A GENERALIZED DIRAC'S QUANTIZATION CONDITION FOR PHENOMENOLOGICAL NON ABELIAN MAGNETIC MONOPOLES

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ABSTRACT – We extend the Dirac's quantization condition for the phenomenological non-abelian magnetic monopoles whose theory was developed in a series of papers in the last few years.

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A Generalized Dirac's quantization condition for phenomenological non - abelian magnetic monopoles

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I. Introduction

Since the pioneering work of Dirac[1], the magnetic monopole have been intensively studied in order to try to discover, or perhaps to guess, the role of this elusive object in nature.

One of the striking consequences of the existence of magnetic monopoles is the quantization of electric charge. In this work we obtain a quantization condition for phenomenological non-abelian magnetic monopoles. This kind of monopole has been described in the literature in a series of papers in the last few years[2–7]. The main characteristic of these monopoles is that, in contrast with the topological ones, they are introduced together with the electric charge as a composed (graded) object in a Grassmann Algebra.

Although there are others quantization conditions, like the Schwinger's one, we obtain here a generalized Dirac quantization condition. In our point of view Dirac's condition is the only quantization condition compatible with the path dependent method introduced by Mandelstam[8] and used by Cabibbo and Ferrari in their, now, classic work[9].

Here we obtain a general quantization condition for an arbitrary Lie group and with this we extend the result already obtained for to the abelian groups[4,5]. Our theory of phenomenological magnetic monopoles is a attempt to develop a consistent description of these objects without considering them as topological objects. The price to be paid for this is that these objects "live" in a Grassmanian World, where potentials, curvatures, etc are elements of a Grassmann Algebra instead of the algebra of real numbers. Nevertheless, it is a geometric theory, because it is founded on a principal fibre-bundle setting over space-time. Roughly speaking this is a more algebraic than a topological theory.

To be self-contained we give in part II a short review of the theory of phenomenological magnetic monopoles. In part III we obtain our quantization condition. Finally in part IV we conclude with some general observations.

II - General theory

Let $P(M,G,\pi)$ be a principal fiber-bundle over space-time M here considered as a Lorentzian manifold where the metric is taken with signature $(+1,-1,-1,-1)$. Let α and α' be two connections defined in $P(M,G,\pi)$ with values in the Lie-algebra $\mathcal{G}$ of G, and such that the pull-backs to M are respectively the gauge potentials A and B.

Also, let $\Lambda(M,\mathcal{G}) = \sum_{k=0}^{n}\Lambda_k(M,\mathcal{G})$ be the Grassmann Algebra of the $\mathcal{G}$-Valued differential k-forms, over space-time. We can choose in $\mathcal{G}$ a orthonormal basis $E_1, E_2, \ldots, E_n$ and we denote the inner product by

$$\langle E_i, E_j\rangle = \delta_{ij} \tag{1}$$

Definition 1. The Generalized Potential is defined by

$$\omega = A + *B \in \Lambda^1(M,\mathcal{G}) \oplus \Lambda^3(M,\mathcal{G}) \tag{2}$$

Definition 2. The Generalized Dirac operator associated to ω is

$$\mathbb{D}^{\omega} = D^{A} + \delta^{B} \tag{3}$$

where D^A and δ^B are the usual covariant derivatives and coderivatives of the usual gauge theories with gauge group G^{10}. Next, we need

Definition 3. The generalized field is defined by

$$\begin{aligned} \Omega &\equiv \mathbb{D}^{\omega}\omega = (D^{A} + \delta^{B})(A + *B) \\ &+ \underbrace{\Omega^{A} + *\Omega^{B}}_{2-form} + \underbrace{\delta^{B}A}_{0-form} + \underbrace{D^{A}(*B)}_{4-form} \end{aligned} \tag{4}$$

where $\Omega^A = D^A A$ and $\Omega^B = D^B B$ as usual.

Eqs. (2) and (3) show that in the general theory the potential is an element of the odd part of the Grassmann algebra of space-time $\Lambda(M, g)$ and the generalized field is an element of the even part of $\Lambda(M, g)$.

The first important remark is that contrary to the usual gauge theories here we do not have the validity of Bianchi's identity. Instead we have

$$\mathbb{D}^{\omega}\Omega = (D^{A}\delta^{B} + \delta^{B}D^{A})(A + *B) \tag{5}$$

which is in general different from zero. The tripotential $*B$ allows degrees of freedom to describe a generalized magnetic monopole. If $B = 0$ we recover the Bianchi's identity, since (5) vanishes identically.

In order to present the field equations of the general theory we need the following definitions:

Definition 4. The dual operator to $\mathbb{D}^{\omega}$ is defined by

$$\Delta^{\omega} = *\mathbb{D}^{\omega}* = *(D^{A} + \delta^{B})* = D^{B} + \delta^{A} \tag{6}$$

We have,

$$\begin{aligned} \Delta^{\omega}\Omega &= D^{B}\Omega^{A} + D^{B}(*\Omega^{B}) + D^{B}\delta^{B}A + D^{B}D^{A}(*B) + \delta^{A}\Omega^{A} + \\ &+ \delta^{A}(*\Omega^{B}) + \delta^{A}\delta^{B}A + \delta^{A}D^{A}(*B) \end{aligned} \tag{7}$$

Eq (7) may be simplified since dim $M = 4$ and $A, B \in \Lambda^1(M, g)$ we have identically

$$D^B D^A(*B) = 0, \qquad \delta^A \delta^B(A) = 0 \tag{8}$$

We now introduce the field equations throught:

Postulate 1 The field equation of the general theory is

$$\Delta^\omega \Omega = J_{\rm el} + *J_{\rm mag} \tag{9}$$

where $J_{\rm el}$ and $*J_{\rm mag}$ describe the sources of the generalized field.

In what follows we call *dual charges* the charges associated to the current $*J_{\rm mag}$.

Eq. (9) may be written, using Eq. (7) and Eq. (8), as

$$\begin{cases} \delta^A \Omega^A + *D^B \Omega^A + D^B \delta^B A &= J_{\rm el} \\ \delta^B \Omega^B + *D^A \Omega^B + D^A \delta^A B &= J_{\rm mag} \end{cases} \tag{10}$$

In the usual gauge theory based on a principal fiber bundle we have associated to the connections α and α' the field equations

$$\delta^A \Omega^A = J_{\rm el} \qquad \text{and} \qquad \delta^B \Omega^B = J_{\rm mag} \tag{11}$$

The additional terms on the left-hand side of Eqs. (10) show that the general theory contains a non-trivial interaction between the potentials A and B, which are represented by interaction currents.

Finally we make an interesting remark: We would like the potentials A and B to be independent of each other, to a certain degree. This is provided by the Generalized Lorentz Gauge:

$$\delta^A B = \delta^B A = 0 \tag{12}$$

Using (12), the Equation (10) can be written as

$$\delta^A \Omega^A + *D^A \Omega^B = J_{\rm el} \tag{13a}$$

$$\delta^B \Omega^B + *D^B \Omega^A = J_{\rm mag} \tag{13b}$$

These equations can be obtained from a spliced bundle formalism in a very elegant way[6].

Definition 5. The interaction currents are

$$\begin{cases} J_{\text{int}}^{(A,B)} = *D^B\Omega^A + D^B\delta^B A & (14a) \\ J_{\text{int}}^{(B,A)} = *D^A\Omega^B + D^A\delta^A B. & (14b) \end{cases}$$

If we use the Generalized Lorentz Gauge, Eq.(12), these equations are simplified to:

$$J_{\text{int}}^{(A,B)} = *D^B\Omega^A \tag{15a}$$

$$J_{\text{int}}^{(B,A)} = *D^A\Omega^B. \tag{15b}$$

We can see that in the usual electrodynamics:

$$J_{\text{int}}^{(A,B)} = *D^A\Omega^B = *d(dB) = 0$$

since $D^A = d$ for any potential A with values in an abelian group such as $U(1)$. This shows that the photon field does not have self-interactions in the present theory as in the usual electrodynamics on a classical level. Eqs. (15) may have a role in strong interaction theories in which magnetic monopoles are inclued since they establish a kind of "minimal coupling".

Using Cartan's structural equation $\Omega^\alpha = d\alpha + \frac{1}{2}[\alpha, \alpha]$ for $\alpha = A, B$, we can obtain an expression in components of the generalized field. We get

$$\begin{aligned} \Omega^i_{\mu\nu} &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \epsilon_{\mu\nu\rho\sigma}\partial^\rho (B^i)^\sigma + \\ &+ \frac{1}{2}A^k_\mu A^j_\nu C^i_{kj} - \epsilon_{\mu\nu\rho\sigma}\frac{1}{2}(B^k)^\rho (B^j)^\sigma C^i_{kj} \end{aligned} \tag{16}$$

where $[E_k, E_j] = C^i_k, E_i$, with $E_i \in g$. Eq. (16) is a generalization of the Cabibbo-Ferrari relation[9] for a non-abelian group G.

III - Non - abelian quantization condition

We define the non-abelian electric charge, e, of a particle as the "charge vector"

$$e \equiv (e_1, e_2, \ldots, e_n) = \sum_{i=1}^{n} e_i E_i \tag{17}$$

where each e_i is a real number. For example, in QCD the charge vector can be interpreted as the electric-colour charges.

Using the Generalized Lorentz gauge (12) the generalized curvature Ω in (4) is reduced to a usual 2-form

$$\Omega = \Omega^A + *\Omega^B. \tag{18}$$

As done in References 4 and 5 and following Mandelstam[8], we can define a path dependent field $\phi(x, P)$ that satisfies

$$\phi(x, P') = \phi(x, P) \ \exp\Big[\int_S -i\langle\, e, \Omega\, \rangle\Big] \tag{19}$$

where P and P' are two curves that are different only on a finite lenght and S is any surface that has the closed curve $C = P - P'$ as a oriented boundary. In other words, (19) is independent of the surface satisfying $\partial S = C$.

The inner product in (19) means

$$\langle\, e, \Omega\, \rangle \equiv \delta_{ij} e^i \Omega^j = \sum_{j=1}^{n} e_j \Omega^j. \tag{20}$$

Then, (19) can be written as

$$\begin{aligned} \phi(x, P') &= \phi(x, P) \ \exp\Big[\int_S -i \sum_{j=1}^{n} e_j \Omega^j\Big] = \\ &= \phi(x, P) \ \exp\Big[\sum_{j=1}^{n} \int_S -i e_j \Omega^j\Big] \end{aligned} \tag{21}$$

where each Ω_j is a $I\!R$-valued 2-form, satisfying the Cartan structural Equation.

Now, as (19) is independent of the surface satisfying $\partial S = C$, for two surfaces S_1 and S_2 with $\partial S_1 = \partial S_2 = C$ we have

$$\phi(x, P) \ \exp\Big[\sum_{j=1}^{n} \int_{S_1} -i e_j \Omega^j\Big] = \phi(x, P) \ \exp\Big[\sum_{j=1}^{n} \int_{S_2} -i e_j \Omega^j\Big] \tag{22}$$

and then

$$\exp\Big[\sum_{j=1}^{n}\int_{S_1} -ie_j\Omega^j - \sum_{j=1}^{n}\int_{S_2} -ie_j\Omega^j\Big] = 1 \tag{23}$$

As S_1 and S_2 have the same boundary, they together define a compact oriented surface without boundary, say S_0. Formally we can write

$$S_0 = S_1 - S_2 \tag{24}$$

From (23) and (24) we obtain

$$\exp\Big[\sum_{j=1}^{n}\oint_{S_0} -ie_j\Omega^j\Big] = 1 \tag{25}$$

By Stoke's theorem, this is equivalent to

$$\exp\Big[\sum_{j=1}^{n}\int_{V} -ie_j d\Omega^j\Big] = 1 \tag{26}$$

where V is the volume enclosed by S_0.

Using again the inner product (1) we can write this equation as

$$\exp\Big[\int -i\langle\, e, d\Omega\,\rangle\Big] = 1 \tag{27}$$

Now, in order to obtain a more familiar expression for $d\Omega$ we calculate $d\Omega$ explicitly

$$\begin{aligned} d\Omega &= d(\Omega^A + *\Omega^B) = \\ &= d[dA + (A\wedge A) + *dB + *(B\wedge B) = \\ &= d(A\wedge A) + d*dB + d*(B\wedge B) = \\ &= *\delta dB + d(A\wedge A) + *\delta(B\wedge B) = \end{aligned} \tag{28}$$

Now, we especialize for the case where we have just one magnetic monopole. In this case for the generalized potential (2) we have $A \equiv 0$ which implies $\Omega^A = 0$.

So, in this particular case (28) is reduced to

$$d\Omega = *\delta dB + *\delta(B\wedge B) \tag{29}$$

On the other hand the inhomogeneous field equation (9) is reduced to

$$J_{\mathrm{mag}} = \delta^B \Omega^B \tag{30}$$

because $A \equiv 0$ and $J_{el} \equiv 0$.

The right hand side of (30) is

$$\begin{aligned} J_{\mathrm{mag}} &= *D^B * [dB + B \wedge B] = *D^B[*dB + *(B \wedge B)] = \\ &= *[d[*dB + *(B \wedge B)] + B \wedge [*dB + *(B \wedge B)]] = \\ &= \delta dB + \delta(B \wedge B) + *(B \wedge dB) + *[B \wedge *(B \wedge B)] \end{aligned}$$

then

$$* \,\delta dB + *\delta(B \wedge B) = *J_{\mathrm{mag}} - B \wedge *dB - B \wedge *(B \wedge B) \tag{31}$$

Comparing (29) and (31) we have:

$$\begin{aligned} d\Omega &= *J_{\mathrm{mag}} - (B \wedge *dB) - B \wedge *(B \wedge B) \\ &= *J_{\mathrm{mag}} - B \wedge *[dB + B \wedge B] \\ &= *J_{\mathrm{mag}} - B \wedge * \mathbb{D}^B B = *J_{\mathrm{mag}} - B \wedge *\Omega^B \end{aligned} \tag{32}$$

Now, the easiest way to get magnetic monopoles in this approach is to make the following constraint-anzatz

$$B \wedge *\Omega^B = 0 \tag{33}$$

From (32) and (33) we get

$$d\Omega = *J_{\mathrm{mag}} \tag{34}$$

substituting (34) in (27) we have:

$$\begin{aligned} 1 &= \exp\left[\int_V -i\langle\, e, d\Omega\,\rangle\right] = \exp\left[-i \int \sum_{j=1}^{n} e_j(*J^j_{\mathrm{mag}})\right] = \\ &= \exp\left[-i \sum_{j=1}^{n} e_j \int_V *J^j_{\mathrm{mag}}\right] = \exp\left[-i \sum_{j=1}^{n} e_j g_j\right] \end{aligned} \tag{35}$$

where we have used that the charge g_j inside the volume V is given by

$$g_j = \int_V *J^j_{\mathrm{mag}} \tag{36}$$

Thus, (35) implies that

$$\sum_{j=1}^{n} e_j g_j = 2\pi m \quad , m = 0, 1, 2, ... \tag{37}$$

or

$$\frac{1}{4\pi} \sum_{j=1}^{n} e_j g_j = \frac{m}{2} \quad , m = 0, 1, 2, ... \tag{38}$$

This is the Generalized Dirac's quantization condition for non-abelian magnetic monopoles.

IV - Conclusion

It's worthy to note that in our theory, such as in the Cabibbo-Ferrari one, there exist a certain asymmetry in the formulation of the coupling of electric charges with magnetic monopoles. The electric charge appears as a coupling constant in the Mandelstam path integral and the magnetic monopoles appears as source, or better, as a phenomenological current. This is because we are considering, as Cabibbo and Ferrari[9], the electric charge as a test particle in the field of a monopole. In our theory this current is represented by a 3-form. We think that this, as well the constraint condition (33), is a shortcoming in these kind of theories. An ideal theory should eliminate them.

So, we can consider our theory as a provisional step to find an algebraic (or non-topological) theory that can place magnetic monopoles and electric charges on the same footing.

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