AN ITERATIVE METHOD FOR THE NUMERICAL INVERSION OF LAPLACE TRANSFORMS

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RELATÓRIO TÉCNICO Nº 66/91

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Dezembro - 1991

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Abstract: We present an algorithm for the numerical inversion of Laplace transforms that is a particular case of the iterated regularization method proposed in [6]; to construct the finite dimensional space we use the Laguerre polynomials. Error bounds for the approximations are derived.

1 - Introduction

In a general framework, let $X = L^2_w(R^+)$ be the weighted Lebesgue space associated with $w(t) = e^{-t}$, $Y = L^2(c,d)$ and $A : X \to Y$ the Laplace transform operator,

$$(Ax)(s) = \int_0^\infty e^{-st}x(t)dt = y(s) \tag{1}$$

As is known, the problem of solving (1.1), for a given $y \in Y$, is ill-posed. The problem of determining A^+y, where A^+ is the generalized inverse of A, is still ill-posed: the solution depends discontinously upon y.
If we only know the perturbed data y_δ, with

$$||y - y_\delta||_Y < \delta \tag{2}$$

then one must use "regularization methods". This is a family of operators $R_N : Y \to X$, indexed by some regularization parameters N, together with some strategy to choose the parameter such that $R_N y_\delta$ is an approximation to A^+y. There are also other kinds of perturbations when, instead of the operator A, we use an approximation A_N such that $||A_N - A|| \leq \beta_N$.

In this paper, we use the arguments presented by Vainikko, in [6], to design an algorithm for the inversion of the Laplace transforms of data with noise. As it is well known, the Laplace transform methods are helpful techniques for differential and integral equations; however when discretization is required to solve the problem in the Laplace domain, errors are introduced. Similar situations arise when we deal with the Laplace inversion of scientific measurements or observations.

2 - The Laguerre approximations:

Let $V_1, \subseteq V_2 \subseteq \ldots$ be a sequence of finite-dimensional subspaces of X where V_N is spanned by the Laguerre polynomials of degree $\leq N$ [1]. The Laguerre polynomials, $\phi_i(t)$, are such that

$$\int_0^\infty e^{-t}\phi_i(t)\phi_j(t)dt = \delta_{ij}$$

and they form a complet set in $L^2_\omega(R^+)$ [2, for ex]. We will denote by P_N the orthogonal projection of X onto V_N and $A_N = AP_N$.

We can obtain an approximation solution to (1.1), for a given y_δ as in (1.2), using

$$x^k = (\lambda I + A^*A)^{-1}(\lambda x^{k-1} + A^*y_\delta) \tag{3}$$

where A^* is the adjoint operator of A; this is the well-known implicit sucessive approximation method. In the finite dimensional subspace V_N, we define the approximation

$$x_N = \sum_{i=1}^{N} a_i \phi_i(t)$$

such that

$$\langle (\lambda I + A_N^* A_N) x_N^{k+1},\ \phi_j \rangle = \langle \lambda x_N^k + A_N^* y_\delta,\ \phi_j \rangle, j = 1, \ldots, N, \lambda > 0.$$

Now, let $\psi_i(s) \in Y$ be the Laplace transform of $\phi_i(t)$, $\psi_i(s) = \int_0^\infty e^{-st} \phi_i(t) dt$; with these functions we construct a matrix M,

$$M_{ij} = \int_c^d \psi_i(s) \psi_j(s) ds = \frac{c^{i+j+1} - d^{i+j+1}}{i+j+1}$$

and a vector f,

$$f_i = \int_c^d y_\delta(s) \psi_i(s) ds.$$

Therefore, the variational formulation of the implicit scheme (3), in V_N, will be

$$(\lambda \mathbf{I} + \mathbf{M}) \mathbf{a}^k = \lambda \mathbf{a}^{k-1} + \mathbf{f}, \tag{4}$$

and, for a given λ, we state the

Procedure

1. Do the Cholesky decomposition

$$\mathbf{L}\mathbf{L}^T = \mathbf{M} + \lambda \mathbf{I}$$

2. $\mathbf{a}^\circ = 0$

   For $k = 1, 2, \ldots$

   solve the system $\mathbf{L}\mathbf{L}^T \mathbf{a}^k = \lambda \mathbf{a}^{k-1} + \mathbf{f}$

We must observe that, in this process, the regularization is an important feature: the condition number of M become insuppartable as N increase; for example, if $N = 15$ the condition number of M is 10^{19}!
By direct calculations we can show that the adjoint operator A^* is, in this case, $A^* : L^2(c, d) \to L^2_\omega(R^+)$,

$$(A^* v)(t) = e^t \int_c^d e^{-ts} v(s) ds;$$

also, we can see that $z(t) = (A^*v)(t),\ v(s) \in Y$, is an analytical function and for $k = 0, 1, \ldots$

$$z^{(k)}(t) = \int_c^d e^{-t(s-1)}(1-s)^k v(s)ds. \tag{5}$$

3. Error bound estimates: Assume that the data are on the interval (c, d) and let $\tilde{c} = \dfrac{1-c}{c}$ and $\tilde{d} = \dfrac{1-d}{d}$; it is evident that $|\tilde{c}| < 1$ and $|\tilde{d}| < 1$.

Lemma: Let $\alpha = \max\{|\tilde{c}|, |\tilde{d}|\}$; then

$$\beta_N = ||A - A_N|| \le \sqrt{2}\frac{\alpha^{(N+1)}}{(1-\alpha^2)^{1/2}}.$$

Proof: As we known

$$||A - A_N|| = ||A(I - P_N)|| = ||(I - P_N)A^*|| = \sup_{||v||=1}\{||(I - P_N)A^*v||_X\}$$

Let $z(t) = (A^*v)(t),\ v \in Y$, s.t. $||v||_Y = 1$. Then

$$||(I - P_N)A^*v||_X = ||(I - P_N)z(t)||_X = ||z(t) - \sum_{i=1}^{N} b_i\phi_i(t)||_X$$

where b_i are the Laguerre-Fourier coefficients of $z(t)$. The next step is to calculate the rate of convergence of the Laguerre-Fourier approximates; we will use a basic property of the Laguerre polynomials [1]

$$e^{-t}\phi_k(t) = \frac{1}{k!}\frac{d^k}{dt^k}(t^k e^{-t})\ \ k = 0, 1, \ldots$$

By sucessive integration by parts and the last equation we get

$$b_k = \int_0^\infty e^{-t}z(t)\phi_k(t)dt = \frac{(-1)^k}{k!}\int_0^\infty e^{-t}t^k z^{(k)}(t)dt;$$

using (5) and the Laplace transform of t^k,

$$b_k = \frac{(-1)^k}{k!}\int_c^d v(s)(1-s)^k\int_0^\infty e^{-st}t^k dt\ ds = \tag{6}$$

$$= (-1)^k\int_c^d \frac{1}{s}\left(\frac{1-s}{s}\right)^k v(s)ds. \tag{7}$$

By the Schwarz's inequality,

$$|b_k|^2 \leq \{\int_c^d \frac{1}{s^2}(\frac{1-s}{s})^{2k} ds\}||v||_Y^2 = \tag{8}$$

$$= \{\frac{1}{2k+1}(\tilde{c}^{2k+1} - \tilde{d}^{2k+1})\} = [\gamma(k)]^{2k}(\tilde{c} - \tilde{d}), \tag{9}$$

where, by the mean value theorem, $\gamma(k) \in (\tilde{d},\ \tilde{c}))$. But $-1 < \frac{1-x}{x} < 1$, if $x > \frac{1}{2}$, then $-1 < \tilde{d} \leq \gamma(k) \leq \tilde{c} < 1$. If $\alpha = \max\{|\ \tilde{c}\ |, |\ \tilde{d}\ |\}$,

$$\sum_{k=N+1}^{\infty} b_k^2 \leq (\tilde{c} - \tilde{d}) \sum_{i=N+1}^{\infty} \alpha^{2i} = (\tilde{c} - \tilde{d}) \frac{\alpha^{2(N+1)}}{1-\alpha^2}$$

and the lemma will follow by the completness of the Laguerre polynomials. □

The method of sucessive approximations (3) is familiar for ill-posed problems [3, 6]. In particular, the theorem 1 in [6] is concerned with "a priori" specification of the regularization parameter. It claims that if

(i) $y \in R(A)$
(ii) $x^+ \in R([A^*A]^{p/2})$, x^+ the solution of (1) closet to 0
(iii) $k = d_1(\delta + \beta)^{-2/(p+1)}$

then

$$||x_k - x||_X \leq d_2(\delta + \beta)^{p/p+1}, \quad d_2 = \text{const } (p, d_1).$$

Our final conclusion follows directly from this result and the previous lemma.

Proposition: Under the conditions (i) - (iii), the sucessive approximations (4), with

$$k = d_1(\delta + \frac{\alpha^{N+1}}{(1-\alpha^2)^{1/2}})^{-2/p+1}, \quad \alpha = \max\{|\ \tilde{c}\ |, |\ \tilde{d}\ |\}, \quad d_1 = \text{const}$$

will give x_k such that

$$||x_k - x^+||_X \leq d_2(\delta + \frac{\alpha^{N+1}}{(1-\alpha^2)^{1/2}})^{p/p+1}$$

where $d_2 = d_2(p, d_1)$.

4. Numerical Experiments and Conclusions: The examples of this section will show a qualitative idea of the performance of the proposed scheme. We choose λ in such way that the first iterate is an approximation for x^+; this is possible since the first iterate is the Tikkonov regularization solution. In this case there are "a priori" estimates for λ as is showed in [3]. To stop the iterative process we use the residue limitation:

$$\mathbf{Res} = \mathbf{f} - \mathbf{M}\mathbf{a}^k = \lambda(\mathbf{a}^k - \mathbf{a}^{k-1}) \tag{10}$$

$$||\mathbf{Res}||_\infty < TOL. \tag{11}$$

Example 1. If $y(s) = \tan^{-1}(\frac{1}{s})$ then $x(t) = \frac{1}{t}$ sint. In the figure 1 we plot $x(t)$ and $x_N(t)$ computed using TOL$= 10^{-15}$ and $(a) N = 10$, $(b) N = 20$. The simbol Δ is used to show $x_N(t)$ and the full-line for $x(t)$.

Example 2. Now we introduce a noise in the $y(s)$ used in the previous example, adding $p(s) = 10^{-6} \sin(10^{12} s), s \in (1, 5)$; in this case we used TOL$= 10^{-8}$. The x_N computed are plotted in the figures $2 : (a)$ with $N = 10$ and (b) with $N = 20$.

The error bound presented here, as well as the above numerical computations, motivate the use of the successive approximation method in the Laplace inversion problem. In different tests we got similar results but specially good results were obtained when we used polynomials for $x(t)$.
The increase of the error for $t > 5$, exhibited in the the figures, is compatible with the norm used to measure the error: the weight e^{-t} allows these large absolute errors. On the other hand, Laguerre polynomials exhibit strong oscilations when N and t increase [1]; we believe that this fact also produces damaging effects. In the future we intend to test functions $\tilde{\phi}_i(t) = e^{-t/2}\phi_i(t)$, for $\phi_i(t)$ Laguerre polynomials; these functions form a complet set in $L^2(R^+)$ and $|\tilde{\phi}_i(t)| \leq 1$, $t > 0$, $i = 0, 1, \ldots$.

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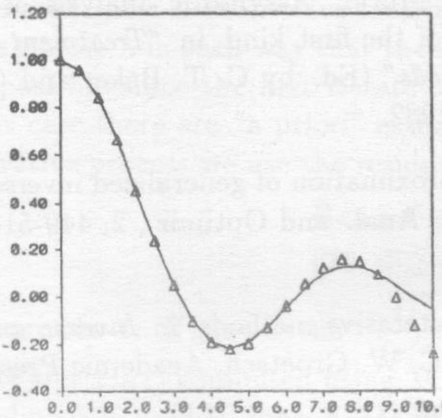

Fig. 1 - (a)

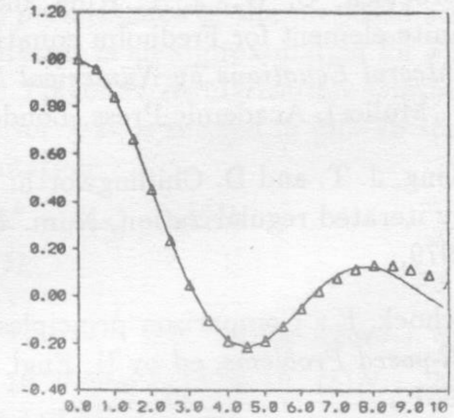

Fig. 1 - (b)

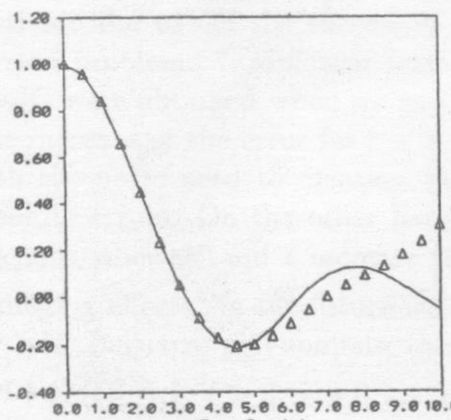

Fig. 2 - (a)

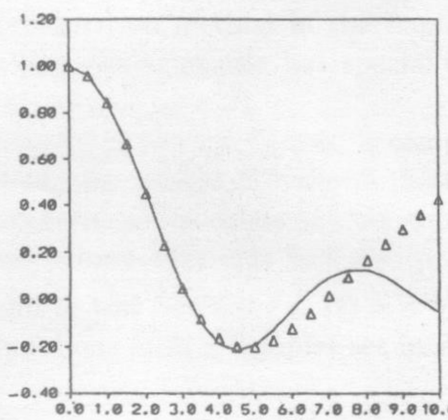

Fig. 2 - (b)

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