MONOTONICITY AND SYMMETRY OF SOLUTIONS OF ELLIPTIC SYSTEMS IN GENERAL DOMAINS

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Novembro RP 49/92

Relatório de Pesquisa

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ABSTRACT – A new Maximum Principle for elliptic equations has appeared recently in works of Berestycki, Nirenberg and Varadhan, see for instance [BN]. The aim of this note is to extend this Maximum Principle to cooperative elliptic systems and then to apply it to monotonicity and symmetry properties of nonlinear elliptic systems. In this way we get more general results than the ones in [T] and [B], with even simpler proofs. The interest in these results comes from their use in obtaining a priori bounds for positive solutions of semilinear elliptic systems [CFM].

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1. Introduction. A new Maximum Principle for elliptic equations has appeared recently in works of Berestycki, Nirenberg and Varadhan, see for instance [BN]. The aim of this note is to extend this Maximum Principle to cooperative elliptic systems and then to apply it to monotonicity and symmetry properties of nonlinear elliptic systems. In this way we get more general results than the ones in [T] and [B], with even simpler proofs. The interest in these results comes from their use in obtaining a priori bounds for positive solutions of semilinear elliptic systems [CFM].

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain and consider the differential operators

$$L_k(D) = \sum b_{ij}^k(x) D_i D_j + \sum b_i^k(x) D_i \,, \quad 1 \leq k \leq n$$

which are assumed to be uniformly elliptic with L^∞ coefficients. We may assume without loss of generality that for a suitable constant $c_0 > 0$ we have

$$c_0^{-1}|\xi|^2 \leq \sum b_{ij}^k(x)\xi_i x_j \leq c_0|\xi|^2 \,, \qquad \forall x \in \Omega \quad \forall k \qquad \forall \xi \in R^N$$

and

$$|b_i^k(x)| \leq c_0 \,, \qquad \forall i, k \,.$$

We consider these operators acting on functions which are in $E =: W^{2,N}_{\text{loc}}(\Omega) \cap C^0(\overline{\Omega})$. So the matricial operator

$$\mathcal{L} = \begin{bmatrix} L_1 & & \\ & \ddots & \\ & & L_n \end{bmatrix}$$

acts on $E\times \overset{n}{\dots} \times E$.
Let $A = (a_{ij}(x))$ be a cooperative $n \times n$ matrix with entries $a_{ij} \in L^\infty$. Cooperativeness means $a_{ij}(x) \geq 0$ for $i \neq j$. We use capital letters to denote vector functions $U(x) = (u_1(x), \dots, u_n(x))$.
We write $U(x) \geq 0$ to mean $u_j(x) \geq 0, \quad \forall j = 1, \dots, n$.
Following [BN] we introduce the

Definition. A Maximum Principle holds for $\mathcal{L} + A$ in Ω if

$$\begin{aligned} &(\mathcal{L} + A)U \geq 0 \quad \text{in} \quad \Omega \\ &\limsup_{x \to \partial\Omega} U(x) \geq 0 \end{aligned}$$

implies $U \leq 0$ in Ω.
We start by proving the Maximum Principle for bounded domains Ω.

Proposition 1. Assume that diam $\Omega \leq d$. Then there exists $\delta > 0$ depending on N, n, d and c_0, such that the maximum principle holds if the Lebesgue measure $|\Omega|$ of Ω is less than δ.

Proof. Consider the k^{th} equation

$$L_k u_k + \sum a_{kj}(x) u_j \geq 0$$

and rewrite it in the form

$$L_k u_k - a_{kk}^- u_k \geq -a_{kk}^+ u_k - \sum_{j \neq k} a_{kj} u_j \,.$$

Since A is cooperative we get

$$L_k u_k - a_{kk}^- u_k \geq -a_{kk}^+ u_k^+ - \sum_{j \neq k} a_{kj} u_j^+ \,.$$

Applying the Aleksandrov maximum principle, see for instance [GT] for statement and proof, we get

$$\sup u_k \leq C || a_{kk}^+ u_k^+ + \sum_{j \neq k} a_{kj}^+ u_j^+ ||_{L^N} \,,$$

where C depends on N, d and c_0. Replacing $\sup u_k$ by $\sup u_k^+$ and adding up these n inequalities we get

$$\sum \sup u_k^+ \leq Cnc_0|\Omega|^{1/N} \sum \sup u_k^+ .$$

So $\sum \sup u_k^+ \leq 0$ if we choose $\delta = (Cnc_0)^{-N}$. ■

2. Semilinear elliptic systems. Now we consider the system

$$\mathcal{L}U + F(U) = 0 \quad \text{in} \quad \Omega \tag{2.1}$$

where $\mathcal{L} = \begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix}$, and $F(U) = (f_1(u_1, \ldots, u_n), \ldots, f_n(u_1, \ldots, u_n))$.

We assume that $F : R^n \to R^n$ is C^1 and its Jacobian has all off-diagonal entries ≥ 0, i.e.

$$\frac{\partial f_i}{\partial \xi_j}(\xi) \geq 0 \qquad \forall \xi \in R^n, \ i \neq j .$$

We consider nonnegative solutions of system (1.2) which vanish on $\partial\Omega$. We remark that Ω is not necessarily bounded.

Fix a direction γ, i.e., $\gamma \in R^N$, $|\gamma| = 1$. We suppose that there is $a > -\infty$ such that

$$\gamma \cdot x > a \qquad \forall x \in \Omega .$$

This means that Ω is at one side of some hyperplane normal to γ. For simplicity, assume $\gamma = (1, 0, \ldots, 0)$. We recall the following standard notation, [GNN]. For $\lambda \in R$ and writing $x = (x_1, y)$:

$$T_\lambda = \{x \in R^N : \quad \gamma \cdot x = \lambda\}$$
$$\Sigma(\lambda) = \{x \in \Omega : \quad \gamma \cdot x < \lambda\}$$
$$\Sigma'(\lambda) = \{x \in R^N : \quad x^\lambda \in \Sigma(\lambda)\}$$

where x^λ is the reflection of x with respect to T_λ. I.e., if $x = (x_1, y)$ then $x^\lambda = (2\lambda - x_1, y)$. Let $\lambda_0 = \sup\{\lambda : T_\lambda \cap \Omega = \emptyset\}$. Now we make the following assumption on the domain

$$(D) \qquad \exists \varepsilon > 0 \text{ s.t. for } \lambda_0 < \lambda < \lambda_0 + \varepsilon \text{ , } \Sigma'(\lambda) \subset \Omega \text{ , and } \textstyle\sum(\lambda) \text{ is bounded.}$$

This assumption is satisfied in the case considered in [T]: Ω bounded, $\partial\Omega$ of class C^2. (D) is also satisfied on the assumptions in [B].

Now define

$$\overline{\lambda} = \sup\{\lambda : \Sigma'(\mu) \subset \Omega\ ,\ \Sigma(\mu) \text{ bounded}, \quad \forall\, \mu < \lambda\}\,.$$

Theorem. Let $U \geq 0$ be a solution of (2.1) in $W^{2,N}_{\text{loc}}(\Omega) \cap C^0(\overline{\Omega})$ with $U = 0$ on $\partial\Omega$. Then, each $u_k(x)$ is monotonically increasing with respect to x_1 for $x \in \Sigma(\overline{\lambda})$.

Remark 1. On the assumptions of the theorem, each u_k satisfies

$$\Delta u_k - a^-_{kk} u_k \geq 0 \quad \text{and} \quad u_k \geq 0$$

in the whole of Ω. Since $u_k = 0$ on $\partial\Omega$, we have by the maximum principle that either $u_k \equiv 0$ or $u_k > 0$ in Ω.

2. Without loss of generality we assume that $u_k \not\equiv 0$ for all k. Indeed if for one such a k, u_k where $\equiv 0$, we reduce the system to the remaining $k-1$ functions and equations. *So we may assume that* $U > 0$ in Ω, i.e. $u_k > 0$ in Ω for all k.

Proof. The proof follows the usual ideas of the moving plane method, as used by [GNN], [BN] and later papers. For each λ, with $\lambda_0 < \lambda < \overline{\lambda}$ we define

$$v_k(x;\lambda) = v_k(x_1, y; \lambda) =: u_k(2\lambda - x_1, y; \lambda)\,, \qquad \forall\, x \in \Sigma(\lambda)$$

and $w_k(x;\lambda) = v_k(x;\lambda) - u_k(x)$. Using the invariance of the Laplacian we get

$$\Delta w_k + f_k(v_1, \ldots, v_n) - f_k(u_1, \ldots, u_n) = 0$$

or

$$\Delta w_k + \sum \frac{\partial f_k}{\partial \xi_j}(\theta^k_1(x), \ldots, \theta^k_n(x)) w_j = 0 \tag{2.2}$$

where $\theta^k_i(x)$ is between $u_i(x)$ and $v_i(x)$. So the coefficients of (2.2) are bounded, and a little argument shows that they are measurable. Now if $\lambda - \lambda_0$ is small enough the Lebesgue measure of $\sum(\lambda)$ is small and we are in condition to apply Proposition 1 to (2.2) and conclude that $W \geq 0$ in $\sum(\lambda)$. Now let

$$\lambda^* = \sup\{\lambda < \overline{\lambda} : W(x;\lambda) \geq 0 \quad \text{in} \quad \Sigma(\lambda)\}.$$

If $\lambda^* = \overline{\lambda}$, the proof is complete. Suppose by contradiction that $\lambda^* < \overline{\lambda}$. By continuity $W(x;\lambda^*) \geq 0$ and using Remark 1 above we see that $W(x,\lambda^*) > 0$ in $\sum(\lambda^*)$. Let now

d be the diameter of $\sum(\lambda^*)$. By Proposition 1 there is a $\delta > 0$ such that a Maximum Principle holds for system (2.2) in domains of measure less than δ. Take a compact set $K \subset \sum(\lambda^*)$ s.t. $|\sum(\lambda^*) \setminus K| < \delta/2$. Let $a > 0$ be such that $w_k(x;\lambda^*) \geq a$ for $x \in K$ and $k = 1, \ldots, n$. By continuity we can find λ, with $\lambda^* < \lambda < \overline{\lambda}$, such that $|\sum(\lambda) \setminus K| < \delta$ and $w_k(x;\lambda) \geq a/2$ for $x \in K$ and $k = 1, \ldots, n$. So applying Proposition 1 we see that $W(x;\lambda) \geq 0$ in $\sum(\lambda) \setminus K$. So $W(x;\lambda) \geq 0$ in $\sum(\lambda)$, contradicting the definition of λ^* ■

The next two results are immediate consequences of the above theorem. The first one is an extension of Theorem 1.3 [BN]. In particular, it gives the radial symmetry of positive solutions of (2.1) subject to Dirichlet boundary conditions in the case that Ω is a ball. The second one is proved in [B]. It gives monotonicity of solutions of (2.1) in certain directions, for domains like cones, paraboloids, cylinders.

Corollary 1. Let Ω be an arbitrary bounded domain in R^n which is convex on the x_1 direction and symmetric with respect to the plane $x_1 = 0$. Let $U \geq 0$ be a solution of the system (2.1) subject to Dirichlet boundary conditions. F satisfies the conditions put in the beginning of Section 2. Then U is symmetric with respect to x_1 and $u_{x_1} < 0$ for $x_1 < 0$ in Ω.

Corollary 2. Suppose Ω is unbounded domain satisfying the following property: there are a $\gamma \in I\!R^n$, $|\gamma| = 1$ and a number $a > -\infty$ such that

i) $x \cdot \gamma > a$, $\forall x \in \Omega$,

ii) $\forall x \in \Omega \Rightarrow x + t\gamma \in \Omega$, $\forall t \geq 0$

iii) $\forall \lambda$ the set $\{x \in \Omega : x \cdot \gamma < \lambda\}$ is bounded.

Let U be a positive solution of (2.1) under Dirichlet boundary conditions. Then U is (strictly) increasing in the direction γ.

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