UNIFORM APPROXIMATION THE: NON-LOCALLY CONVEX CASE

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Uniform approximation: the non-locally convex case

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Abstract : If S is a compact Hausdorff space of finite covering dimension and (E, τ) is a real or complex topological vector space (not necessarily locally convex), we prove a Weierstrass-Stone theorem for subsets of $C(S; E)$, the space of all continuous functions from S into E, equipped with the topology of uniform convergence over S.

§1. Definitions and Lemmas

Throughout this paper S is a non–empty compact Hausdorff spase of finite covering dimension and (E,τ) is a non-trivial real or complex topological vector space. $C(S;E)$ is the linear space of all continuous functions from S into E, equipped with the topology of uniform convergence determined by the fundamental system of neighborhoods of the origin given by the family of all subsets of $C(S;E)$ of the form

$$\{f \in C(S;E);\ \ f(s) \in V, s \in S\}$$

when V ranges over a fundamental system of neighborhoods of the origin in the space E.

The purpose of this paper is to prove a Weierstrass-Stone theorem for subsets, and in particular linear subspaces, of $C(S;E)$. As a corollary we get A.H. Shuchat's result that $C(S;\mathbb{R}) \otimes E$ is uniformly dense in $C(S;E)$. (See Corollary 2 below and Shuchat [6]). The main idea of the proof is analogous to that of Theorem 1 of Prolla [5], in which E is a normed space. The loss of local convexity for E is compensated by the finite covering dimension of S. Notice that going from the normed case to the locally convex case is straightforward and presents no new difficulty: only the non-locally convex case presents new difficulties. In fact, many results proved in [5] in the normed case, and which could be easily extended to the locally convex case, are without analogues in the present paper. They remain open problems in the non-locally convex case.

A particular and interesting case is the one in which E is a Ψ-monotone quasi-normed space,i.e. the topology considered on E comes from a Ψ-monotone quasi-norm $\|v\|$ defined on it. We recall that a Ψ-monotone quasi-norm on E is a real-valued function $v \to \|v\|$ such that

(1) $\|v\| \geq 0$,
(2) $\|u+v\| \leq \|u\| + \|v\|$,
(3) $\|-u\| = \|u\|$,
(4) $\|\lambda\, v\| \leq \Psi(\ |\lambda|\ ) \cdot \|v\|$,

for all $u, v \in E$ and $\lambda \in \mathbb{K}$, where Ψ is a non-decreasing function from $[0,\infty]$ into itself such that $\Psi(\alpha) \to 0$ as $\alpha \to 0$, and

$$\Psi(\,|\lambda\mu|\,) \leq \Psi(\,|\lambda|\,) \cdot \Psi(\,|\mu|\,)$$

for all $\lambda, \mu \in \mathbb{K}$. Clearly, every normed space is quasi-normed, with $\Psi(\alpha) = \alpha$, for all $\alpha \geq 0$. Another example is $L^p[0,1]$ with $0 < p < 1$ and

$$\|f\| = \int_0^1 |f(x)|^p dx.$$

Here $\Psi(\alpha) = \alpha^p$, for all $\alpha \geq 0$.

A more interesting example is given by the block spaces introduced by Taibleson and Weiss [7], in which

$$\Psi(\alpha) = \alpha\Big(1 + \log^+\Big(\frac{1}{\alpha}\Big)\Big)$$

for all $\alpha \geq 0$. The metric d making E a metric linear space is

$$d(u,v) = \|u - v\|$$

for all $u, v \in E$. Our Theorem 1 below is then true for functions with values in these spaces and say $S \subset \mathbb{R}^n$, a compact subset. But whether the formula

$$\operatorname{dist}(f; W) = \sup_{x \in S} \operatorname{dist}(f(x); W(x))$$

(which is true in the case of normed spaces) remains true for quasi-normed spaces is an open problem.
When $E = \mathbb{R}$, we denote by $C(S; [0, 1])$ the subset of $C(S; \mathbb{R})$ consisting of those functions from S into the unit interval $[0, 1] \subset \mathbb{R}$. Following Jewett [6] we state the following definition.

Definition 1. A non-empty subset $M \subset C(S; [0, 1])$ is said to have property V, if
(1) $\varphi \in M$ implies $1 - \varphi$ belongs to M;
(2) $\varphi \in M$ and $\psi \in M$ implies $\varphi \psi \in M$.

Following Feyel and De La Pradelle [3] and Chao-Lin [2], we state our next definition.

Definition 2. Let $W \subset C(S; E)$ be a non-empty subset. A function $\varphi \in C(S; [0, 1])$ is called a **multiplier** of W if $\varphi f + (1 - \varphi) g$ belongs to W, for every pair, f and g, of elements of W.

Let M be the set of all multipliers of W. It is easy to see that the set M has property V.

Definition 3. A subset $X \subset C(S; \mathbb{R})$ is a said **to separate the points of** S if, given any two distinct points, s and t, of S, there is a function $\varphi \in X$ such that $\varphi(s) \neq \varphi(t)$.

Our first two lemmas are taken from Jewett [4]. (See also Burckel [1].)

Lemma 1. *Let $0 < a < b < 1$ and $0 < \delta < 1/2$ be given. There exists a polynomial $p(x) = (1 - x^m)^n$, such that*

(1) $$p(t) > 1 - \delta, \quad \textit{for all} \quad 0 \leq t \leq a,$$
(2) $$p(t) < \delta, \quad \textit{for all} \quad b \leq t \leq 1.$$

Proof. See Lemma 2, Jewett [4]. □

Lemma 2. *If $M \subset C(S; [0, 1])$ has property V, and φ and ψ belong to M, then* $\max(\varphi, \psi)$ *belongs to the uniform closure of M.*

Proof. See Theorem 1, Jewett [4]. Just notice that the uniform closure of M in $C(S; [0, 1])$ has property V too. □

Lemma 3. *Let $M \subset C(S; [0, 1])$ be a non-empty separating subset with property V. Let $x \in S$ and let N be an open neighborhood of x in S. There exists $\varphi \in M$ such that*

(1) $$\varphi(x) > 3/4,$$

(2) $$\varphi(t) < 1/4, \qquad \textit{for all} \qquad t \notin N.$$

Proof. Let K be the complement of N. For each $t \in K$, there is $\varphi_t \in M$ such that $\varphi_t(t) < \varphi_t(x)$. Choose real numbers a and b such that $\varphi_t(t) < a < b < \varphi_t(x)$. By Lemma 1, there is a polynomial $p_t(u) = (1-u^m)^n$ such that $p_t(u) < 1/4$ for $b \leq u \leq 1$, and $p_t(u) > 3/4$ for $0 \leq u \leq a$. Hence $p_t(\varphi_t(x)) < 1/4$ and $p_t(\varphi_t(t)) > 3/4$. Let $\mathcal{U}(t) = \{s \in S; p_t(\varphi_t(s)) > 3/4\}$. Then $\mathcal{U}(t)$ is an open neighborhood of t. By compactness, there are $t_1, \ldots, t_m \in K$ such that $K \subset \mathcal{U}(t_1) \cup \mathcal{U}(t_2) \cup \cdots \cup \mathcal{U}(t_m)$. For each $i = 1, \ldots, m$ let $\varphi_i(s) = p_{t_i}(\varphi_{t_i}(s))$, $s \in S$. Clearly, $\varphi_i \in M$, for all $i = 1, \ldots, m$. Let $\psi(s) = \max(\varphi_1(s), \ldots, \varphi_m(s))$, $s \in S$.

By Lemma 2 the function ψ belongs to the uniform closure of M. Notice that $\psi(x) < 1/4$ and $\psi(t) > 3/4$, for all $t \in K$. Choose $\varepsilon > 0$ so that $\psi(x) + \varepsilon < 1/4$ and $\psi(t) - \varepsilon > 3/4$ for all $t \in K$. Let $\eta \in M$ be such that $\|\psi - \eta\| < \varepsilon$ and let $\varphi = 1 - \eta$. The $\varphi \in M$ and $\varphi(x) > 3/4$, while $\varphi(t) < 1/4$ for all $t \in K$. □

§2. The Weierstrass–Stone Theorem

Theorem 1. *Let W be a non-empty subset of $C(S;E)$ such that the set M of all multipliers of W separates the points of S. Let $f \in C(S;E)$ be given. The following are equivalent:*

(1) f belongs to the uniform closure of W;

(2) for each $x \in S$, the vector $f(x)$ belongs to the uniform closure of $W(x) = \{g(x); g \in W\}$ in E.

Proof. Clearly (1) ⇒ (2). Conversely, assume (2) is true. Let n be the covering dimension of S and let U be on open neighborhood of the origin in E. Choose an open and balanced neighborhood V of 0 in E such that the $(n+3)$-fold sum $V + \cdots + V$ is contained in U. By (2), for each $x \in S$, there is some $w_x \in W$ such that $f(x) - w_x(x) \in V$. Consider the open covering $\mathcal{U}$ of S given by

$$U_x = \{t \in S; f(t) - w_x(t) \in V\}, x \in S.$$

By our hypothesis, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is of order at most $n+1$. Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in \Lambda}$. For each $\alpha \in \Lambda$, let $S_\alpha = \{x \in S; V_\alpha \subset U_x\}$.

Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, the set S_α is non-empty. By the Axiom of choice, there is a mapping $s : \Lambda \Rightarrow S$ such that $s(\alpha) \in S_\alpha$, for all $\alpha \in \Lambda$. Hence $V_\alpha \subset U_{s(\alpha)}$.

For each $x \in S$, let $\Lambda_x = \{\alpha \in \Lambda; x \in V_\alpha\}$. Since $\mathcal{V}$ is a covering of S, the set Λ_x is non-empty. By the Axiom of choice, there is a mapping $\lambda : S \Rightarrow \Lambda$ such that $\lambda(x) \in \Lambda_x$, for all $x \in S$. Hence

(1) $$x \in V_{\lambda(x)} \subset U_{s(\lambda(x))}, \quad \text{for all } x \in S.$$

By lemma 3, applied to $N = V_{\lambda(x)}$, there is $\varphi_x \in M$ such that $0 \leq \varphi_x \leq 1$, $\varphi_x(x) > \frac{3}{4}$ and $\varphi_x(t) < \frac{1}{4}$ for all $t \notin V_{\lambda(x)}$. Define $W_x = \{t \in S; \varphi_x(t) > \frac{2}{3}\}$. Then W_x is open and contains x. Moreover, $w_x \subset V_{\lambda(x)}$. By compactness of S, there are $x_1, \ldots, x_m \in S$ such that S is contained in the union $W_{x_1} \cup \ldots \cup W_{x_m}$. Notice that

(2) $$W_{x_i} \subset V_{\lambda(x_i)} \subset U_{s(\lambda(x_i))}, i = 1, \ldots, m.$$

To simplify notation, let us define

$$\begin{aligned}
\varphi_i &= \varphi_x \quad \text{, where } \mathrm{x} = \mathrm{x_i} \\
W_i &= W_x \quad \text{, where } \mathrm{x} = \mathrm{x_i} \\
V_i &= V_\alpha \quad \text{, where } \alpha = \lambda(\mathrm{x_i}) \\
U_i &= U_x \quad \text{, where } \mathrm{x} = \mathrm{s}(\lambda(\mathrm{x_i})) \\
w_i &= w_x \quad \text{, where } \mathrm{x} = \mathrm{s}(\lambda(\mathrm{x_i})).
\end{aligned}$$

Choose an open neighborhood V' of 0 in E such that the m-fold sum $V' + \cdots + V'$ is contained in V. Let

$$B = \bigcup_{i=1}^{m} \Big[(f - w_i)(S) \Big].$$

The set B is compact. Hence it is bounded in E. Choose $0 < \delta < 1/2$ so that $|\lambda| < \delta$ and $v \in B$ implies $\lambda v \in V'$.

By Lemma 1, there is a polynomial $q : I\!R \to I\!R$ of the form $q(t) = 1 - (1 - t^s)^k, t \in I\!R$, such that $0 \leq q(t) \leq 1$, for all $t \in [0, 1]$,

(a) $0 \leq q(t) < \delta$, for all $0 \leq t \leq 1/3$,

(b) $1 \geq q(t) > 1 - \delta$, for all $2/3 \leq t \leq 1$.

Let $\psi_i = q \circ \varphi_i, i = 1, 2, \ldots, m$. Then $0 \leq \psi_i \leq 1$ and $\psi_i \in M$. Moreover

(3) $0 \leq \psi_i(t) < \delta$, if $t \notin V_i$,

(4) $1 \geq \psi_i(t) > 1 - \delta$, if $t \in W_i$.

Define

$$\begin{aligned}
&\varphi_1 = \psi_1 \\
&\varphi_2 = (1 - \psi_1)\, \psi_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\varphi_m = (1 - \psi_1)(1 - \psi_2) \cdots (1 - \psi_{m-1})\, \psi_m.
\end{aligned}$$

Then

$$\varphi_1 + \varphi_2 + \cdots + \varphi_m = 1 - (1 - \psi_1)(1 - \psi_2) \cdots (1 - \psi_m). \tag{5}$$

Let $\varphi_{m+1} = (1 - \psi_1)(1 - \psi_2) \cdots (1 - \psi_m)$. Then $\varphi_{m+1} \in M$.

Given $x \in S$, there is some index i such that $x \in W_i$. By (4), $\psi_i(x) > 1 - \delta$, and therefore $\delta > 1 - \psi_i(x)$. Hence

$$\varphi_{m+1}(x) = (1 - \psi_i(x)) \cdot \prod_{j \neq i} (1 - \psi_j(x)) < \delta. \tag{6}$$

On the other hand, (3) and $\varphi_i(t) \leq \psi_i(t)$ imply

$$0 \leq \varphi_i(t) < \delta, \quad \text{if } t \notin V_i \tag{7}$$

for each $i = 1, 2, \ldots, m$.

Let $w = \sum_{i=1}^{m+1} \varphi_i w_i$ where $w_{m+1} = w_1$. Then $w \in W$, because $w = \psi_1 w_1 + (1-\psi_1)[\psi_2 w_2 + (1-\psi_2)[\psi_3 w_3 + \cdots + (1-\psi_{m-1})[\psi_m w_m + (1-\psi_m)w_1]\cdots]]$.

Let $x \in S$ be given. Define two sets of indices $I(x)$ and $J(x)$ by

$$I(x) = \{1 \le i \le m;\ x \in V_i\},$$
$$J(x) = \{1 \le i \le m;\ x \notin V_i\}.$$

Notice that the cardinality of $I(x)$ is at most $n+1$. Now, for each $i \in I(x)$ we have

$$\varphi_i(x)(f(x) - w_i(x)) \in V$$

because $V_i \subset U_i = U_{s(\lambda(x_i))}, w_i = w_{s(\lambda(x_i))}, 0 \le \varphi_i(x) \le 1$, and V is balanced. Hence

$$\sum_{i \in I(x)} \varphi_i(x)(f(x) - w_i(x)) \in V + \cdots + V, ((n+1)\text{-fold sum}).$$

On the other, for each $i \in J(x)$, we have $x \notin V_i$ and by (7), $\varphi_i(x) < \delta$. The cardinality of $J(x)$ is at most m. Hence

$$\sum_{i \in J(x)} \varphi_i(x)(f(x) - w_i(x)) \in V' + \cdots + V', (m\text{-fold sum}).$$

Consequently,

$$\sum_{i=1}^{m} \varphi_i(x)(f(x) - w_i(x)) \in V + \cdots + V, ((n+2)\text{-fold sum}). \tag{8}$$

Finally, $\varphi_{m+1}(x)(f(x) - w_{m+1}(x)) \in V'$, because by (6) $\varphi_{m+1}(x) < \delta$, and $f(x) - w_{m+1}(x) = f(x) - w_1(x)$ belongs to B. Hence $V' \subset V$ implies

$$\sum_{i=1}^{m+1} \varphi_i(x)(f(x) - w_i(x)) \in V + \cdots + V, ((n+3)\text{-fold sum}). \tag{9}$$

It remains to notice that (5) implies that $\varphi_1(x) + \varphi_2(x) + \cdots + \varphi_m(x) + \varphi_{m+1}(x) = 1$ and

$$f(x) - w(x) = \sum_{i=1}^{m+1} \varphi_i(x)(f(x) - w_i(x)) \in V + \cdots + V, ((n+3)\text{-fold sum}).$$

Since the $(n+3)$-fold sum $V + \cdots + V$ is contained in U and U was arbitrary, this ends the proof that W is uniformly dense in $C(S; E)$. □

Corollary 1. *Let W be a non-empty subset of $C(S; E)$ such that*

(1) *for each pair of distinct points, x and y, of S there is some multiplier φ of W such that $\varphi(x) \ne \varphi(y)$;*

(2) *for each $x \in S, W(x)$ is dense in E.*

Then W is dense in $C(S; E)$.

Proof. By (1), the set M of all multipliers of W is separating over S. By (2), the set $\{g(x);\ g \in W\}$ is dense in E. Hence, every $f \in C(S;E)$ verifies (2) of Theorem 1, and therefore belongs to the uniform closure of W in $C(S;E)$. □

Our next result is a Weierstrass–Stone theorem for **linear subspaces** of $C(S;E)$.

Theorem 3. *Let $W \subset C(S;E)$ be a vector subspace such that*

$$A = \{\varphi \in C(S;\mathbb{R}); \varphi g \in W, \quad \textit{for all} \quad g \in W\}$$

separates the point of S, and for each $x \in S, W(x)$ is dense in E.
Then W is uniformly dense in $C(S;E)$.

Proof. Notice that the set A is a subalgebra of $C(S;\mathbb{R})$ containing the constants. The set $M = \{\varphi \in A;\ 0 \leq \varphi \leq 1\}$ is the set of all multipliers of W. Given $x \neq y$, by hypothesis there is some $\varphi \in A$ such that $\varphi(x) \neq \varphi(y)$. Since A is an algebra containing the constants, a standard argument shows that we may assume that $\varphi(x) = 0$ and $\varphi(y) = 1$. Let $\psi = \varphi^2/||\varphi^2||$. Then $\psi \in M$, and $\psi(x) = 0, \psi(y) = 1$. Hence M separates the points of S, and condition (1) of Corollary 1 is verified. By hypothesis, condition (2) of Corollary 1 is verified also. □

Corollary 2. *Let $A \subset C(S;\mathbb{K})$ be a dense linear subspace, which is assumed to be self-adjoint in the complex case and let $W = A \otimes E \subset C(S;E)$. Then W is uniformly dense in $C(S;E)$. In particular, $C(S;\mathbb{K}) \otimes E$ is uniformly dense in $C(S;E)$.*

Proof. It is clear that the real part of A is contained in

$$\{\varphi \in C(S;\mathbb{R}) \ ;\ \varphi g \in W, \text{for all } g \in W\},$$

and therefore the result follows from Theorem 3. □

Corollary 3. *Let A be a dense linear subspace of $C(S;\mathbb{K})$. Then $A \otimes E$ is uniformly dense in $C(S;E)$.*

Proof. Let $f \in C(S;E)$ and let V be an open neighborhood of 0 in E. Choose an open and balanced neighborhood U of 0 in E such that $U + U \subset V$. By Corollary 2, there is some $g \in C(S;\mathbb{K}) \otimes E$ such that $f(s) - g(s) \in U$, for all $s \in S$. Let

$$g = \sum_{i=1}^{m} h_i v_i$$

where $h_i \in C(S;\mathbb{K})$, and $v_i \in E, i = 1, \ldots, m$. Choose another neighborhood W of 0 in E such that the m-fold sum $W + \cdots + W$ is contained in U. Choose $\delta > 0$ so that $\lambda v \in W$, for all $|\lambda| \leq \delta$ and $v \in \{v_1, v_2, \cdots, v_m\}$. Let $a_1, \cdots, a_m \in A$ be such that $\|a_i - h_i\| < \delta$. Then $w = a_1 v_1 + \cdots + a_m v_m$ belongs to $A \otimes E$ and $g(s) - w(s) \in U$ for all $s \in S$. Hence $f(s) - w(s) \in V$, for all $s \in S$. □

§3. Simultaneous approximation and interpolation

We can apply our Theorem 1 to get results on simultaneous approximation and interpolation of **vector-valued** functions. Let us say that a subset $A \subset C(S;E)$ is an **interpolating family for** $C(S;E)$ if, given any finite subset $F \subset S$ and any $f \in C(S;E)$, there exists $g \in A$ such that $f(x) = g(x)$ for all $x \in F$.

Theorem 4. *Let $A \subset C(S;E)$ be an interpolating family such that the set of multipliers of A separates the points of S. Then, for every $f \in C(S;E)$, every open neighborhood V of the origin in E and every finite subset $F \subset S$, there exists $g \in A$ such that $f(s) - g(s) \in V$ for all $s \in S$, and $f(x) = g(x)$ for all $x \in F$. In particular, A is uniformly dense in $C(S;E)$.*

Proof. Define $W = \{g \in A;\ f(x) = g(x) \text{ for all } x \in F\}$. Since A is an interpolating family, $W \neq \emptyset$. Now it is easy to verify that each multiplier of A is also a multiplier of W. Let $x \in S$ be given. Consider the finite set $F \cup \{x\}$. Since A is an interpolating family for $C(S;E)$, there exists $g_x \in A$ such that $f(t) = g_x(t)$ for all $t \in F \cup \{x\}$. In particular, $f(t) = g_x(t)$ for all $t \in F$. Hence $g_x \in W$. On the other hand $f(x) = g_x(x)$ shows that $f(x) \in W(x)$. By Theorem 1, there exists $g \in W$ such that $f(s) - g(s) \in V$, for all $s \in S$, and $g \in W$ implies $g \in A$ and $g(t) = f(t)$ for all $t \in F$. □

Theorem 5. *Let A be a dense linear subspace of $C(S;\mathbb{K})$. Let $A \otimes E = W$. Then, for every $f \in C(S;E)$, every open neighborhood V of 0 in E and every finite subset $F \subset S$, there exists $g \in W$ such that $f(s) - g(s) \in V$ for all $s \in S$, and $f(x) = g(x)$ for all $x \in F$.*

Proof. Case 1: $F = \emptyset$.

By Corollary 3, $W = A \otimes E$ is dense in $C(S;E)$ and therefore $f(s) - g(s) \in V$ for all $s \in S$, for some $g \in W$.

Case 2: $F = \{x_1, \ldots, x_n\} \neq \emptyset$.

We first remark that A is an interpolating family for $C(S;\mathbb{K})$. Indeed, if we define $T : C(S;\mathbb{K}) \to \mathbb{K}^n$ by

$$Tg = (g(x_1), \ldots, g(x_n))$$

for each $g \in C(S;\mathbb{K})$, then by density of A and continuity of T, we have

$$T(C(S;\mathbb{K})) = T(\overline{A}) \subset \overline{T(A)} = T(A),$$

where the last equality is a consequence of the fact that $T(A)$ is a linear subspace of $\mathbb{K}^n$, because A is a linear subspace of $C(S;\mathbb{K})$.

Let $a_1, \ldots, a_n \in A$ be such that

$$a_i(x_j) = \delta_{ij}, \quad 1 \leq i,j \leq n.$$

Choose N an open and balanced neighborhood of 0 in E such that the $(n+1)$-fold sum $N + \cdots + N$ is contained in V. Let $\lambda = \max\{\|a_i\|; 1 \leq i \leq n\}$. Let U be the open neighborhood $\lambda^{-1}N$. Notice that $\lambda \geq 1$ implies $U = \lambda^{-1}N \subset N$, because N is balanced. By

Case 1, W is dense in $C(S;E)$. Hence there is some $g_1 \in W$ such that $f(s) - g_1(s) \in U$, for all $s \in S$. Let

$$v_i = f(x_i) - g_1(x_i)\ ,\ \ 1 \leq i \leq n.$$

Since $A \otimes E = W$, it follows that

$$g_2(x) = \sum_{i=1}^{n} a_i(x) v_i\ \ ,\ \ x \in S\ ,$$

belongs to W. Notice that $g_2(x_j) = v_j$ for all $1 \leq j \leq n$. Hence $g(x_j) = f(x_j)$, for all $1 \leq j \leq n$, if $g \in W$ is defined to be $g_1 + g_2$. On the other hand,

$$g_2(s) = \textstyle\sum_{i=1}^{n} a_i(s) v_i = \sum_{i=1}^{n} [a_i(s)\lambda^{-1}]\lambda v_i \in N + \cdots + N, (n\text{-fold sum}),$$

because $|a_i(s)\lambda^{-1}| \leq 1$, and $\lambda v_i \in \lambda\, U = N$, for all $i = 1, 2, \ldots, n$, and N is balanced. Hence

$$f(s) - g(s) = f(s) - g_1(s) - g_2(s) \in N + N + \cdots + N, ((n+1)\text{-fold sum}).$$

Since the $(n+1)$-fold sum $N + N + \cdots + N$ is contained in V, we have $f(s) - g(s) \in V$, for all $s \in S$. □

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