ON THE SOLUTION OF HOMOGENEOUS GENERALIZED WAVE EQUATION

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RELATÓRIO TÉCNICO Nº 01/92

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Janeiro - 1992

On The Solution of Homogeneous Generalized Wave Equation

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I. Introduction

In a recent paper Hillion[1] studies the Goursat problem (boundary value problem with data on the characteristics) for the tridimensional wave equation where he said that the modal waves seem to be the most natural solutions. Bélanger[2] obtained modal waves for the free-space homogeneous wave equation, writing the solutions in terms of Laguerre functions which propagates in a straight line with the velocity of light and remaining focused for all time. Brittingham[3] showed that packetlike beams are solutions to the homogeneous Maxwell equations.

Recently[4] we have discussed and solve the generalized Laplace equation extended to the De Sitter universe by means of Jacobi polynomials, which depends of one parameter N, known as, the degree of homogeneity of the function. Yet, in another recent paper[5] we have solved the D'Alembert wave equation extend to the De Sitter universe by means of the same polynomials depending also on the parameter N. In both cases we have the classical result when the radius of the De Sitter universe goes to infinite.

In this paper, we discuss the homogeneous D'Alembert generalized wave equation when we have a small distance (local problem) and we obtain certain solutions for the homogeneous D'Alembert generalized wave equation by means of Hillion procedure. When the radius of the De Sitter universe goes to infinite we obtain exactly Hillion's solutions.

II. Generalized D'Alembert Wave Equation

The generalized D'Alembert wave equation is given by[6]

$$\Box\psi(\overline{x}_A) = \overline{\partial}_A \overline{\partial}_A \psi(\overline{x}_A) = 0 \tag{1}$$

where $A = 0, 1, \dots 4$. Writing this equation in cartesiane coordinates we have the following differential equation

$$\begin{aligned} & R^2 \Box\, \psi(x_i) = \\ = \quad & A^2(R^2\partial_i^2 + x_i x_j \partial_i \partial_j + 2x_i\partial_i)\psi(x_i) + N(N+K-1)\psi(x_i) = 0 \end{aligned} \tag{2}$$

where $i, j = 0, 1, \dots 3$; $A^2 = 1+\alpha^2-\gamma^2 = 1+\alpha_j\alpha_j$ with $\alpha_j = x_j/R \;\; \gamma = ct/R, R$ is the radius of the De Sitter universe; N is a parameter, known as, the degree of homogeneity of the function and $1 \leq K \leq 4$ because the function $\psi(x_i)$ has a K-number of independent variables.

Introducing the function $\varphi = A^N\psi$ and considering $K = 4$ we obtain the following differential equation

$$R^2 \Box\, \varphi(x_i) = A^{2-N}\{R^2\partial_i^2 + x_i x_j \partial_i \partial_j + (N+3)(N - 2x_i\partial_i)\}\varphi(x_i) = 0 \tag{3}$$

where $i, j = 0, 1 \dots 3$.

Now, we discuss the above differential equation when we have a local problem $(\frac{x_i}{R} << 1)$ then we obtain the following D'Alembert wave equation

$$\{\partial_i^2 + \frac{1}{R^2}N(N+3)\}\varphi(x_i) = 0 \tag{4}$$

That writing in the explicit form is given by

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{1}{R^2}N(N+3)\right\}\varphi_N(x, y, z, t) = 0 \tag{5}$$

we note that when we take the limit $R \to \infty$ we obtain exactly the classical tridimensional D'Alembert wave equation studies by Hillion[1].

III. Solution of the D'Alembert Wave Equation

Introducing the characteristics

$$\xi = z - ct \quad \text{and} \quad \eta = z + ct$$

in eq.(5) and considering the variables x and y as parameters we have

$$\left\{4\frac{\partial^2}{\partial\xi\partial\eta} + \nabla_2 + \frac{1}{R^2}N(N+3)\right\}\varphi_N(\xi,\eta,x,y) = 0 \tag{6}$$

where $\nabla_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the bidimensional Laplacian. To solve eq.(6) we introduce polar coordinates, orthogonal to η and ξ, by means of $x = \rho\cos\theta$ and $y = \rho \sin\theta$, and we have the following differential equation

$$\left\{4\frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{R^2}N(N+3)\right\}\varphi_N(\xi,\eta,\rho) = 0 \tag{7}$$

Now, we consider only the completly separable problem defined by

$$\varphi_N(\xi,\eta,\rho) = H_N(\xi,\eta)I(\rho) \tag{8}$$

and we have the following differential equations

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + k^2\right)I(\rho) = 0 \tag{9a}$$

$$\left(4\frac{\partial^2}{\partial\xi\partial\eta} - \lambda^2\right)H_N(\xi,\eta) = 0 \tag{9b}$$

where K^2 is a constant and we have defined $\lambda^2 = K^2 - N(N+3)/R^2$.

Considering a Fourier expansion we can write the solution of the eq.(9a) by means of

$$I(p) = \sum_{m=1}^{\infty} A_m J_0(kj_m\rho) \tag{10}$$

where $j_1, j_2 \ldots$ are the positive zeros of the Bessel function, $J_0(\mu)$, given in a crescent order of amplitudes.

To solve the eq.(9b) we use a Green's function technique associated with the Riemann method[7]. Then the Green's function is given by the following expression

$$G(\xi,\xi',\eta,\eta') = I_0\left(\lambda\sqrt{\xi-\xi'}\sqrt{\eta-\eta'}\right) \tag{11}$$

where $I_0(\mu)$ is a modified Bessel function. Considering $f_1(\xi)$ and $f_2(\eta)$ as the data of the Gousart problem and using the Riemann method we obtain the solution given as an integral representation,

$$H_N(\xi,\eta) = \int_0^\xi I_0\left\{\lambda\sqrt{\eta(\xi-s)}\right\}\frac{\partial f_1(s)}{\partial s}ds + \int_0^\eta I_0\left\{\lambda\sqrt{\xi(\eta-s)}\right\}\frac{\partial f_2(s)}{\partial s}ds \tag{12}$$

Using the eq.(8) we have the solution of the eq.(7) given by

$$\varphi_N(\xi,\eta,\rho) = \sum_{m=1}^{\infty} A_m J_0(Kj_m\rho)H_N(\xi,\eta)$$

which in the original variables is given by the following expression

$$\varphi_N(x,y,z,t) = \sum_{m=1}^{\infty} A_m J_0\left(Kj_m\sqrt{x^2+y^2}\right).$$

$$\cdot\left\{\int_0^{z-ct} I_0\left\{\lambda\sqrt{(z+ct)(z-ct-s)}\right\}\frac{\partial f_1(s)}{\partial s}ds + \right. \tag{13}$$

$$\left. + \int_0^{z+ct} I_0\left\{\lambda\sqrt{(z-ct)(z+ct-s)}\right\}\frac{\partial f_2(s)}{\partial s}ds\right\}$$

where $\lambda^2 = K^2 - N(N+3)/R^2$.

IV. Particular Case (Hillion Results)

When we have $\partial f_1(s)/\partial s = \partial f_2(s)/\partial s = \delta(s)$ where $\delta(s)$ is the Dirac distribuition we get an exactly solution of the eq.(13) as follow

$$\varphi_N(x,y,z,t) = \sum_{\mu,\nu} A_{\mu\nu}e^{(\mu+\nu)z-(\mu-\nu)ct}J_0\left(K\sqrt{x^2+y^2}\right)$$

where $4\mu\nu = K^2 - N(N+3)/R^2$. Now, taking the limit $R \to \infty$ we have exactly the Hillion Results.

Aknowledgements

We are grateful to professor W. A. Rodrigues Jr. for many and useful discussions and L. Martins for stimulus. We are also grateful to CNPq for a research grants.

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