|\nabla u_1|)\right]|\nabla w|.$$

Thus, if $\frac{2}{\nu_0(T_0)k_0(T_0)}||T_1||_2(\alpha c|g| + C_1|\nabla u_2|) + \frac{C_B}{\nu_0(T_0)}|\nabla u_1| < 1$, we have $|\Delta w| = |\Delta\xi| = 0$. Since $w \in V$ and $\xi \in H_0^1(\Omega) \cap H^2(\Omega)$, we see $w = 0$, $\xi = 0$ in Ω.

Therefore, $u_1 = u_2$, $T_1 = T_2$ in Ω $\square$

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