ON THE PAPER "CAUCHY COMPLETENESS OF ELEMENTARY LOGIC" OF D. MUNDICI AND A.M. SETTE

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Relatório de Pesquisa

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Abstract - In [M – S] it was shown that the space $Str(\tau)$ of all structures of finite type τ has a natural pseudometric which generates the elementary topology and which is "Cauchy Complete". In this note we generalize this result to arbitrary type τ, eliminating the hypothesis of total boundedness of the pseudometric. Finally, a new interpretation of Los' Theorem is given.

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O conteúdo do presente Relatório de Pesquisa é de única responsabilidade do autor.

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Abstract - In [M – S] it was shown that the space $Str(\tau)$ of all structures of finite type τ has a natural pseudometric which generates the elementary topology and which is "Cauchy Complete". In this note we generalize this result to arbitrary type τ, eliminating the hypothesis of total boundedness of the pseudometric. Finally, a new interpretation of Łos' Theorem is given.

The elementary topology of $Str(\tau)$, the space of all structures of arbitrary similarity type τ, is generated by the following basis: $\{\text{Mod }(\varphi) \mid \varphi \in L^{\tau}_{\omega\omega}\}$, which does not necessarily satisfies the second axiom of countability. We show in this note that this topology is uniformizable for every type τ in such a way that the resulting uniform space is totally bounded.

As we shall see, the definition of uniformity given here is very natural, by the reasons explained bellow.

Let $\{\mathcal{U}_\alpha\}$ be a uniform structure underlying $Str(\tau)$ (not necessarily totally bounded). In particular for every α there exists β such that $\mathcal{U}_\beta \circ \mathcal{U}_\beta \subseteq \mathcal{U}_\alpha$, and for every $A \in Str(\tau)$, the collections $\mathcal{U}_\alpha[A] = \{B \mid (A, B) \in \mathcal{U}_\alpha\}$ are open sets in the elementary topology.

Definitions.

1. Let $(D, \leq)$ be a *directed set*; a *net* in $Str(\tau)$ is any family of structures $(A_i)_{i\in D}$.
2. $(A_i)_{i\in D}$ is a *Cauchy net* if for every α there exists $k \in D$ such that for every $i, j \geq k$, $(A_i, A_j) \in \mathcal{U}_\alpha$.
3. $\lim_i A_i = A$ if for every α there exists $k \in D$ such that for every $i \geq k$, $(A, A_i) \in \mathcal{U}_\alpha$.
4. Let U be an ultrafilter over D; then $lim_U A_i = A$ if for every α there exists $X \in U$ such that for every $i \in X$, $(A, A_i) \in \mathcal{U}_\alpha$; or equivalently, if for every α, $\{i \in D \mid (A, A_i) \in \mathcal{U}_\alpha\} \in U$.
5. An ultrafilter U over D is called *free* if it contains all the subsets $Y_k = \{i \in D \mid i \geq k\}$, $k \in D$.

Observe that the notion of free ultrafilter over a directed set generalizes the notion of non-principal ultrafilter over ω. Note also that $\{Y_k\}_{k\in D}$ enjoys the finite intersection property.

Theorem. Let $(A_i)_{i\in D}$ be a Cauchy net and U a free ultrafilter over D; if $A = \Pi_U A_i$ represents the ultraproduct of A_i modulo U, then $A = \lim_i A_i$.

Proof. We show first that $A = \lim_U A_i$. Suppose by contradiction that there exists α such that $\{i \in D \mid (A, A_i) \notin \mathcal{U}_\alpha\} = \{i \in D \mid A_i \notin \mathcal{U}_\alpha[A]\} \in U$.

Since $\mathcal{U}_\alpha[A]$ is open in the elementary topology and $A \in \mathcal{U}_\alpha[A]$ then there exists

ψ_A such that $A \in \text{Mod }(\psi_A) \subseteq \mathcal{U}_\alpha[A]$, i.e. $A \models \psi_A$ and $\mathcal{U}_\alpha[A] \supseteq \text{Mod }(\psi_A)$, hence $\{i \in D \,/ A_i \notin \mathcal{U}_\alpha[A]\} \subseteq \{i \in D \,/ A_i \notin \text{Mod }(\psi_A)\}$, i.e. $\{i \in D \,/\, A_i \models \neg\psi_A\} \in U$; therefore, by Los' Theorem $A \models \neg\psi_A$, a contradiction. It is interesting to note that this part of the proof makes no appeal to the fact that uniformity is totally bounded; the hypothesis of total boundedness is also unnecessary in the proof given in [M – S].

We prove now the main claim, namely, that $A = \lim_i A_i$.

We have already proved that given α there exists $X_\alpha \in U$ such that for every $i \in X_\alpha$, $(A, A_i) \in \mathcal{U}_\alpha$. Moreover, the fact that $(A_i)_{i \in D}$ is a Cauchy net implies that there exists $k_\alpha \in D$ such that for every $i, j \geq k_\alpha$, $(A_i, A_j) \in \mathcal{U}_\alpha$.

For a given α consider β such that $\mathcal{U}_\beta \circ \mathcal{U}_\beta \subseteq \mathcal{U}_\alpha$, then, there are X_β and k_β as above. Since U is free, we have that $Z = X_\beta \cap Y_{k_\beta} \in U$. Let k be any element of Z.

Claim. For every $i \geq k$, $(A, A_i) \in \mathcal{U}_\alpha$.

Indeed, if $i \geq k$, then as $k \in X_\beta$ we have that $(A, A_k) \in \mathcal{U}_\beta$. But as $i, k \geq k_\beta$ we also have that $(A_k, A_i) \in \mathcal{U}_\beta$, hence $(A, A_i) \in \mathcal{U}_\beta \circ \mathcal{U}_\beta \subseteq \mathcal{U}_\alpha$. This proves that $A = \lim_i A_i$.

QED

The previous proof shows the Cauchy completeness of the uniform space $Str(\tau)$ independently whether or not that space being totally bounded. Hence Los Theorem can be interpreted as a proof of completeness. Compactness is then a trivial topological consequence in the case the space is totally bounded.

We note finally that $Str(\tau)$ possesses a natural uniform structure which is totally bounded: for every sentence $\varphi \in L^\tau_{\omega\omega}$ we define

$$\mathcal{U}_\varphi = \{(A, B) \,/\, A \models \varphi \Leftrightarrow B \models \varphi\}.$$

It is easy to see that the collection $\{\mathcal{U}_\varphi\}$ is a subbasis for the intended uniformity, having the additional property that for every structure A,

$$\mathcal{U}_\varphi[A] = \begin{cases} \text{Mod }(\varphi), & \text{if } A \models \varphi \\ \text{Mod }(\neg\varphi), & \text{if } A \not\models \varphi; \end{cases}$$

this proprerty guarantees that the elementary topology is generated by the uniformity and that uniformity is totally bounded.

REFERENCE

[M - S] Mundici, D. and Sette A. M. **Cauchy Completeness of Elementary Logic.** To appear.

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