ON THE VELOCITY INDEPENDENT POTENTIALS

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Abstract. We construct one dimensional velocity independent potentials for which the Schrödinger differential equation can be solved by means of hypergeometric functions and confluent hypergeometric functions.

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On The Velocity Independent Potentials

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1. Introduction

Analytical functions are used to furnish numerical approximations. Thus, for simple sistems - soluble potential - the analytical solutions have an important role. Soluble potentials are the ones for which the Schrödinger differential equation can be solved by means of analytical functions.

Analytical solutions for soluble potentials are used, e.g., for the development of various approximation techniques; perturbation theory, Padé approximations, Hill determinants, continued fractions, and variational principles. Unfortunately there are few soluble potentials, almost all found without a single theoretical scheme.

In a recent paper, Aly and Barut [1] obtained exact solutions for the one dimensional (radial Schrödinger) equation with various classes of anharmonic oscillator potentials. Using an ansatz for the eigenfunctions, Kaushan [2] obtained an exact analytic solution for the Schrödinger equation for the doubly anharmonic potential. The coulomb atom in the presence of some anharmonic oscillators have been discussed by Dutra [3].

Many years ago Battacharjie and Sudarshan [4] presented a systematic method for constructing velocity independent potentials for which the Schrödinger differential equation can be solved by means of known functions. The authors have considered a general linear second order differential equa-

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tion which can be reduced to the one dimensional Schrödinger differential equation.

Another method to construct soluble potentials have been presented by Cordero and Ghirardi [5] using techniques of Lie algebras.

The treatments made by these authors are not within a single scheme. Firstly they studied the potentials which comes from a hypergeometric equation when it is reduced to a Schrödinger's like equation, and secondly they studied the potentials which comes from a confluent hypergeometric equation when it is reduced to a Schrödinger's like equation. We have made a confrontation of the methods and we conclude that it is not necessary to study both cases separately [6].

The purpose of this paper is to point out the fact that we can obtain potentials coming from the confluent hypergeometric equations when they are reduced to Schrödinger's like equations by means of potentials coming from the hypergeometric equations when they are reduced to a Schrödinger's like equation.

This paper is organized as follows: in section 2 we present the reduction of a general linear second order differential equation to a Schrödinger's like equation; in section 3 we show a particular case which reduces the equation to a hypergeometric equation; in section 4 we obtain the confluent hypergeometric equation by means of a limit process and finally we present an example, discussing the isotonic oscillator.

2. Transformation on the General Linear Second Order Differential Equation

In this section we show how to transform a general linear second order differential equation into a Schrödinger's like equation where we can identify the potential.

We consider the following linear second order differential equation

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0 \qquad (1.1)$$

where $u \equiv u(x)$.

Introducting the following transformations

$$x = f(r) \qquad\qquad u(x) = \rho(r)\phi(r) \tag{1.2}$$

where $\rho(r) \neq 0$, in the eq. (1.1) we obtain a differential equation for $\phi(r)$, as follows

$$\frac{d^2\phi(r)}{dr^2} + A(r)\frac{d\phi(r)}{dr} + B(r)\phi(r) = 0 \tag{1.3}$$

where $A(r)$ and $B(r)$ are given by

$$A(r) = 2\frac{d}{dr}\Big\{\ell n[\rho(r)]\Big\} + P(r)\frac{df(r)}{dr} - H(r) \tag{1.4}$$

and

$$\begin{aligned} B(r) = \frac{1}{\rho(r)}\frac{d^2\rho(r)}{dr^2} + Q(r)\Big\{\frac{df(r)}{dr}\Big\}^2 + \\ + \quad \frac{d}{dr}\Big\{\ell n[\rho(r)]\Big\}\cdot\Big[P(r)\frac{df(r)}{dr} - H(r)\Big] \end{aligned} \tag{1.5}$$

where $P(r) = p[f(r)]$, $Q(r) = q[f(r)]$ and $H(r) = \dfrac{d^2f(r)}{dr^2}\Big/\dfrac{df(r)}{dr}$.

Now, taking $A(r) = 0$ and $B(r) = k^2 - V(r)$ with $V(r)$ velocity independent we obtain the Schrödinger differential equation

$$-\frac{d^2\phi(r)}{dr^2} + V(r)\phi(r) = k^2\phi(r) \tag{1.6}$$

where $V(r)$ is the velocity independent potential and k^2 is a term involving the energy.

Then, the general form for the potential is the following

$$V(r) = k^2 - B(r) \tag{1.7}$$

where $B(r)$ is given by eq. (1.5). We note that the potential can be energy dependent. Aly and Barut [1] have obtained several energy dependent potentials.

3. Transformation on the Hypergeometric Equation

As a first particular case we have the hypergeometric differential equation

$$z(1-z)\frac{d^2F(z)}{dz^2}+[c-(a+b+1)z]\frac{dF(z)}{dz}-abF(z)=0 \qquad (2.1)$$

where a, b, c are constants.

In this case we obtain the potential which comes from a hypergeometric differential equation. Taking

$$P(r)=\frac{c-(a+b+1)f(r)}{f(r)[1-f(r)]} \quad \text{and} \quad Q(r)=\frac{-ab}{f(r)[1-f(r)]}$$

and using the condition $A(r)=0$ we obtain the following differential equation,

$$2\frac{d}{dr}\Big\{\ell n[\rho(r)]\Big\}+\frac{c-(a+b+1)f(r)}{f(r)[1-f(r)]}\frac{df(r)}{dr}-H(r)=0 \qquad (2.3)$$

The above equation can be integrated for $\rho(r)$ and then we get

$$\rho^2(r)=\frac{1}{M}\Big\{\frac{df(r)}{dr}f^{-c}(r)\,[1-f(r)]^{c-a-b-1}\Big\} \qquad (2.4)$$

where M is a constant.

Introducing the eq. (2.2) and eq (2.4) in the eq. (1.7) we obtain the following non linear third order differential equation

$$\frac{1}{2}\frac{f'''}{f'}-\frac{3}{4}\Big(\frac{f''}{f'}\Big)^2+\frac{1}{2}\Big\{\frac{c(2-c)}{2f^2}+\frac{a+b+1-c}{(1-f)^2}-\frac{(a+b+1-c)^2}{2(1-f)^2}-$$
$$-\frac{2ab-c(a+b+1-c)}{f(1-f)}\Big\}(f')^2=k^2-V(r) \qquad (2.5)$$

where $f\equiv f(r)$ and the prime denotes differentiation.

Now, to obtain soluble potentials in terms of hypergeometric functions we must find particular functions $f(r)$ and $\rho(r)$ which lead to the Schrödinger differential equation. Battacharjie and Sudarshan [4] have obtained functions for the Pöschl-Teller potential, Bargmann's potentials of the linear types and others. The above equation can be obtained by means of Lie algebra [5] with a convenient choice of the parameters [6].

4. Transformation on the Confluent Hypergeometric Equation

In this section we find the general form for the potentials associated with the confluent hypergeometric equation.

We know that the confluent hypergeometric equation is obtained by a limit process of the eq. (2.1) and results in

$$z\ \frac{d^2F(z)}{dz^2}+(c-z)\ \frac{dF(z)}{dz}-aF(z)=0 \tag{3.1}$$

where a, c are constants.

Introducing the function $f(r)=\varepsilon f(\varepsilon r)$ in eq.(2.5) and take the limit when $\varepsilon\rightarrow 0$ we obtain the following non linear third order differential equation

$$\frac{1}{2}\frac{f'''}{f'}-\frac{3}{4}\left(\frac{f''}{f'}\right)^2+\frac{c(2-c)}{4}\left(\frac{f'}{f}\right)^2+\left(\frac{c}{2}-a\right)\frac{(f')^2}{f}-\frac{1}{4}(f')^2=k^2-V(r) \tag{3.2}$$

which is the general form for the potential which is reduces a confluent hypergeometric form.

This result is the same obtained by Battacharjie and Sudarshan [4] but only after repeating the whole procedure which have been made for the case of the hypergeometric equation.

5. The Isotonic Oscillator

As an example we discuss the isotonic oscillator [7] which results in the same equation as the one for the two-body problems discussed by Calogero [8].

Taking the function $f(r)$ as

$$f(r)=\sqrt{\frac{w}{2}}\,r^2$$

where w is a constant, and substituing in the eq. (3.2) we obtain for the potential

$$V(r)=\frac{\hbar^2}{2\mu}\left\{\frac{wr^2}{2}+\frac{(2c-1)(2c-3)}{4r^2}\right\}$$

which is the potential of the isotonic oscillator. The solution of the eq.(1.6) with the above potential can be find in terms of the confluent hypergeometric function, which are solutions of the eq. (3.1).

Conclusion

In this paper we obtained the general case of the potential associated to the confluent hypergeometric equation when it is reduced to a Schrödinger's like equation by means of a limit process without making the exhaustive calculation done, e.g. by [4].

Many others potentials can be obtained for a convenient choice of the function $f(r)$. The important fact is that, when we have $f(r)$, we obtain the potential (depending or not on the energy) which implies that the solution of the corresponding Schrödinger equation is a confluent hypergeometric equation. The same is valid for the potential associated to the hypergeometric equation.

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